Zhang and Sun *Advances in Difference Equations* 2014, 2014:194
http://www.advancesindifferenceequations.com/content/2014/1/194

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of mild solutions for the impulsive semilinear nonlocal problem with random effects

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Abstract

In this paper, we study the existence of impulsive semilinear nonlocal problems with random effects. Random perturbations are taken into consideration in abstract spaces. By applying the measure of noncompactness and a random fixed point theorem with stochastic domain, we get some existence results which improve and generalize many known results. Some weaker assumptions are established. Besides, we do not claim the semigroup to be compact.

Keywords: random fixed point; existence; nonlocal conditions; measure of noncompactness; impulsive differential equations

1 Introduction

During the past few decades impulsive differential equations have been widely studied and significant progress has been made in [1–8] and references therein. Since many real world phenomena exhibit the presence of sudden state changes, the study of impulsive differential equations is becoming more important nowadays. The theory of impulsive differential equations now not only is being recognized richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modeling of many real world phenomena.

On the other hand, due to a combination of uncertainties and complexities, deterministic equations can hardly describe a real system precisely. In order to take random factors into account, many stochastic models were proposed and various achievements were obtained; see [5, 9–13] and references therein. Between them differential equations with random coefficients (see, *e.g.*, [13–15]) offer a natural and rational approach (see [16, Chapter 1]), since sometimes we can get the random distributions of some main disturbances by historical experiences and data rather than take all random disturbances into account and assume the noise to be white noises.

The main purpose of this paper is to discuss the existence of mild solutions for the nonlocal initial value problem with impulses and random effects of the form

$$\begin{cases} x'(t,\omega) = Ax(t,\omega) + f(t,x(t,\omega),\omega), & t \neq t_k, k = 1,2,\ldots,m, \\ x(t_k^+,\omega) - x(t_k^-,\omega) = I_k(x(t_k,\omega),\omega), & k = 1,2,\ldots,m, \\ x(0,\omega) = g(x,\omega) \end{cases} \tag{1}$$

(with some notations to be given later), where $(\Omega, \mathcal{F}, P)$ is a complete probability space, $\omega \in \Omega$, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (*i.e.*, C_0-semigroup) $T(t)$ in a Banach space X, $f : I \times X \times \Omega \to X$, $g : PC(I,X) \times \Omega \to X$, $I_k : X \times \Omega \to X$, $k = 1, 2, \dots, m$, are given random functions which represent random nonlinear effects of the system.

The investigation of nonlocal initial value problems was initiated by Byszewski in [17]. Later on, Byszewski's work was followed by many others and a lot of work was presented in [7, 8, 18–23], since in some physics models the nonlocal problems play a better role than the classical ones. The corresponding deterministic case of problem (1), with or without impulses, has been studied by many authors. For instance, Ntouyas and Tsamatos in [18] investigated the case of compactness conditions of f and $T(t)$. Xue in [19] studied the problem when f is compact. For impulsive semilinear nonlocal problems, in [20], Liang *et al.* obtained existence and uniqueness results by various assumptions of compactness or Lipschitz on $T(t)$, f, g or I_k. Very recently, in [7, 8] the authors put forward sufficient conditions on the existence of nonlocal impulsive differential equations by supposing the semigroup $T(t)$ to be equicontinuous, but in their papers impulsive functions are always assumed to be compact or Lipschitzian. The goal of this paper is to continue the study of these authors by making full use of the Hausdorff measure of noncompactness and take random effects into consideration. The nonlocal problem with impulses and differential equations with random effects were studied in, *e.g.*, [8, 15], respectively. In this paper, we study the nonlocal problem with impulses combined with random effects. Though the nonlocal problem with impulses and random effects was studied in, *e.g.*, [24] and references therein, in this paper we study impulsive semilinear nonlocal problem with random effects of another type. We prove existence results without compactness on $T(t)$ or f and the Lipschitz condition on f, and we weaken assumptions on impulsive functions I_k.

The rest of this paper is organized as follows. In Section 2, we recall some notations and important lemmas. In Section 3, we give sufficient conditions on the existence of mild solutions of problem (1). Finally, the conclusion is draw in Section 4.

2 Notations and preliminaries

In this section, we introduce some basic definitions and results which will be used in the paper.

Let $(X, \|\cdot\|)$ be a real separable Banach space, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $I = [0, T]$ be a bounded closed interval in $\mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. By $C(I,X)$ we denote the Banach space of all continuous mappings from I into X with the norm $\|f\|_C = \sup\{\|f(t)\| : t \in I\}$. Let $PC(I,X) = \{y : I \to X : y \text{ is continuous for } t \neq t_k, y(t_k^-) \text{ and } y(t_k^+) \text{ exist with } y(t_k^-) = y(t_k), k = 1, \dots, m\}$. It is easy to verify that $PC(I,X)$ is a separable Banach space with the norm $\|f\|_{PC} = \sup\{\|f(t)\| : t \in I\}$. We denote by $L^1(I,X)$ the space of X-valued Bochner functions on I with the norm $\|x\| = \int_0^T \|x(s)\|\, ds$. Thus $C(I,X) \subseteq PC(I,X) \subseteq L^1(I,X)$.

A C_0-semigroup $T(t)$ is said to be compact if $T(t)$ is compact for any $t > 0$. If $t \to T(t)x$ is equicontinuous at all $t > 0$ with respect to x in all bounded subsets of X, then the semigroup $T(t)$ is called equicontinuous. Note that if the semigroup $T(t)$ is compact, then it must be equicontinuous (*cf.* [25]).

Let Y be a separable Banach space with the Borel σ-algebra $\mathcal{B}_Y$. A mapping y: $\Omega \to Y$ is said to be a random variable with values in Y if for each $B \in \mathcal{B}_Y$, $y^{-1}(B) \in \mathcal{F}$. A mapping $T : \Omega \times Y \to Y$ is called a random operator if $T(\cdot, y)$ is measurable for each $y \in Y$ and is generally expressed as $T(\omega, y) := T(\omega)y$; we will use these two expressions alternatively.

Next, we will give a very useful random fixed point theorem with stochastic domain.

Definition 1 (see [26]) Let C be a mapping from Ω into 2^Y. A mapping $T : \{(\omega, y) | \omega \in \Omega \wedge y \in C(\omega)\} \to Y$ is called 'random operator with stochastic domain C' iff C is measurable (*i.e.*, for all closed $A \subseteq Y$, $\{\omega \in \Omega | C(\omega) \cap A \neq \emptyset\} \in \mathcal{F}$) and for all open $D \subseteq Y$ and all $y \in Y$, $\{\omega \in \Omega | y \in C(\omega) \wedge T(\omega, y) \in D\} \in \mathcal{F}$. T will be called 'continuous' if every $T(\omega)$ is continuous. For a random operator T, a mapping $y : \Omega \to Y$ is called 'random (stochastic) fixed point of T' iff for P-almost all $\omega \in \Omega$, $y(\omega) \in C(\omega)$ and $T(\omega)y(\omega) = y(\omega)$ and for all open $D \subseteq Y$, $\{\omega \in \Omega | y(\omega) \in D\} \in \mathcal{F}$ ('y is measurable').

Remark 1 If $C(\omega) \equiv Y$, then the definition of random operator with stochastic domain coincides with the definition of random operator.

Lemma 1 (see [26]) *Let $C : \Omega \to 2^Y$ be measurable with $C(\omega)$ closed, convex and solid* (*i.e.*, $\operatorname{int} C(\omega) \neq \emptyset$) *for all $\omega \in \Omega$. We assume that there exists measurable $y_0 : \Omega \to Y$ with $y_0 \in \operatorname{int} C(\omega)$ for all $\omega \in \Omega$. Let T be a continuous random operator with stochastic domain C such that for every $\omega \in \Omega$, $\{y \in C(\omega) | T(\omega)y = y\} \neq \emptyset$. Then T has a stochastic fixed point.*

Let ξ be a mapping of $I \times \Omega$ into X. ξ is said to be a stochastic process if for each $t \in I$, $\xi(t, \cdot)$ is measurable and if, in addition, for each $\omega \in \Omega$, $\xi(\cdot, \omega) \in PC(I, X)$, ξ is said to satisfy condition (PC, Ω). If ξ satisfies condition (PC, Ω), then ξ is considered to be a mapping of Ω into $PC(I, X)$.

By some similar techniques in [27], without proof we state the following proposition.

Proposition 1 *ξ satisfies condition (PC, Ω) if and only if ξ is measurable as a mapping of Ω into $PC(I, X)$.*

Definition 2 A stochastic process x is said to be a mild random solution of problem (1) if it satisfies condition (PC, Ω) and for $\omega \in \Omega$,

$$x(t, \omega) = T(t)g\big(x(\cdot, \omega)\big) + \int_0^t T(t-s)f\big(s, x(s, \omega), \omega\big)\, ds + \sum_{0<t_k<t} T(t-t_k)I_k\big(x(t_k, \omega), \omega\big)$$

for all $t \in I$.

Let α denote the Hausdorff measure of noncompactness both on X and $PC(I, X)$, then we have the following fixed point theorem.

Lemma 2 ([28]) *Let F be a closed and convex subset of a real Banach space, let $A : F \to F$ be a continuous operator and $A(F)$ be bounded. For each bounded subset $B \subseteq F$, set*

$$A^1(B) = A(B), \qquad A^n(B) = A\big(\overline{\operatorname{conv}}\big(A^{n-1}(B)\big)\big), \quad n = 2, 3, \ldots.$$

If there exist a constant $0 \leq k < 1$ and a positive integer n_0 such that for each bounded subset $B \subseteq F$,

$$\alpha\big(A^{n_0}(B)\big) \leq k\alpha(B),$$

then A has a fixed point in F.

For specific properties of measures of noncompactness, we refer the readers to [29].

To prove the existence results in this paper, we need the following lemmas concerning the relationship in measures of noncompactness.

Lemma 3 ([30]) *If Y is a bounded subset of Banach space X, then for each $\epsilon > 0$, there is a sequence $\{y_k\}_{k=1}^{\infty} \subseteq Y$ such that*

$$\alpha(Y) \leq 2\alpha\big(\{y_k\}_{k=1}^{\infty}\big) + \epsilon.$$

Lemma 4 ([31]) *If $\{u_k\}_{k=1}^{\infty} \subseteq L^1(I,X)$ is uniformly integrable, then $\alpha(\{u_k(t)\}_{k=1}^{\infty})$ is measurable and*

$$\alpha\left(\left\{\int_0^t u_k(s)\,ds\right\}_{k=1}^{\infty}\right) \leq 2\int_0^t \alpha\big(\{u_k(s)\}_{k=1}^{\infty}\big)\,ds \quad \text{for } t \in [0,T].$$

Lemma 5 ([32]) *If the semigroup $T(t)$ is equicontinuous and $\eta \in L^1(I,\mathbb{R}^+)$, then the set $\{t \to \int_0^t T(t-s)x(s)\,ds; \|x(s)\| \leq \eta(s)$, for a.e. $s \in [0,T]\}$ is equicontinuous on $[0,T]$.*

Lemma 6 ([7]) *If $W \subseteq PC(I,X)$ is bounded, then $\alpha(W(t)) \leq \alpha(W)$ for all $t \in I$, where $W(t) = \{u(t); u \in W\} \subseteq X$. Furthermore, suppose that the following conditions are satisfied*:

(1) *W is equicontinuous on $J_0 = [0,t_1]$ and each $J_i = (t_i,t_{i+1}]$, $i = 1,\dots,m$;*

(2) *W is equicontinuous at $t = t_i^+$, $i = 1,\dots,m$.*

Then $\sup_{t\in I}\alpha(W(t)) = \alpha(W)$.

3 Main results

In this section, we give existence results for the nonlocal problem (1). First we list the following conditions.

(HA) The C_0-semigroup $T(t)$ generated by A is equicontinuous and we denote $N = \sup\{\|T(t)\| : t \in I\}$.

(Hg) $g : PC(I,X) \times \Omega \to X$. For each $\omega \in \Omega$, $g(\cdot,\omega)$ is continuous and compact.

(Hf)

(i) $f : I \times X \times \Omega \to X$. For each $\omega \in \Omega$, $f(\cdot,\cdot,\omega) : I \times X \to X$ satisfies the Caratheodory condition, *i.e.*, $f(\cdot,x,\omega)$ is measurable for all $x \in X$, and $f(t,\cdot,\omega)$ is continuous for a.e. $t \in I$.

(ii) There exist functions $\phi : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$ and $m : I \times \Omega \to \mathbb{R}^+$ such that for each $\omega \in \Omega$, $\phi(\cdot,\omega)$ is nondecreasing and continuous and $m(\cdot,\omega)$ integrable with

$$\big\|f(t,x,\omega)\big\| \leq m(t,\omega)\phi\big(\|x\|,\omega\big)$$

for all $x \in X$ and a.e. $t \in I$.

(iii) There exists a function $L : I \times \Omega \to \mathbb{R}^+$ with $L(\cdot, \omega) \in L^1(I, \mathbb{R}^+)$ for each $\omega \in \Omega$ such that for any bounded $B \subseteq X$,

$$\alpha\big(f(t, B, \omega)\big) \le L(t, \omega)\alpha(B) \tag{2}$$

for a.e. $t \in I$.

(HI) $I_k : X \times \Omega \to X$ is sample path continuous and there exists $\mathcal{L}_k : \Omega \to \mathbb{R}^+$ such that for $\omega \in \Omega$ and all bounded subset $B \subseteq X$,

$$\alpha\big(I_k(B, \omega)\big) \le \mathcal{L}_k(\omega)\alpha(B) \tag{3}$$

for $k = 1, 2, \dots, m$.

(Hm) For each $t \in I$, $x \in X$, $f(t, x, \cdot)$ and $I_k(x, \cdot)$, $k = 1, 2, \dots, m$, are measurable; for each $x \in PC(I, X)$, $g(x, \cdot)$ is measurable.

(HR) There exists a random function $R : \Omega \to \mathbb{R}^+ \setminus \{0\}$ such that

$$\sup_{x \in B_{R(\omega)}} \big\|g(x, \omega)\big\| + \phi\big(R(\omega), \omega\big) \int_0^T m(s, \omega)\, ds + \sum_{k=1}^{m} \sup_{x \in B_{R(\omega)}} \big\|I_k\big(x(t_k), \omega\big)\big\| \le \frac{R(\omega)}{N}$$

for each $\omega \in \Omega$, where B_r denotes the closed ball in $PC(I, X)$ centered at zero and with radius r.

(H*)

$$N \sum_{k=1}^{m} \mathcal{L}_k(\omega) < 1, \quad \forall \omega \in \Omega.$$

Remark 2 Conditions (2) or (3) can be derived by the Lipschitz conditions on f or I_k, $k = 1, 2, \dots, m$, see [23].

Theorem 1 *Suppose that hypotheses* (HA), (Hf), (Hg), (HI), (Hm), (HR) *and* (H*) *are valid, then the nonlocal random impulsive problem* (1) *has at least one mild random solution.*

Proof Consider the random operator $\mathscr{P} : \Omega \times PC(I, X) \to PC(I, X)$ defined by

$$\begin{aligned} \big(\mathscr{P}(\omega)x\big)(t) &= T(t)g(x, \omega) + \int_0^t T(t-s)f\big(s, x(s), \omega\big)\, ds \\ &\quad + \sum_{0<t_k<t} T(t-t_k)I_k\big(x(t_k), \omega\big), \quad t \in I, \end{aligned} \tag{4}$$

and we can divide $\mathscr{P}$ into three parts,

$$\begin{aligned} F(\omega)x(t) &= T(t)g(x, \omega), \\ G(\omega)x(t) &= \int_0^t T(t-s)f\big(s, x(s), \omega\big)\, ds, \\ H(\omega)x(t) &= \sum_{0<t_k<t} T(t-t_k)I_k\big(x(t_k), \omega\big). \end{aligned}$$

For each $\omega \in \Omega$ and each $x \in PC(I, X)$, by the definition of $\mathscr{P}$, clearly $\mathscr{P}(\omega)(x) \in PC(I, X)$. Thus the mapping is well defined.

Then we show that the mapping defined by (4) is a random operator. To do this, we need to prove that for any $x \in PC(I,X)$ given, $\mathscr{P}(\cdot)(x) : \Omega \to PC(I,X)$ is a random variable. By Proposition 1, $\mathscr{P}(\cdot)(x) : \Omega \to PC(I,X)$ is measurable if and only if $\mathscr{P}(\cdot,x)$, as a mapping of $I \times \Omega$ into X, satisfies condition (PC,Ω), *i.e.*,

(a) for any $\omega \in \Omega$, $\mathscr{P}(\omega,x)(t) \in PC(I,X)$;
(b) for any $t \in I$, $\mathscr{P}(\cdot,x)(t) : \Omega \to X$ is measurable;

(a) is satisfied since the mapping is well defined, (b) can be fulfilled by assumption (Hm).

By some usual techniques (see, *e.g.*, [7]), it is easy to show that the random operator $\mathscr{P}$ is continuous.

Let $W : \Omega \to 2^{PC(I,X)}$ be defined by $W(\omega) = \{x \in PC(I,X) : \|x\|_{PC} \le R(\omega)\}$ with $W(\omega)$ bounded, closed, convex and solid for all $\omega \in \Omega$. Then W is measurable by Lemma 17 in [26].

Let $\omega \in \Omega$ be fixed, then for any $x \in W(\omega)$, we have

$$\begin{aligned}\big\|\big(\mathscr{P}(\omega)x\big)(t)\big\| \le{}& \big\|T(t)g(x,\omega)\big\| + \left\|\int_0^t T(t-s)f\big(s,x(s),\omega\big)\,ds\right\| \\ &+ \sum_{0<t_k<t}\big\|T(t-t_k)I_k\big(x(t_k),\omega\big)\big\| \\ \le{}& N\sup_{x\in W(\omega)}\big\|g(x,\omega)\big\| + N\phi\big(R(\omega),\omega\big)\int_0^T m(s,\omega)\,ds \\ &+ N\sum_{k=1}^m \sup_{x\in W(\omega)}\big\|I_k\big(x(t_k),\omega\big)\big\| \\ \le{}& R(\omega).\end{aligned}$$

This implies that $\mathscr{P}$ is a random operator with stochastic domain W and $\mathscr{P}(\omega) : W(\omega) \to W(\omega)$ for each $\omega \in \Omega$.

Next, let $\omega \in \Omega$ be fixed (therefore we do not write 'ω' in the sequel) but arbitrary.

For any $B \subseteq W$, $\forall t \in I$,

$$\alpha\big(FB(t)\big) = \alpha\big(T(t)g(B)\big) = 0.$$

From Lemma 3 and Lemma 4, for any $\epsilon > 0$, there is a sequence $\{x_n\}_{n=1}^\infty \subseteq B$ such that

$$\begin{aligned}\alpha\big(GB(t)\big) &= \alpha\left(\left\{\int_0^t T(t-s)f\big(s,x(s)\big)\,ds, x \in B\right\}\right) \\ &\le 2\alpha\left(\left\{\int_0^t T(t-s)f\big(s,x_n(s)\big)\,ds\right\}_{n=1}^\infty\right) + \epsilon \\ &\le 4\int_0^t \alpha\big(\big\{T(t-s)f\big(s,x_n(s)\big)\big\}_{n=1}^\infty\big)\,ds + \epsilon \\ &\le 4N\int_0^t L(s)\alpha\big(\big\{x_n(s)\big\}_{n=1}^\infty\big)\,ds + \epsilon \\ &\le 4N\int_0^t L(s)\,ds \cdot \alpha\big(\{x_n\}_{n=1}^\infty\big) + \epsilon \\ &\le 4N\int_0^t L(s)\,ds \cdot \alpha(B) + \epsilon,\end{aligned}$$

$$\begin{aligned}\alpha\bigl(HB(t)\bigr) &= \alpha\left(\left\{\sum_{0<t_k<t} T(t-t_k)I_k\bigl(x(t_k)\bigr), x\in B\right\}\right)\\ &\le N\alpha\left(\left\{\sum_{k=1}^{m} I_k\bigl(x(t_k)\bigr), x\in B\right\}\right)\\ &\le N\sum_{k=1}^{m}\mathcal{L}_k\cdot\alpha(B).\end{aligned}$$

Thus, since $\epsilon > 0$ is arbitrary,

$$\alpha\bigl(\mathscr{P}B(t)\bigr) \le \left(N\sum_{k=1}^{m}\mathcal{L}_k + 4N\int_0^t L(s)\,ds\right)\cdot\alpha(B).$$

We know that there is a continuous function $p : I \to \mathbb{R}^+$ ($M = \max\{|p(t)| : t\in I\}$) such that for any $\gamma > 0$ ($\gamma < \frac{1-N\sum_{k=1}^{m}\mathcal{L}_k}{4N}$),

$$\int_0^T \bigl|L(s)-p(s)\bigr|\,ds < \gamma.$$

So

$$\begin{aligned}\alpha\bigl(\mathscr{P}B(t)\bigr) &\le \left(N\sum_{k=1}^{m}\mathcal{L}_k + 4N\left(\int_0^t \bigl|L(s)-p(s)\bigr|\,ds + \int_0^t \bigl|p(s)\bigr|\,ds\right)\right)\cdot\alpha(B)\\ &\le \left(N\sum_{k=1}^{m}\mathcal{L}_k + 4N(\gamma + Mt)\right)\cdot\alpha(B)\\ &= (a+bt)\alpha(B),\end{aligned}$$

where $a = N\sum_{k=1}^{m}\mathcal{L}_k + 4N\gamma$, $b = 4NM$.

From Lemma 5 and Lemma 6, it is easy to verify the equicontinuity of $\mathscr{P}B$, thus

$$\alpha(\mathscr{P}B) \le (a+bT)\alpha(B). \tag{5}$$

Again, in view of (3), Lemma 3 and Lemma 4, by some similar argument, for any $\epsilon > 0$, there is a sequence $\{y_n\}_{n=1}^{\infty} \subseteq \overline{\operatorname{conv}}(\mathscr{P}^1B)$ such that

$$\begin{aligned}&\alpha\bigl(\mathscr{P}^2B(t)\bigr)\\ &\quad= \alpha\bigl(\mathscr{P}\bigl(\overline{\operatorname{conv}}\bigl(\mathscr{P}^1B\bigr)\bigr)(t)\bigr)\\ &\quad\le 2\alpha\left(\left\{\int_0^t T(t-s)f\bigl(s,y_n(s)\bigr)\,ds\right\}_{n=1}^{\infty}\right) + \epsilon + N\sum_{k=1}^{m}\mathcal{L}_k\alpha(\mathscr{P}B)\\ &\quad\le N\sum_{k=1}^{m}\mathcal{L}_k(a+bT)\alpha(B) + 4N\int_0^t L(s)\alpha\bigl(\mathscr{P}B(s)\bigr)\,ds + \epsilon\\ &\quad\le N\sum_{k=1}^{m}\mathcal{L}_k(a+bT)\alpha(B) + 4N\int_0^t \bigl(\bigl|L(s)-p(s)\bigr| + \bigl|p(s)\bigr|\bigr)(a+bs)\alpha(B)\,ds + \epsilon\end{aligned}$$

$$\begin{aligned}&\le N\sum_{k=1}^{m}\mathcal{L}_k(a+bT)\alpha(B)+4N\left[(a+bT)\gamma+M\left(aT+\frac{b}{2}T^2\right)\right]\alpha(B)+\epsilon\\&=\left[\left(N\sum_{k=1}^{m}\mathcal{L}_k+4N\gamma\right)a+\left(N\sum_{k=1}^{m}\mathcal{L}_k+4N\gamma\right)bT+4NM\left(aT+\frac{b}{2}T^2\right)\right]\alpha(B)+\epsilon\\&=\left(a^2+2abT+\frac{(bT)^2}{2}\right)\alpha(B)+\epsilon.\end{aligned}$$

Since ϵ is arbitrary,

$$\alpha\big(\mathscr{P}^2B(t)\big)\le\left(a^2+2abT+\frac{(bT)^2}{2!}\right)\alpha(B).$$

Hence, by mathematical induction, for any positive integer n and $t\in[0,T]$, we obtain

$$\alpha\big(\mathscr{P}^nB(t)\big)\le\left(a^n+C_n^1a^{n-1}bT+C_n^2a^{n-2}\frac{(bT)^2}{2!}+\cdots+\frac{(bT)^n}{n!}\right)\alpha(B).$$

Since $\mathscr{P}^nB$ is equicontinuous on I, consequently we have

$$\alpha\big(\mathscr{P}^nB\big)\le\left(a^n+C_n^1a^{n-1}bT+C_n^2a^{n-2}\frac{(bT)^2}{2!}+\cdots+\frac{(bT)^n}{n!}\right)\alpha(B).$$

Since $a<1$, from Lemma 2.7 in [33], there exists a positive integer n_0 such that

$$a^{n_0}+C_{n_0}^1a^{n_0-1}bT+C_{n_0}^2a^{n_0-2}\frac{(bT)^2}{2!}+\cdots+\frac{(bT)^{n_0}}{n_0!}=r<1.$$

Then

$$\alpha\big(\mathscr{P}^{n_0}B\big)\le r\alpha(B).$$

It follows from Lemma 2 that for each $\omega\in\Omega$, $\mathscr{P}$ has at least one fixed point in W. Since $\bigcap_{\omega\in\Omega}\operatorname{int}W(\omega)\neq\emptyset$, the hypothesis that a measurable selector of $\operatorname{int}W$ exists holds. By Lemma 1, $\mathscr{P}$ has a stochastic fixed point, *i.e.*, the nonlocal initial value problem has at least one mild solution which completes the proof. □

Remark 3 If I_k is sample path continuous and compact, then $\mathcal{L}_k=0$ for $k=1,2,\ldots,m$. Thus hypotheses (HI) and (H*) hold naturally.

Actually, (HR) is a very rough condition. If we specify some features on I_k and g, $k=1,2,\ldots,m$, we have the following corollary.

(Hg′) $g:PC(I,X)\times\Omega\to X$. For each $\omega\in\Omega$, $g(\cdot,\omega)$ is continuous and compact, there exists $c_0:\mathbb{R}^+\times\Omega\to\mathbb{R}^+$ such that $\|g(x,\omega)\|\le c_0(\|x\|_{PC},\omega)$, $\forall x\in PC(I,X)$.

(HI′) $I_k:X\times\Omega\to X$ is sample path continuous and there exist $c_k:\mathbb{R}^+\times\Omega\to\mathbb{R}^+$ and $\mathcal{L}_k:\Omega\to\mathbb{R}^+$ such that $\|I_k(x,\omega)\|\le c_k(\|x\|,\omega)$, $\forall x\in X$, and for all bounded subset $B\subseteq X$,

$$\alpha\big(I_k(B,\omega)\big)\le\mathcal{L}_k(\omega)\alpha(B)$$

for $k=1,2,\ldots,m$.

(HR′) There exists a random function $R' : \Omega \to \mathbb{R}^+ \setminus \{0\}$ such that

$$c_0\big(R'(\omega), \omega\big) + \phi\big(R'(\omega), \omega\big) \int_0^T m(s, \omega)\, ds + \sum_{k=1}^{m} c_k\big(R'(\omega), \omega\big) \le \frac{R'(\omega)}{N}$$

for each $\omega \in \Omega$.

Corollary 1 *Suppose that hypotheses* (HA), (Hf), (Hg′), (HI′), (Hm), (HR′) *and* (H*) *are valid, then the nonlocal random impulsive problem* (1) *has at least one mild random solution.*

Theorem 2 *Suppose that hypotheses* (HA), (Hf), (Hg′), (HI′), (Hm) *and* (H*) *are valid, then the nonlocal random impulsive problem* (1) *has at least one mild random solution provided that*

$$\int_0^T m(s, \omega)\, ds \le \liminf_{r \to +\infty} \frac{r - N \sum_{k=0}^{m} c_k(r, \omega)}{\phi(r, \omega)}. \tag{6}$$

Furthermore, if I_k *is compact for* $k = 1, 2, \ldots, m$, *then* (H*) *is not essential.*

Proof Since the proof is similar to Theorem 1, we omit it. □

Remark 4 If we do not consider the random effects, *i.e.*, taking $\Omega = \{\omega_0\}$ as a single point set, since hypotheses on f and I_k are weaker, we generalize the results in [7]. Furthermore, if let the impulsive functions I_k be 0, for $k = 1, 2, \ldots, m$, then since we relax restrictions on g, our results generalize the results in [23]. Besides, since there are no compactness assumptions on the semigroup $T(t)$ or f, our work extends and improves many main results such as those in [19–21].

4 An example

Consider the impulsive partial differential system with nonlocal conditions and random effects of the form

$$\begin{aligned}
&\frac{\partial}{\partial t} x(t, \xi, \omega) = \frac{\partial}{\partial \xi} x(t, \xi, \omega) + m(\xi) u(t, \xi, \omega) + F\big(t, x(t, \xi, \omega), \omega\big) \\
&\quad \text{for } \xi \in [0, \pi], t \in [0, T], t \ne t_k, k = 1, \ldots, m, \\
&x(t, 0, \omega) = x(t, \pi, \omega) = 0, \quad t \in [0, T], \\
&x\big(t_k^+, \xi, \omega\big) - x\big(t_k^-, \xi, \omega\big) = I_k\big(x\big(t_k^-, \xi, \omega\big), \omega\big), \quad k = 1, 2, \ldots, m, \\
&x(0, \xi, \omega) = \int_0^T h(s, \omega) \log\big(1 + \big|x(s, \xi, \omega)\big|\big)\, ds.
\end{aligned} \tag{7}$$

Suppose that $X = L^2[0, \pi]$, $(\Omega, \mathcal{F}, P)$ is a complete probability space. Let $A : X \to X$ be defined by $Ay = y'$ with the domain $D(A) = \{y \in X : y \text{ is absolutely continuous } y' \in X, y(0) = y(\pi) = 0\}$. It is well known that A is an infinitesimal generator of a semigroup $T(t)$ defined by $T(t)y(s) = y(t + s)$ for each $y \in X$. $T(t)$ is not a compact semigroup on X and $\alpha(T(t)D) \le \alpha(D)$, where α is the Hausdorff MNC. We assume the following.

(H1) $f : [0,T] \times X \times \Omega \to X$ is a continuous random function defined by

$$f(t,x,\omega)(\xi) = F\big(t,x(\xi),\omega\big), \quad t \in [0,T], \xi \in [0,\pi].$$

We take $F(t,x(\xi),\omega) = c_0(\omega)\sin(x(\xi))$, c_0 is a real-valued random variable. F is Lipschitz continuous for the second variable. Then f satisfies hypothesis (Hf).

(H2) $I_k : X \times \Omega \to X$ is a continuous random function for $k = 1,2,\ldots,m$, defined by

$$I_k(x,\omega)(\xi) = I_k\big(x(\xi),\omega\big).$$

We take $I_k(x(\xi),\omega) = \int_0^\pi d_0(\omega)\rho_k(\xi,y)\cos^2(x(y))\,dy$, $x \in X$, $\rho_k \in C([0,\pi] \times [0,\pi],R)$ for $k = 1,2,\ldots,m$, d_0 is a real-valued random variable. Then I_k is compact thus satisfies hypothesis (HI).

(H3) $g : PC([0,T];X) \times \Omega \to X$ is a continuous random function defined by

$$g(\varphi,\omega)(\xi) = \int_0^T h(s,\omega)\log\big(1 + \big|\varphi(s)(\xi)\big|\big)\,ds, \quad \varphi \in PC\big([0,T];X\big)$$

with $\varphi(s)(\xi) = x(s,\xi)$. Then g is a compact operator and satisfies hypothesis (Hg). Under these assumptions, the partial differential system (7) can be reformulated as the abstract problem (1). By Theorem 1 we conclude that there exists at least one random solution for this system.

5 Conclusion

In this paper, we have studied existence results for impulsive semilinear nonlocal problems with random effects. Random perturbations have been taken into consideration in abstract spaces. By applying the measure of noncompactness and a random fixed point theorem with stochastic domain, we have obtained some existence results for impulsive semilinear nonlocal problems with random effects. Our results have improved and generalized many known results. Some weaker assumptions have been established.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SZ completed the proof and wrote the initial draft. JS provided the problem and gave some suggestions on the amendment. SZ then finalized the manuscript. Correspondence was mainly done by JS. All authors read and approved the final manuscript.

Acknowledgements

This work is supported by the NSF of China under grant No. 61174039.

Received: 29 October 2013 Accepted: 9 July 2014 Published: 23 Jul 2014

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10.1186/1687-1847-2014-194
Cite this article as: Zhang and Sun: **Existence of mild solutions for the impulsive semilinear nonlocal problem with random effects.** *Advances in Difference Equations* 2014, 2014:194