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Iterated remainder operator, tests for multiple convergence of series, and solutions of difference equations

Janusz Migda*

*Correspondence:
migda@amu.edu.pl
Faculty of Mathematics and
Computer Science, A. Mickiewicz
University, Umultowska 87, Poznań,
61-614, Poland

Abstract

We establish some properties of iterations of the remainder operator which assigns to any convergent series the sequence of its remainders. Moreover, we introduce the spaces of multiple absolute summable sequences. We also present some tests for multiple absolute convergence of series. These tests extend the well-known classical tests for absolute convergence of series. For example we generalize the Raabe, Gauss, and Bertrand tests. Next we present some applications of our results to the study of asymptotic properties of solutions of difference equations. We use the spaces of multiple absolute summable sequences as the measure of approximation.

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1 Introduction

Let r denote the operator which assigns to any convergent series the sequence of its remainders. The purpose of this paper is to study the basic properties of iterations r^m of r and apply these results to the study of asymptotic properties of solutions of difference equations. We also study the spaces of multiple absolute summable sequences. Moreover, we obtain extensions of some classical tests for absolute convergence of series.

The paper is organized as follows. In Section 2, we introduce our notation and terminology. In Section 3, we define the iterations of the remainder operator and the spaces $S(m)$ of m -times summable sequences. Moreover, we establish some basic properties of r^m . For example, we show that

$$\Delta^m r^m x = (-1)^m x \quad \text{and} \quad r^m \Delta^m z = (-1)^m z \quad (1)$$

for any sequence $x \in S(m)$ and any sequence z convergent to zero. This means that the operator $(-1)^m r^m$ is inverse to the restriction $\Delta^m|Z$ where Z denotes the space of all convergent to zero sequences.

In Section 4, we introduce the spaces $A(m)$ of absolutely m -times summable sequences and establish the relationships between $A(m)$ and the spaces $O(n^s)$. We also extend some

classical tests for absolute convergence of series. For example, the Raabe test states that

$$\text{if } \liminf n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > 1, \quad \text{then } \sum_{n=1}^{\infty} |a_n| < \infty.$$

In Lemma 4.5 we show that

$$\text{if } \liminf n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > 2, \quad \text{then } \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} |a_k| < \infty$$

and, more generally,

$$\text{if } \liminf n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > m, \quad \text{then } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}| < \infty.$$

On the other hand,

$$\text{if } n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) \leq m \quad \text{for large } n, \quad \text{then } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}| = \infty.$$

Similarly in Lemma 4.4 we show that

$$\limsup \frac{\log |a_n|}{\log n} < -m \quad \text{implies } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}| < \infty$$

and

$$\frac{\log |a_n|}{\log n} \geq -m \quad \text{for large } n \quad \text{implies } \sum_{i_1=1}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} |a_{i_m}| = \infty.$$

For $x \in S(m)$, by definition, we have

$$r^m(x)(n) = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \cdots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}. \tag{2}$$

If $|x| \in S(m)$, i.e., $x \in A(m)$, then $x \in S(m)$ and

$$r^m(x)(n) = \sum_{k=0}^{\infty} \binom{m-1+k}{m-1} x_{n+k} = \sum_{i=n}^{\infty} \binom{m+i-n-1}{m-1} x_i. \tag{3}$$

We use (3) to obtain basic properties of the operator r^m . At the end of Section 4, using all possible $\Delta^p A(m)$ and $r^p A(m)$, we obtain an infinite stratification of the space of all sequences convergent to zero. This stratification induces a substratification of $A(m)$ for any m .

In Section 5, we present some applications of our results to the study of asymptotic properties of solutions of difference equations. We use the spaces $A(p)$ as the measure of approximation. For example, we apply our results and fixed point theorems to the study of

solutions with prescribed asymptotic behavior. More precisely, using the Schauder fixed point theorem and the Knaster-Tarski fixed point theorem, we establish conditions under which for a given sequence b and a solution y of the equation $\Delta^m y = b$ there exists a solution x of the equation

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n \quad (4)$$

such that

$$x \in y + A(p).$$

We also show that if $a, b \in A(m+p)$ and x is a solution of (4) such that the sequence $(f(x_n))$ is bounded, then

$$x \in \text{Ker } \Delta^m + A(p).$$

Hence x is asymptotically polynomial. The equality

$$\Delta^m ((-1)^m r^m x) = x$$

for $x \in S(m)$, which is a consequence of (1), plays a crucial role in the application of fixed point theorems to the study of solutions of difference equations. The value $r^m x$ is used, mainly implicitly, in the study of solutions with prescribed asymptotic behavior. In some papers the multiple sums (2) appear explicitly; see for example [1–4] or [5]. This paper is a continuation of the papers [4, 6] and [7]. Our studies were inspired by the papers [8–12] and the papers [13–17], and [18].

Some applications of our results to the study of asymptotic properties of solutions of nonautonomous difference equations are presented in [19].

2 Notation and terminology

Let \mathbb{N} , \mathbb{Z} , \mathbb{R} denote the set of positive integers, the set of all integers and the set of real numbers, respectively. The space of all sequences $x : \mathbb{N} \rightarrow \mathbb{R}$ we denote by SQ.

If $p, k \in \mathbb{Z}$, $p \leq k$, then $\mathbb{N}(p)$, $\mathbb{N}(p, k)$ denote the sets defined by

$$\mathbb{N}(p) = \{p, p+1, \dots\}, \quad \mathbb{N}(p, k) = \{p, p+1, \dots, k\}.$$

If x, y in SQ, then xy denotes the sequence defined by pointwise multiplication

$$xy(n) = x_n y_n.$$

Moreover, $|x|$ denotes the sequence defined by $|x|(n) = |x_n|$ for every n .

We use the symbols ‘big O’ and ‘small o’ in the usual sense but for $a \in \text{SQ}$ we also regard $o(a)$ and $O(a)$ as subspaces of SQ. More precisely, let

$$o(1) = \{x \in \text{SQ} : x \text{ is convergent to zero}\}, \quad O(1) = \{x \in \text{SQ} : x \text{ is bounded}\}$$

and for $a \in \text{SQ}$ let

$$o(a) = a o(1) = \{ax : x \in o(1)\}, \quad O(a) = a O(1) = \{ax : x \in O(1)\}.$$

For $m \in \mathbb{N}(0)$ we define

$$\text{Pol}(m - 1) = \text{Ker } \Delta^m = \{x \in \text{SQ} : \Delta^m x = 0\}.$$

Then $\text{Pol}(m - 1)$ is the space of all polynomial sequences of degree less than m . For $m \in \mathbb{N}(0)$, $k \in \mathbb{N}$ we define numbers s_k^m by

$$s_k^m = \binom{m+k-1}{m} = \frac{k(k+1) \cdots (k+m-1)}{m!}.$$

For $p \in \mathbb{N}$ we define

$$\text{Fin}(p) = \{x \in \text{SQ} : x_n = 0 \text{ for } n \geq p\}.$$

We say that a subset U of a metric space X is a uniform neighborhood of a subset Y of X if there exists a positive number ε such that

$$\bigcup_{y \in Y} B(y, \varepsilon) \subset U,$$

where $B(y, \varepsilon)$ denotes an open ball of radius ε about y .

A sequence $x \in \text{SQ}$ is called nonoscillatory if $x_n x_{n+1} \geq 0$ for large n .

Assume $f : \mathbb{R} \rightarrow \mathbb{R}$. We say that a sequence $x \in \text{SQ}$ is f -bounded if the sequence $f \circ x$ is bounded. Note that $x \circ \sigma$ is f -bounded for any f -bounded sequence x and any sequence $\sigma : \mathbb{N} \rightarrow \mathbb{N}$.

Example 2.1 If $f(t) = e^t$, then f -boundedness of a sequence x is equivalent to the boundedness above of x . Assume $f(t) = t^{-1}$ for $t \neq 0$, $f(0) = 3$ and let $L(x)$ denote the set of limit points of a given sequence x . Then f -boundedness of x is equivalent to the condition $0 \notin L(x)$.

3 Iterated remainder operator

In this section we introduce the remainder operator r , the spaces $S(m)$ of m -times summable sequences and iterations $r^m : S(m) \rightarrow S(0)$. Next in Lemma 3.1 we establish some basic properties of r^m and the relationships between r^m and Δ^m . Let

$$S(0) = o(1) = \{x \in \text{SQ} : \lim x_n = 0\},$$

$$S(1) = \left\{ x \in \text{SQ} : \text{the series } \sum x_n \text{ is convergent} \right\}.$$

For $x \in S(1)$, we define the sequence $r(x)$ by the formula

$$r(x)(n) = \sum_{j=n}^{\infty} x_j.$$

Then $r(x) \in S(0)$ and we obtain the remainder operator

$$r : S(1) \rightarrow S(0).$$

For $m \in \mathbb{N}$ we define, by induction, the linear space $S(m + 1)$ and the linear operator $r^{m+1} : S(m + 1) \rightarrow S(0)$ by

$$S(m + 1) = \{x \in S(m) : r^m(x) \in S(1)\}, \quad r^{m+1}(x) = r(r^m(x)).$$

The value $r^m(x)(n)$ we denote also by $r_n^m(x)$ or simply $r_n^m x$.

Lemma 3.1 *Assume $x, y \in SQ$, $m, k \in \mathbb{N}$, $p \in \mathbb{N}(0)$ and $q \in \mathbb{N}(0, m - 1)$. Then*

- (01) *if $|x| \in S(m)$, then $x \in S(m)$ and $|r^m x| \leq r^m |x|$,*
- (02) *$|x| \in S(m)$ if and only if $\sum_{n=1}^{\infty} s_n^{m-1} |x_n| < \infty$,*
- (03) *$|x| \in S(m)$ if and only if $\sum_{n=1}^{\infty} n^{m-1} |x_n| < \infty$,*
- (04) *$|x| \in S(m)$ if and only if $O(x) \subset S(m)$,*
- (05) *if $|x| \in S(m)$, then $r_n^m x = s_1^{m-1} x_n + s_2^{m-1} x_{n+1} + s_3^{m-1} x_{n+2} + \dots$,*
- (06) *if $|x| \in S(m)$, then $r_k^m |x| \leq \sum_{n=k}^{\infty} n^{m-1} |x_n|$,*
- (07) *if $x \in S(m)$, then $\Delta^m r^m x = (-1)^m x$,*
- (08) *if $x = o(1)$, then $\Delta^m x \in S(m)$ and $r^m \Delta^m x = (-1)^m x$,*
- (09) $\Delta^m S(0) = S(m)$, $r^m S(m) = S(0)$,
- (10) $\Delta^p S(m) = S(m + p)$, $r^p S(m + p) = S(m)$,
- (11) *if $x, y \in S(m)$ and $x_n \leq y_n$ for $n \geq p$, then*

$$r_n^m x \leq r_n^m y \quad \text{for } n \geq p,$$

- (12) *if $x \in S(m)$ and $y_n = x_n$ for $n \geq p$, then $y \in S(m)$ and*

$$r_n^m y = r_n^m x \quad \text{for } n \geq p,$$

- (13) *if $y \in S(m)$ and $0 \leq x \leq y$, then $x \in S(m)$,*
- (14) *if $|x| \in S(m)$ and y is bounded, then $yx \in S(m)$ and*

$$|r^m(yx)| \leq |y| r^m |x|,$$

- (15) *if $|yx| \in S(m)$, $|y|$ is nondecreasing and positive, then $|x| \in S(m)$,*

$$|y| r^m |x| \leq r^m |yx| \quad \text{and} \quad \Delta^q r^m x = o(n^{-q} y^{-1}),$$

- (16) *if $t \in (-\infty, 0]$ and $\sum_{n=1}^{\infty} n^{m-t-1} |x_n| < \infty$, then $\Delta^q r^m x = o(n^{t-q})$.*

Proof Assertion (01) is proved in Lemma 1 of [6]. Assertion (02) is proved in Lemma 2 of [6]. (03) is proved in Lemma 3 of [6]. Since $|x| \in O(x)$ we see that (04) is a consequence of (03). (05) is proved in Lemma 2 of [6]. Assertion (06) follows from (05). Assertion (07) is proved in Lemma 5 of [6]. Assertion (08) follows from Lemma 6 of [6]. Assertion (09) is an easy consequence of (08). Assertion (10) is a consequence of (09) and (08). Assertion (11) is obvious for $m = 1$. For $m > 1$ it can easily be proved by induction. Assertion (12) is

an easy consequence of (11). Assertion (13) is well known for $m = 1$. Assume it is true for certain $m \geq 1$. Let $0 \leq x \leq y$ and $y \in S(m+1)$. Then $y \in S(m)$ and, by assumption, $x \in S(m)$. Moreover, using (11) we have

$$0 \leq r^m x \leq r^m y$$

and, by (10), $r^m y \in S(1)$. Hence $r^m x \in S(1)$ and we obtain $x \in S(m+1)$. Assertion (14) follows from (03) and Lemma 4 of [6]. Assume $|y|$ is nondecreasing and positive and $|yx| \in S(m)$. Then the sequence y^{-1} is bounded and using (04) we have $|x| = |y|^{-1}|yx| \in S(m)$. Moreover, using (05), we have

$$\begin{aligned} |y_n| r_n^m |x| &= |y_n| s_1^{m-1} |x_n| + |y_n| s_2^{m-1} |x_{n+1}| + |y_n| s_3^{m-1} |x_{n+2}| + \dots \\ &\leq s_1^{m-1} |y_n x_n| + s_2^{m-1} |y_{n+1} x_{n+1}| + s_3^{m-1} |y_{n+2} x_{n+2}| + \dots = r_n^m |yx| \end{aligned}$$

for any n . Hence $|y| r^m |x| \leq r^m |yx|$. Using (01), we obtain

$$|y| |r^m x| \leq |y| r^m |x| \leq r^m |yx| = o(1) \quad \text{and} \quad r^m x = o(y^{-1}). \tag{5}$$

By (03), $|y n^q x| \in S(m-q)$. Hence, replacing y by $n^q y$ and m by $m-q$ in (5), we obtain $r^{m-q} x = o(n^{-q} y^{-1})$. Therefore

$$\Delta^q r^m x = \Delta^q r^q r^{m-q} x = (-1)^q r^{m-q} x = o(n^{-q} y^{-1})$$

and we obtain (15). Using (03) and taking $y_n = n^{-t}$ in (15) we obtain (16). □

Remark 3.1 For $x \in S(m)$ and $n \in \mathbb{N}$, by definition of r^m , we have

$$r_n^m x = \sum_{i_1=n}^{\infty} \sum_{i_2=i_1}^{\infty} \dots \sum_{i_m=i_{m-1}}^{\infty} x_{i_m}.$$

Moreover, if $|x| \in S(m)$, then, by Lemma 3.1(05), we have

$$r_n^m x = \sum_{k=0}^{\infty} \binom{m-1+k}{m-1} x_{n+k}.$$

Remark 3.2 By Lemma 3.1(07) and (08) the restriction $\Delta^m |S(0) : S(0) \rightarrow S(m)$ is a bijection with inverse $(-1)^m r^m$.

Remark 3.3 If X is a linear subspace of $S(0)$ and Y is a linear subspace of $S(m)$, then, using Lemma 3.1(08) and (07), we have

$$r^m \Delta^m X = X, \quad \Delta^m r^m Y = Y.$$

For any $m \in \mathbb{N}$ we have $S(m) \subset S(0) = o(1) = o(n^0)$. In the following example we show that for any $m \in \mathbb{N}$ and any $\varepsilon > 0$ there exists a sequence x such that

$$x \in S(m) \setminus O(n^{-\varepsilon})$$

and, moreover, $\lim_{n \rightarrow \infty} n^\varepsilon |x_n| = \infty$.

Example 3.1 Let $m \in \mathbb{N}$ and $\varepsilon > 0$. Choose $\delta \in (0, \varepsilon)$. Let $y_n = (-1)^n n^{-\delta}$ and $x = \Delta^m y$. Then $y \in S(0)$, $x \in S(m)$ and

$$\begin{aligned} |x_n| &= |\Delta^m y_n| = \left| \sum_{i=0}^m (-1)^{m+i} \binom{m}{i} y_{n+i} \right| \\ &= \left| \sum_{i=0}^m (-1)^{m+i} \binom{m}{i} (-1)^{n+i} (n+i)^{-\delta} \right| = \left| (-1)^{m+n} \sum_{i=0}^m \binom{m}{i} (n+i)^{-\delta} \right| \geq n^{-\delta}. \end{aligned}$$

Hence $n^\varepsilon |x_n| \geq n^{\varepsilon-\delta} \rightarrow \infty$ and $x \in S(m) \setminus O(n^{-\varepsilon})$.

4 Absolute summable sequences

In this section we introduce the spaces $A(m)$ of absolutely m -times summable sequences. We establish the relationships between $A(m)$ and the spaces $o(n^s)$ and $O(n^s)$. There exist many tests for the absolute convergence of series. Most of them may be extended to the case $A(m)$. For example we present five of them in Lemmas 4.3-4.7. At the end of the section, using all possible $\Delta^p A(m)$ and $r^p A(m)$ we obtain an infinite stratification of the set $S(0) \setminus A(\infty)$.

For $m \in \mathbb{N}(0)$ we define the set $A(m)$ by

$$A(m) = \{a \in SQ : |a| \in S(m)\}.$$

Moreover, let

$$A(\infty) = \bigcap_{k=0}^{\infty} A(k), \quad o(n^{-\infty}) = \bigcap_{s \in \mathbb{R}} o(n^s), \quad \text{Fin} = \bigcup_{p=1}^{\infty} \text{Fin}(p).$$

Remark 4.1 Note that $A(0) = S(0)$ and the condition $a \in A(1)$ is equivalent to the absolute convergence of the series $\sum_{n=1}^{\infty} a_n$. Moreover, $A(m)$ is a linear subspace of $S(m)$ for any $m \in \mathbb{N}(0)$. Note also that if $p \in \mathbb{N}$ and $\lambda \in (-1, 1)$, then

$$\text{Fin}(p) \subset \text{Fin} \subset o(\lambda^n) \subset A(\infty).$$

Lemma 4.1 Assume $a \in SQ$, $m \in \mathbb{N}(0)$ and $s \in \mathbb{R}$. Then

- (a) $a \in A(m+1) \Leftrightarrow (n^m a_n) \in A(1)$,
- (b) $a \in A(m+1) \Leftrightarrow (na_n) \in A(m)$,
- (c) $(n^s) \in A(m) \Leftrightarrow s < -m$.

Proof Assertion (a) is a consequence of Lemma 3.1(03), (b) is a consequence of (a). Assertion (c) follows from (b) and from the fact that the condition $(n^s) \in A(1)$ is equivalent to the condition $s < -1$. □

Example 4.1 Let $m \in \mathbb{N}(0)$. If we define a sequence x by

$$x_n = \frac{1}{n^m \log n},$$

then, using Lemma 4.1(a), we obtain $x \in o(n^{-m}) \setminus A(m)$.

Example 4.2 Assume $m \in \mathbb{N}$ and $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $p^{-1} < \varepsilon$. Let

$$A = \{k^{2p} : k \in \mathbb{N}\}, \quad B = \mathbb{N} \setminus A, \quad a_n = \begin{cases} n^{-p^{-1}} & \text{for } n \in A, \\ n^{-2} & \text{for } n \in B. \end{cases}$$

Then

$$\sum_{n \in A} a_n = \sum_{k=1}^{\infty} (k^{2p})^{-p^{-1}} = \sum_{k=1}^{\infty} k^{-2} < \infty, \quad \sum_{n \in B} a_n \leq \sum_{n=1}^{\infty} n^{-2} < \infty,$$

and $n^\varepsilon a_n = n^{\varepsilon-p^{-1}}$ for $n \in A$. Hence $\limsup_{n \rightarrow \infty} n^\varepsilon a_n = \infty$ and we obtain $a \in A(1) \setminus O(n^{-\varepsilon})$. Let $b_n = n^{1-m} a_n$. Then, by Lemma 4.1(a),

$$b \in A(m) \setminus O(n^{-m+1-\varepsilon}).$$

Remark 4.2 Note that for real s, t the conditions $o(n^s) \subset o(n^t)$ and $O(n^s) \subset O(n^t)$ are equivalent to the condition $s \leq t$.

Lemma 4.2 Assume $m \in \mathbb{N}$, $\varepsilon > 0$ and $s \in \mathbb{R}$. Then

- (a) $O(n^{-m-\varepsilon}) \subset A(m) \subset o(n^{-m+1})$,
- (b) $o(n^s) \subset A(m) \Leftrightarrow s < -m$,
- (c) $A(m) \subset O(n^s) \Leftrightarrow s \geq -m + 1$,
- (d) $A(\infty) = o(n^{-\infty})$.

Proof Assertion (a) follows from Lemma 4.1, (b) is a consequence of (a) and Example 4.1, (c) is a consequence of (a) and Example 4.2. Let $m \in \mathbb{N}(0)$. If $s < -m$, then, by (b), $o(n^s) \subset A(m)$. Hence $o(n^{-\infty}) \subset A(m)$ for any $m \in \mathbb{N}(0)$. Therefore

$$o(n^{-\infty}) \subset A(\infty).$$

Let $s \in \mathbb{R}$. Choose $m \in \mathbb{N}(0)$ such that $m > 1 - s$. If $a \in A(m)$, then, by the convergence of the series $\sum_{n=1}^{\infty} n^{m-1} |a_n|$, we have $n^{m-1} a_n = o(1)$ and so $a \in o(n^{1-m}) \subset o(n^s)$. Hence $A(m) \subset o(n^s)$. Therefore $A(\infty) \subset o(n^s)$ for any $s \in \mathbb{R}$ and we obtain

$$A(\infty) \subset o(n^{-\infty}). \quad \square$$

Lemma 4.3 (Comparison test) Assume $a, b \in \text{SQ}$ and $m \in \mathbb{N}$. Then

- (a) if $|a_n| \leq |b_n|$ for large n and $b \in A(m)$, then $a \in A(m)$,
- (b) if $|a_n| \geq |b_n|$ for large n and $b \notin A(m)$, then $a \notin A(m)$,
- (c) if $\limsup |a_n/b_n| < \infty$ and $b \in A(m)$, then $a \in A(m)$,
- (d) if $\liminf |a_n/b_n| > 0$ and $a \in A(m)$, then $b \in A(m)$,
- (e) if $|a_{n+1}/a_n| \leq |b_{n+1}/b_n|$ for large n and $b \in A(m)$, then $a \in A(m)$.

Proof Assertion (a) follows from Lemma 3.1(13). Assertions (b), (c), (d), and (e) are consequences of (a). □

Lemma 4.4 (Generalized logarithmic test) *Assume $a \in \text{SQ}$, $m \in \mathbb{N}$ and*

$$u_n = \frac{\log |a_n|}{\log n}.$$

Then

- (a) *if $\limsup u_n < -m$, then $a \in A(m)$,*
- (b) *if $u_n \geq -m$ for large n , then $a \notin A(m)$,*
- (c) *if $\liminf u_n > -m$, then $a \notin A(m)$,*
- (d) *if $\lim u_n = -\infty$, then $a \in A(\infty)$.*

Proof If $\limsup u_n < -m$, then there exists a number $s < -m$ such that $u_n < s$ for large n . Then $|a_n| \leq n^s$ for large n . Hence, using Lemma 4.1 and Lemma 4.3, we obtain (a). If $u_n \geq -m$ for large n , then $|a_n| \geq n^{-m}$ for large n and (b) follows from Lemma 4.3 and the fact that $(n^{-m}) \notin A(m)$. Assertion (c) follows immediately from (b), and (d) is a consequence of (a). \square

Lemma 4.5 (Generalized Raabe test) *Assume $a \in \text{SQ}$, $m \in \mathbb{N}$ and*

$$u_n = n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right).$$

Then

- (a) *if $\liminf u_n > m$, then $a \in A(m)$,*
- (b) *if $u_n \leq m$ for large n , then $a \notin A(m)$,*
- (c) *if $\limsup u_n < m$, then $a \notin A(m)$,*
- (d) *if $\lim u_n = \infty$, then $a \in A(\infty)$.*

Proof For $m = 1$ assertion (a) follows from the usual Raabe test. Assume it is true for certain $m \geq 1$ and $\liminf u_n > m + 1$. Let

$$b_n = na_n, \quad w_n = n \left(\frac{|b_n|}{|b_{n+1}|} - 1 \right).$$

Then

$$w_n = n \left(\frac{n|a_n|}{(n+1)|a_{n+1}|} - \frac{n+1}{n+1} \right) = \frac{n}{n+1} \left(\frac{n|a_n|}{|a_{n+1}|} - n - 1 \right) = \frac{n}{n+1} (u_n - 1).$$

Hence $\liminf w_n = \liminf u_n - 1 > m$ and, by inductive hypothesis, $b \in A(m)$. Hence, by Lemma 4.1(b), $a \in A(m + 1)$ and we obtain (a). Similarly we may obtain (b), by taking $b_n = na_n$, using the usual Raabe test and Lemma 4.1(b). Assertion (c) follows from (b), and (d) is a consequence of (a). \square

Lemma 4.6 (Generalized Gauss test) *Let $a \in \text{SQ}$, $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $s < -1$ and*

$$\frac{|a_n|}{|a_{n+1}|} = \alpha + \frac{\beta}{n} + O(n^s).$$

Then

- (a) *if $\alpha > \lambda > 1$, then $a \in o(\lambda^n)$,*

- (b) if $\alpha < 1$, then $a \notin o(1)$,
- (c) if $\alpha = 1$ and $\beta > m$, then $a \in A(m)$,
- (d) if $\alpha = 1$ and $\beta \leq m$, then $a \notin A(m)$.

Proof Note that

$$\lim_{n \rightarrow \infty} \frac{|a_n|}{|a_{n+1}|} = \alpha.$$

Hence (a) and (b) follow from the d'Alembert ratio test. For $m = 1$ assertions (c) and (d) follow from the usual Gauss test. Assume they are true for certain $m \geq 1$. Let $b_n = na_n$, $u_n = 1/(n + 1)$. Then

$$\begin{aligned} \frac{|b_n|}{|b_{n+1}|} &= \frac{n}{n+1} \frac{|a_n|}{|a_{n+1}|} = (1 - u_n) \left(1 + \frac{\beta}{n} + O(n^s) \right) \\ &= 1 + \frac{\beta}{n} + O(n^s) - \frac{1}{n+1} - \frac{\beta}{n(n+1)} - \frac{1}{n+1} O(n^s) \\ &= 1 + \frac{\beta - 1}{n} + \left(\frac{1}{n} - \frac{1}{n+1} \right) + O(n^s) + O(n^{-2}) + O(n^{s-1}) \\ &= 1 + \frac{\beta - 1}{n} + O(n^{s'}) \end{aligned}$$

for certain $s' < -1$. If $\beta > m + 1$, then $\beta - 1 > m$ and, by inductive hypothesis, $b \in A(m)$. Similarly, if $\beta \leq m + 1$, then $b \notin A(m)$. Now, assertions (c) and (d) follow from Lemma 4.1(b). \square

Lemma 4.7 (Generalized Bertrand test) *Assume $a \in SQ$, $m \in \mathbb{N}$ and*

$$\frac{|a_n|}{|a_{n+1}|} = 1 + \frac{m}{n} + \frac{\lambda_n}{n \log n}. \tag{6}$$

Then

- (a) if $\liminf \lambda_n > 1$, then $a \in A(m)$,
- (b) if $\lambda_n \leq 1$ for large n , then $a \notin A(m)$,
- (c) if $\limsup \lambda_n < 1$, then $a \notin A(m)$.

Proof Let $b_n = na_n$. Then

$$\begin{aligned} \frac{|b_n|}{|b_{n+1}|} &= \frac{n}{n+1} \left(1 + \frac{m}{n} + \frac{\lambda_n}{n \ln n} \right) \\ &= \frac{n}{n+1} + \frac{m}{n+1} + \frac{\lambda_n}{(n+1) \log n} \\ &= 1 + \frac{m-1}{n+1} + \left(\frac{n}{n+1} \right) \frac{\lambda_n}{n \log n} \\ &= 1 + \frac{m-1}{n} - c_n + \frac{z_n \lambda_n}{n \log n}, \end{aligned}$$

where

$$z_n = \frac{n}{n+1}, \quad c_n = \frac{m-1}{n} - \frac{m-1}{n+1} = \frac{m-1}{n(n+1)}.$$

Let

$$w_n = c_n n \log n = \frac{(m-1) \log n}{n+1}, \quad \tau_n = z_n \lambda_n - w_n.$$

Then

$$\lim w_n = 0, \quad \lim z_n = 1, \quad \liminf \tau_n = \liminf \lambda_n \tag{7}$$

and

$$\frac{|b_n|}{|b_{n+1}|} = 1 + \frac{m-1}{n} + \frac{\tau_n}{n \log n}. \tag{8}$$

Moreover,

$$w_n \geq 0, \quad 0 < z_n < 1, \quad \tau_n \leq \lambda_n. \tag{9}$$

For $m = 1$ assertion (a) follows from the classical Bertrand test. Assume it is true for certain $m-1 \geq 1$. Then using the inductive hypothesis, (8), and (7), we have $b \in A(m-1)$. Hence, by Lemma 4.1, $a \in A(m)$ and we obtain (a). Analogously, using (9), we obtain (b). Assertion (c) is a consequence of (b). The proof is complete. \square

Remark 4.3 Computing λ_n from (6) we have

$$\lambda_n = \left(n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) - m \right) \log n. \tag{10}$$

Replacing (6) by (10) in Lemma 4.7 one can obtain another form of the Bertrand test.

Lemma 4.8 Assume $m \in \mathbb{N}$. Then

- (a) $\Delta A(m) \subset A(m)$,
- (b) $A(m) \subset rA(m)$,
- (c) $rA(m) \subset A(m-1)$,
- (d) $A(m) \subset \Delta A(m-1)$.

Proof For $y \in \text{SQ}$ let $Ey \in \text{SQ}$ be defined by

$$Ey(n) = y(n+1).$$

Let $x \in S(m)$. By Lemma 3.1(09) there exists $y \in S(0)$ such that $x = \Delta^m y$. Obviously $E\Delta y = \Delta Ey$ and, by induction, $E\Delta^m y = \Delta^m Ey$. Moreover, $Ey \in S(0)$. Hence

$$Ex = E\Delta^m y = \Delta^m Ey \in S(m).$$

Now assume $x \in A(m)$. Then $|x| \in S(m)$ and $|Ex| = E|x| \in S(m)$. Hence

$$0 \leq |\Delta x| \leq |Ex| + |x| \in S(m).$$

By Lemma 3.1(13) we have $\Delta x \in A(m)$. Hence

$$\Delta A(m) \subset A(m) \quad \text{and} \quad A(m) = r\Delta A(m) \subset rA(m).$$

From $|x| \in S(m)$ we have $r|x| \in S(m-1)$. Moreover, $0 \leq |rx| \leq r|x|$. Hence $|rx| \in S(m-1)$. Therefore $rx \in A(m-1)$ and we obtain

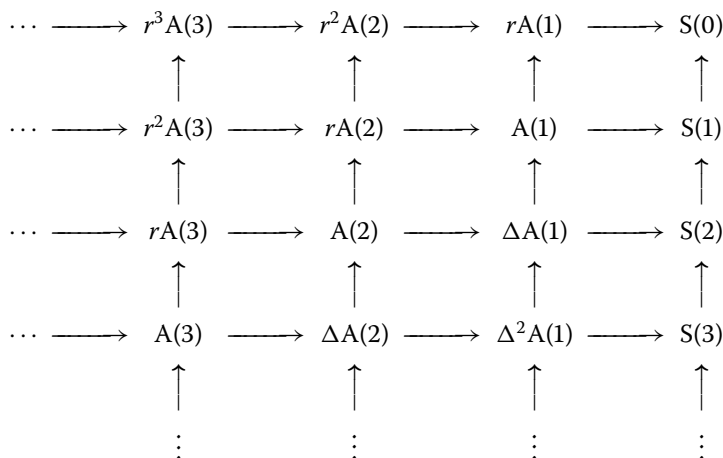
$$rA(m) \subset A(m-1), \quad A(m) = \Delta rA(m) \subset \Delta A(m-1).$$

The proof is complete. □

Example 4.3 Let $m \in \mathbb{N}$, $x_n = n^{-m}$, $y = \Delta x$. By Theorem 2.2 in [7], we have $y \in O(n^{-m-1})$. Hence, by Lemma 4.2, $y \in A(m)$. On the other hand $x \notin A(m)$ and we obtain $y = \Delta x \notin \Delta A(m)$. Therefore $y \in A(m) \setminus \Delta A(m)$.

Example 4.4 Let $m \in \mathbb{N}$, $s \in (m, m+1]$, $x_n = (-1)^n n^{-s}$. Then, by Lemma 4.1, $x \in A(m) \setminus A(m+1)$. Assume $x \in rA(m+1)$. Choose $y \in A(m+1)$ such that $x = ry$. Then, using Lemma 3.1(07), we obtain $\Delta x = \Delta ry = -y \in A(m+1)$. Since x is alternating we have $|\Delta x| \geq |x|$. Hence, by Lemma 4.3, $y = \Delta x \notin A(m+1)$. This contradiction shows that $x \in A(m) \setminus rA(m+1)$.

Remark 4.4 Using Lemma 4.8 we obtain the following infinite diagram, where arrows denote inclusions:



Using Remark 3.2 and Example 4.3 we can see that any vertical arrow represents a proper inclusion. Analogously, using Remark 3.2 and Example 4.4 we can see that any horizontal arrow represents a proper inclusion.

Remark 4.5 Introducing new notation

$$\begin{aligned}
 S_k^{k+p} &= \Delta^p A(k), & S_k^{k-m} &= r^m A(k), \\
 S_k^\infty &= \bigcap_{n=0}^\infty S_k^n, & S_\infty^n &= \bigcap_{k=0}^\infty S_k^n, & S_\infty^\infty &= \bigcap_{k=0}^\infty S_k^k
 \end{aligned}$$

for $k, p \in \mathbb{N}(0)$ and $m \in \mathbb{N}(0, k)$ we can extend the diagram from Remark 4.4 in the following way:

$$\begin{array}{cccccc}
 S_{\infty}^0 & \longrightarrow & \cdots & \longrightarrow & S_3^0 & \longrightarrow & S_2^0 & \longrightarrow & S_1^0 & \longrightarrow & S_0^0 \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 S_{\infty}^1 & \longrightarrow & \cdots & \longrightarrow & S_3^1 & \longrightarrow & S_2^1 & \longrightarrow & S_1^1 & \longrightarrow & S_0^1 \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 S_{\infty}^2 & \longrightarrow & \cdots & \longrightarrow & S_3^2 & \longrightarrow & S_2^2 & \longrightarrow & S_1^2 & \longrightarrow & S_0^2 \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 S_{\infty}^3 & \longrightarrow & \cdots & \longrightarrow & S_3^3 & \longrightarrow & S_2^3 & \longrightarrow & S_1^3 & \longrightarrow & S_0^3 \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \vdots & & & & \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 S_{\infty}^{\infty} & \longrightarrow & \cdots & \longrightarrow & S_3^{\infty} & \longrightarrow & S_2^{\infty} & \longrightarrow & S_1^{\infty} & \longrightarrow & S_0^{\infty}
 \end{array}$$

Note that if $k, n, p \in \mathbb{N}(0)$ and $m \in \mathbb{N}(0, n)$, then

$$\begin{aligned}
 \Delta^p S_k^n &= S_k^{n+p}, & r^m S_k^n &= S_k^{n-m}, & \Delta^p S_{\infty}^n &= S_{\infty}^{n+p}, & r^m S_{\infty}^n &= S_{\infty}^{n-m}, \\
 \Delta^m S_k^{\infty} &= S_k^{\infty} = r^m S_k^{\infty}, & \Delta^m S_{\infty}^{\infty} &= S_{\infty}^{\infty} = r^m S_{\infty}^{\infty}.
 \end{aligned}$$

Moreover,

$$S_{\infty}^{\infty} = A(\infty) = o(n^{-\infty}).$$

5 Approximative solutions of difference equations

In this section we present some applications of our previous results in the study of asymptotic properties of solutions of difference equations. We use the spaces $A(p)$ to measure the ‘degree of approximation’.

Assume $m \in \mathbb{N}$, $a, b \in \text{SQ}$, $f : \mathbb{R} \rightarrow \mathbb{R}$, $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sigma(n) = \infty$. We consider the equation

$$\Delta^m x_n = a_n f(x_{\sigma(n)}) + b_n. \tag{E}$$

By a solution of (E) we mean a sequence $x : \mathbb{N} \rightarrow \mathbb{R}$ satisfying (E) for all large n . Note that the assumption $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ does not exclude the case of equations with delayed argument. For example we may define $\sigma(n) = n - 5$ for $n > 5$ and $\sigma(n) = 1$ for $n \leq 5$.

In Theorem 5.1, using fixed point theorems and the iterated remainder operator, we establish conditions under which there exist solutions of (E) with prescribed asymptotic behavior. In Theorem 5.2 we show that in many cases some assumptions of Theorem 5.1 are necessary. In Theorem 5.3 we establish conditions under which all f -bounded solutions of (E) are asymptotically polynomial.

In the proof of our first theorem we will use the Schauder fixed point theorem and the following version of the Knaster-Tarski fixed point theorem.

Lemma 5.1 *If X is a complete partially ordered set and a map $T : X \rightarrow X$ is nondecreasing then there exists $x_0 \in X$ such that $T(x_0) = x_0$.*

A simple proof of this result can be found in [20] (or in [19]).

Theorem 5.1 *Assume $p \in \mathbb{N}(0)$, $a \in A(m + p)$, $y \in \text{SQ}$, $\Delta^m y = b$, f is bounded on some uniform neighborhood U of the set $y(\mathbb{N})$ and one of the following conditions is satisfied:*

- (a) *f is nondecreasing on U and $(-1)^m a_n \geq 0$ for large n ,*
- (b) *f is nonincreasing on U and $(-1)^m a_n \leq 0$ for large n ,*
- (c) *f is continuous on U .*

Then there exists a solution x of (E) such that

$$x \in y + A(p).$$

Proof For $x \in \text{SQ}$ let $\bar{x} = f \circ x \circ \sigma$. Choose $c > 0$ and $M > 0$ such that

$$\bigcup_{n=1}^{\infty} (y_n - c, y_n + c) \subset U, \quad |f(t)| \leq M \quad \text{for } t \in U. \tag{11}$$

Let

$$\rho = Mr^m |a|.$$

Choose numbers k_0 and k such that $\rho_n < c$ for $n \geq k_0$ and $\sigma(n) \geq k_0$ for $n \geq k$. Let

$$S = \{x \in \text{SQ} : |x - y| \leq \rho \text{ and } x_n = y_n \text{ for } n < k\}.$$

It is easy to see that S , with natural order defined by $x \leq z$ if $x_n \leq z_n$ for every $n \in \mathbb{N}$, is a complete partially ordered set. If $x \in S$ and $n \geq k$, then $|x_{\sigma(n)} - y_{\sigma(n)}| < c$. Hence

$$x_{\sigma(n)} \in U \tag{12}$$

for any $x \in S$ and $n \geq k$. If $x \in S$, then, by (11) and (12), the sequence \bar{x} is bounded and so $a\bar{x} \in A(m + p)$. We define an operator $H : S \rightarrow \text{SQ}$ by

$$H(x)(n) := \begin{cases} y_n & \text{for } n < k, \\ y_n + (-1)^m r_n^m (a\bar{x}) & \text{for } n \geq k. \end{cases}$$

If $n \geq k$, then, using Lemma 3.1(01) and Lemma 3.1(14), we obtain

$$|H(x)(n) - y_n| = |r_n^m (a\bar{x})| \leq r_n^m |a\bar{x}| \leq Mr_n^m |a| = \rho_n.$$

Hence $H(S) \subset S$.

Assume now that the condition (a) is satisfied. We can assume that $(-1)^m a_n \geq 0$ for $n \geq k$. Let $x, z \in S$ and $x \leq z$. Since f is nondecreasing on U we have $f(x_{\sigma(n)}) \leq f(z_{\sigma(n)})$ for $n \geq k$. Hence

$$(-1)^m a_n f(x_{\sigma(n)}) \leq (-1)^m a_n f(z_{\sigma(n)})$$

for $n \geq k$. By Lemma 3.1(11), we have $H(x)(n) \leq H(z)(n)$ for $n \geq k$. Moreover, $H(x)(n) = y_n = H(z)(n)$ for $n < k$. Hence $H(x) \leq H(z)$. Now, using the Knaster-Tarski fixed point theorem we obtain $x \in S$ such that $H(x) = x$. Analogously if condition (b) is satisfied then $H(x) = x$ for certain $x \in S$.

Assume (c) and let d be a metric on S defined by

$$d(x, z) = \|x - z\| = \sup_{n \in \mathbb{N}} |x_n - z_n|.$$

Let BS denote the Banach space of all bounded sequences $x \in SQ$ with the norm $\|x\| = \sup_{n \in \mathbb{N}} |x_n|$ and let

$$T = \{x \in BS : |x| \leq \rho \text{ and } x_n = 0 \text{ for } n < k\}.$$

It is easy to see that T is a convex and closed subset of BS . Choose an $\varepsilon > 0$. Then there exists $q \in \mathbb{N}$ such that $\rho_n < \varepsilon$ for $n \geq q$. For $n = 1, \dots, q$ let G_n denote a finite ε -net for the interval $[-\rho_n, \rho_n]$ and let

$$G = \{x \in T : x_n \in G_n \text{ for } n \leq q \text{ and } x_n = 0 \text{ for } n > q\}.$$

Then G is a finite ε -net for T . Hence T is a complete and totally bounded metric space and so, T is compact. Hence T is a convex and compact subset of the Banach space BS and, by the Schauder fixed point theorem, any continuous map $T \rightarrow T$ has a fixed point. Let $F : T \rightarrow S$ be a map given by $F(x)(n) = x_n + y_n$. Then F is an isometry of T onto S . Assume $U : S \rightarrow S$ is a continuous map and let $W = F^{-1} \circ U \circ F$. Then $W : T \rightarrow T$ is continuous and there exists a point $z \in T$ such that $Wz = z$. Let $x = Fz$. Then

$$x = Fz = FWz = FF^{-1}UFz = Ux.$$

Hence any continuous map $U : S \rightarrow S$ has a fixed point. Let $\varepsilon > 0$. Choose $q \in \mathbb{N}(k)$ and $\alpha > 0$ such that

$$M \sum_{m=q}^{\infty} n^{m-1} |a_n| < \varepsilon \quad \text{and} \quad \alpha \sum_{n=k}^q n^{m-1} |a_n| < \varepsilon.$$

Let

$$W = \bigcup_{n=k}^q [y_n - \rho_n, y_n + \rho_n].$$

Then $W \subset U$. By compactness of W , f is uniformly continuous on W . Hence there exists $\delta > 0$ such that if $s, t \in W$ and $|s - t| < \delta$, then $|f(s) - f(t)| < \alpha$. Assume $x, z \in S$, $d(x, z) < \delta$. Let $u = \bar{x} - \bar{z}$. Then

$$\begin{aligned} d(Hx, Hz) &= \sup_{n \geq k} |r_n^m(a\bar{x}) - r_n^m(a\bar{z})| \\ &= \sup_{n \geq k} |r_n^m(au)| \leq \sup_{n \geq k} r_n^m |au| = r_p^m |au| \end{aligned}$$

$$\begin{aligned} &\leq \sum_{n=k}^q n^{m-1} |a_n u_n| + \sum_{n=q}^{\infty} n^{m-1} |a_n u_n| \\ &\leq \alpha \sum_{n=k}^q n^{m-1} |a_n| + 2M \sum_{n=q}^{\infty} n^{m-1} |a_n| < 3\varepsilon. \end{aligned}$$

Hence H is continuous and there exists $x \in S$ such that $H(x) = x$. Then

$$x_n = H(x)(n) = y_n + (-1)^m r_n^m(a\bar{x})$$

for $n \geq k$. Hence

$$x - (y + (-1)^m r^m(a\bar{x})) \in \text{Fin}(k).$$

Since the sequence \bar{x} is bounded, we have $a\bar{x} \in O(a)$. Hence

$$x \in y + r^m O(a) + \text{Fin}(k) \subset y + A(p).$$

Moreover, for $n \geq k$, by Lemma 3.1(07), we have

$$\Delta^m x_n = \Delta^m y_n + (-1)^m \Delta^m r_n^m(a\bar{x}) = b_n + a_n f(x_{\sigma(n)}).$$

Hence x is a solution of (E). The proof is complete. □

Remark 5.1 The conclusion $x \in y + A(p)$ of Theorem 5.1 may be written in the form

$$y \in x + A(p).$$

We can say that y is an approximative solution of (E) with ‘degree of approximation’ $A(p)$.

Remark 5.2 Using Lemma 4.5 we can replace the assumption $a \in A(m+p)$ of Theorem 5.1 by

$$\liminf n \left(\frac{|a_n|}{|a_{n+1}|} - 1 \right) > m + p.$$

Analogously, the conclusion of this theorem may be replaced: there exist a solution x of (E) and a sequence $z \in \text{SQ}$ such that $x = y + z$ and

$$\limsup n \left(\frac{|z_n|}{|z_{n+1}|} - 1 \right) \geq p.$$

Similarly, using other tests, we can obtain many formulations of this theorem.

Theorem 5.2 Assume $q \in \mathbb{N}$, $p \in \mathbb{N}(0)$, $\varepsilon > 0$, $s \in (-\infty, -p)$, $a, b, y \in \text{SQ}$, $\Delta^m y = b$, a is nonoscillatory, there exists a uniform neighborhood U of the set $y(\mathbb{N}(q))$ such that $f|U \geq \varepsilon$ or $f|U \leq -\varepsilon$, and one of the following conditions is satisfied:

- (a) there exists a solution x of (E) such that $x = y + o(n^s)$,

(b) *there exists a solution x of (E) such that $x \in y + S(p)$.
 Then $a \in A(m + p)$.*

Proof By Lemma 4.2 we have $o(r^s) \subset A(p) \subset S(p)$. Hence, there exists a solution x of (E) and a sequence $z \in S(p)$ such that $x = y + z$. Let $\bar{x} = f \circ x \circ \sigma$. For large n we have

$$a_n \bar{x}_n + b_n = a_n f(x_{\sigma(n)}) + b_n = \Delta^m x_n = \Delta^m y_n + \Delta^m z_n = b_n + \Delta^m z_n.$$

By Lemma 3.1(10), $\Delta^m z \in S(m + p)$. Hence, by Lemma 3.1(12), we have $a\bar{x} \in S(m + p)$. Since $z_n = o(1)$, we have $x_{\sigma(n)} \in U$ for large n . Hence the sequence $a\bar{x}$ is nonoscillatory and we obtain $|a\bar{x}| \in S(m + p)$, which means $a\bar{x} \in A(m + p)$. Moreover,

$$|a_n| = |a_n \bar{x}_n| |\bar{x}_n|^{-1} \leq |a_n \bar{x}_n| \varepsilon^{-1}$$

for large n . Hence, by Lemma 3.1(04), $a \in A(m + p)$. The proof is complete. □

Remark 5.3 Theorem 5.2 extends Theorem 1 of [8] and parts of Theorems 6 and 7 of [6].

Lemma 5.2 *Assume $m \in \mathbb{N}$, $p \in \mathbb{N}(0)$, $x \in SQ$ and $\Delta^m x \in A(m + p)$. Then*

$$x \in \text{Pol}(m - 1) + A(p).$$

Proof Using Lemma 4.8(c) we obtain

$$r^m A(m + p) \subset A(p). \tag{13}$$

Let $z = (-1)^m r^m \Delta^m x$. Then $\Delta^m z = \Delta^m (-1)^m r^m \Delta^m x = \Delta^m x$. Hence

$$x - z \in \text{Ker } \Delta^m = \text{Pol}(m - 1).$$

By (13) we have $z \in A(p)$. Hence

$$x = (x - z) + z \in \text{Pol}(m - 1) + A(p). \tag{14} \quad \square$$

Theorem 5.3 *Assume $p \in \mathbb{N}(0)$, $a, b \in A(m + p)$ and x is an f -bounded solution of (E). Then*

$$x \in \text{Pol}(m - 1) + A(p).$$

Proof The sequence $\bar{x} = f \circ x \circ \sigma$ is bounded. Hence $a\bar{x} \in O(a)$ and we obtain $\Delta^m x \in O(|a| + |b|)$. Therefore $\Delta^m x \in A(m + p)$ and, by Lemma 5.2, we have

$$x \in \text{Pol}(m - 1) + A(p). \tag{15} \quad \square$$

Competing interests

The author declares that he has no competing interests.

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