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The application of trigonal curve theory to the second-order Benjamin-Ono hierarchy

Guoliang He[1*] and Lin He[2]

*Correspondence:
glhemath@163.com
[1]Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, 450002, China
Full list of author information is available at the end of the article

Abstract

By introducing two sets of Lenard recursion equations, the second-order Benjamin-Ono hierarchy is proposed. In view of the characteristic polynomial of Lax matrix, a trigonal curve of arithmetic genus $m-1$ is deduced. Then the trigonal curve theory is used to derive the explicit algebro-geometric solutions represented in theta functions to the second-order Benjamin-Ono hierarchy with the help of the properties of Baker-Akhiezer function, the meromorphic function and the three kinds of Abel differentials.

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1 Introduction

The principal aim of the present paper concerns the algebro-geometric solutions of the second-order Benjamin-Ono hierarchy with the aid of the theory of trigonal curves [1–3]. To the best of the authors' knowledge, there have been no results about the algebro-geometric solutions of the second-order Benjamin-Ono equation [4, 5]

$$u_{tt} = \alpha\left(u^2\right)_{xx} + \beta u_{xxxx}, \tag{1.1}$$

which is used in the analysis of long waves in shallow water and many other physical applications, where α is a constant controlling nonlinearity and the characteristic speed of the long waves, and β is the depth of the fluid, although there are some results about the exact solutions of (1.1), such as the pulse-type and kink-type solutions, periodic solitary wave and double periodic solutions, soliton solutions *etc.*, by using the following methods: the Jacobi elliptic function expansion method, the bilinear method, the extended homoclinic test approach, the homogeneous balance method and the lattice Boltzmann method *etc.* [6–10].

Before turning to the contents of each section, it seems appropriate to review the existing literature on algebro-geometric solutions, which are of great importance for revealing inherent structure mechanism of solutions and describing the quasi-periodic behavior of nonlinear phenomena. During the last few years, there have been fairly mature techniques to construct algebro-geometric solutions of soliton equations associated with 2×2 matrix spectral problems, such as the KdV, nonlinear Schrödinger, sine-Gordon, Toda equations and so on [11–15]. Unfortunately, the situation is not so good for soliton equations associated with 3×3 matrix spectral problems, which are more complicated and more difficult. In [16], a unified framework was proposed to yield all algebro-geometric solutions of the entire Boussinesq hierarchy. Recently, based on the characteristic polynomial of Lax matrix associated with the 3×3 matrix spectral problems, we have developed the method in [16] to deal with some important soliton equations by introducing the trigonal curves of arithmetic genus $m-1$ and deriving the explicit Riemann theta function representations of the entire hierarchies, such as the modified Boussinesq, the Kaup-Kupershmidt hierarchies and others [17–19].

The present paper is organized as follows. In Section 2, based on two kinds of different Lenard recursion equations, we derive the second-order Benjamin-Ono hierarchy, which relates to a 3×3 matrix spectral problem. In Section 3, we introduce the Baker-Akhiezer function and the associated meromorphic function. Then the second-order Benjamin-Ono hierarchy is decomposed into the system of Dubrovin-type ordinary differential equations. In Section 4, the explicit Riemann theta function representations of the Baker-Akhiezer function and the meromorphic function, and especially of the solutions to the entire second-order Benjamin-Ono hierarchy are displayed by resorting to the Riemann theta functions, the holomorphic differentials, and the Abel map.

2 The zero-curvature representation to the second-order Benjamin-Ono hierarchy

In this section, we shall derive the second-order Benjamin-Ono hierarchy associated with the 3×3 matrix spectral problem

$$\psi_x = U\psi, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 & 0 \\ u & 0 & 1 \\ v+\lambda & u & 0 \end{pmatrix}, \tag{2.1}$$

where u and v are two potentials, and λ is a constant spectral parameter. To this end, we introduce two sets of Lenard recursion equations

$$Kg_{j-1} = Jg_j, \quad g_j|_{(u,v)=0} = 0, j \geq 0, \tag{2.2}$$

$$K\hat{g}_{j-1} = J\hat{g}_j, \quad \hat{g}_j|_{(u,v)=0} = 0, j \geq 0 \tag{2.3}$$

with two starting points

$$g_{-1} = (1,0)^T, \qquad \hat{g}_{-1} = (0,1)^T,$$

where the initial conditions mean to identify constants of integration as zero, and two operators are defined as follows:

$$K = \begin{pmatrix} \partial u + u\partial - \partial^3 & \partial v + \frac{1}{2}v\partial \\ 2v\partial + \partial v & \frac{1}{6}\partial^5 - \frac{1}{3}(\partial^3 u + u\partial^3) - \frac{1}{2}(\partial^2 u\partial + \partial u\partial^2) + u^2\partial + \partial u^2 + \frac{2}{3}u\partial u \end{pmatrix},$$

$$J = \begin{pmatrix} 0 & -\frac{3}{2}\partial \\ -3\partial & 0 \end{pmatrix}.$$

Hence g_j and $\hat{g}_j$ are uniquely determined, for example, the first two members read as

$$g_0 = -\frac{1}{3}\begin{pmatrix} v \\ 2u \end{pmatrix}, \qquad \hat{g}_0 = \frac{1}{9}\begin{pmatrix} u_{xx} - 4u^2 \\ -6v \end{pmatrix}.$$

In order to generate a hierarchy of evolution equations associated with the spectral problem (2.1), we solve the stationary zero-curvature equation

$$V_x - [U, V] = 0, \qquad V = (V_{ij})_{3\times 3}, \tag{2.4}$$

which is equivalent to

$$\begin{aligned}
&V_{11,x} + uV_{12} + (v+\lambda)V_{13} - V_{21} = 0,\\
&V_{12,x} + uV_{13} + V_{11} - V_{22} = 0,\\
&V_{13,x} - V_{23} + V_{12} = 0,\\
&V_{21,x} + u(V_{22} - V_{11}) + (v+\lambda)V_{23} - V_{31} = 0,\\
&V_{22,x} + u(V_{23} - V_{12}) + V_{21} - V_{32} = 0,\\
&V_{23,x} - uV_{13} + V_{22} - V_{33} = 0,\\
&V_{31,x} + u(V_{32} - V_{21}) + (v+\lambda)(V_{33} - V_{11}) = 0,\\
&V_{32,x} + u(V_{33} - V_{22}) - (v+\lambda)V_{12} + V_{31} = 0,\\
&V_{33,x} - uV_{23} - (v+\lambda)V_{13} + V_{32} = 0,
\end{aligned} \tag{2.5}$$

where each entry $V_{ij} = V_{ij}(a, b)$ is a Laurent expansion in λ:

$$\begin{aligned}
&V_{11} = \frac{1}{3}\left(\frac{1}{2}\partial^2 - u\right)b - \partial a, \qquad V_{12} = a - \frac{1}{2}\partial b, \qquad V_{13} = b,\\
&V_{21} = \left(\frac{1}{6}\partial^3 - \frac{1}{3}\partial u - \frac{1}{2}u\partial + v + \lambda\right)b + \left(u - \partial^2\right)a, \qquad V_{22} = \frac{1}{3}\left(-\partial^2 + 2u\right)b,\\
&V_{23} = a + \frac{1}{2}\partial b, \qquad V_{31} = \left(\frac{1}{6}\partial^4 - \frac{1}{3}\partial^2 u - \frac{1}{2}\partial u\partial - \frac{1}{2}u\partial^2 + u^2\right)b + (v+\lambda)a,\\
&V_{32} = \left(-\frac{1}{6}\partial^3 + \frac{1}{3}\partial u + \frac{1}{2}u\partial + v + \lambda\right)b + \left(u - \partial^2\right)a,\\
&V_{33} = \frac{1}{3}\left(\frac{1}{2}\partial^2 - u\right)b + \partial a,
\end{aligned} \tag{2.6}$$

$$a = \sum_{j\geq 0} a_{j-1}\lambda^{-j}, \qquad b = \sum_{j\geq 0} b_{j-1}\lambda^{-j}. \tag{2.7}$$

A direct calculation shows that (2.5) and (2.6) imply the Lenard equation

$$KG = \lambda JG, \quad G = (a, b)^T. \tag{2.8}$$

Substituting (2.7) into (2.8) and collecting terms with the same powers of λ, we arrive at the following recursion relation:

$$KG_{j-1} = JG_j, \qquad JG_{-1} = 0, \quad j \geq 0, \tag{2.9}$$

where $G_j = (a_j, b_j)^T$. Since the equation $JG_{-1} = 0$ has the general solution

$$G_{-1} = \alpha_0 g_{-1} + \beta_0 \hat{g}_{-1}, \tag{2.10}$$

then G_j can be expressed as

$$G_j = \alpha_0 g_j + \beta_0 \hat{g}_j + \cdots + \alpha_j g_0 + \beta_j \hat{g}_0 + \alpha_{j+1} g_{-1} + \beta_{j+1} \hat{g}_{-1}, \quad j \geq 0, \tag{2.11}$$

where α_j and β_j are arbitrary constants.

Let ψ satisfy the spectral problem (2.1) and its auxiliary problem

$$\psi_{t_r} = \widetilde{V}^{(r)} \psi, \qquad \widetilde{V}^{(r)} = \left(\widetilde{V}_{ij}^{(r)}\right)_{3\times 3}, \tag{2.12}$$

where each entry $\widetilde{V}_{ij}^{(r)} = \widetilde{V}_{ij}(\tilde{a}^{(r)}, \tilde{b}^{(r)})$,

$$\tilde{a}^{(r)} = \sum_{j=0}^{r} \tilde{a}_{j-1} \lambda^{r-j}, \qquad \tilde{b}^{(r)} = \sum_{j=0}^{r} \tilde{a}_{j-1} \lambda^{r-j}$$

with

$$\widetilde{G}_j = (\tilde{a}_j, \tilde{b}_j)^T = \tilde{\alpha}_0 g_j + \tilde{\beta}_0 \hat{g}_j + \cdots + \tilde{\alpha}_j g_0 + \tilde{\beta}_j \hat{g}_0 + \tilde{\alpha}_{j+1} g_{-1} + \tilde{\beta}_{j+1} \hat{g}_{-1}, \quad j \geq -1.$$

Then the compatibility condition of (2.1) and (2.12) yields the zero-curvature equation, $U_{t_r} - \widetilde{V}_x^{(r)} + [U, \widetilde{V}^{(r)}] = 0$, which is equivalent to the hierarchy of nonlinear evolution equations

$$(u_{t_r}, v_{t_r})^T = \widetilde{X}_r, \quad r \geq 0, \tag{2.13}$$

where the vector fields $\widetilde{X}_j = \widetilde{X}_j(u, v; \underline{\tilde{\alpha}}^{(j)}, \underline{\tilde{\beta}}^{(j)}) = K\widetilde{G}_{j-1} = J\widetilde{G}_j$, and $\underline{\tilde{\alpha}}^{(j)} = (\tilde{\alpha}_0, \ldots, \tilde{\alpha}_j)$, $\underline{\tilde{\beta}}^{(j)} = (\tilde{\beta}_0, \ldots, \tilde{\beta}_j)$. The first nontrivial member in the hierarchy (2.13) is as follows:

$$\begin{aligned} u_{t_0} &= \tilde{\alpha}_0 u_x + \tilde{\beta}_0 v_x, \\ v_{t_0} &= \tilde{\alpha}_0 v_x - \frac{1}{3} \tilde{\beta}_0 (u_{xxx} - 8uu_x). \end{aligned} \tag{2.14}$$

For $\tilde{\alpha}_0 = 0$, $\tilde{\beta}_0 = 1$ ($t_0 = t$), equation (2.14) is reduced to the second-order Benjamin-Ono equation by canceling the variable v

$$u_{tt} = \frac{4}{3}\left(u^2\right)_{xx} - \frac{1}{3} u_{xxxx}. \tag{2.15}$$

The second one in the hierarchy (2.13) (as $\tilde{\alpha}_1 = 0$, $\tilde{\beta}_1 = 0$) can be written as

$$\begin{aligned} u_{t_1} &= \frac{1}{3}\tilde{\alpha}_0(v_{xx} - 4uv)_x - \frac{1}{54}\tilde{\beta}_0\big(6u_{xxxx} - 60uu_{xx} - 45u_x^2 + 40u^3 + 45v^2\big)_x, \\ v_{t_1} &= -\frac{1}{27}\tilde{\alpha}_0\big(3u_{xxxx} - 36uu_{xx} - 18u_x^2 + 32u^3 + 18v^2\big)_x \\ &\quad - \frac{1}{9}\tilde{\beta}_0\big(v_{xxxx} - 5u_{xx}v - 10uv_{xx} - 5u_xv_x + 20u^2v\big)_x. \end{aligned} \tag{2.16}$$

For $\tilde{\alpha}_0 = 0$, $\tilde{\beta}_0 = -9$ ($t_1 = t$), equation (2.16) is reduced to a 5-order coupled equation

$$\begin{aligned} u_t &= u_{xxxxx} - \left(10uu_{xx} + 9u_x^2 - 9v^2 - \frac{20}{3}u^3\right)_x, \\ v_t &= v_{xxxxx} - \big(5u_{xx}v + 10uv_{xx} + 5u_xv_x - 20u^2v\big)_x. \end{aligned} \tag{2.17}$$

3 The meromorphic function and Dubrovin-type equations

In this section, we shall consider the Baker-Akhiezer function and the associated meromorphic function. By introducing the elliptic kind coordinates, we decompose the second-order Benjamin-Ono equation into the system of Dubrovin-type differential equations.

We first introduce the Baker-Akhiezer function $\psi(P,x,x_0,t_r,t_{0,r})$ by

$$\begin{aligned} &\psi_x(P,x,x_0,t_r,t_{0,r}) = U\big(u(x,t_r),v(x,t_r);\lambda(P)\big)\psi(P,x,x_0,t_r,t_{0,r}), \\ &\psi_{t_r}(P,x,x_0,t_r,t_{0,r}) = \widetilde{V}^{(r)}\big(u(x,t_r),v(x,t_r);\lambda(P)\big)\psi(P,x,x_0,t_r,t_{0,r}), \\ &V^{(n)}\big(u(x,t_r),v(x,t_r);\lambda(P)\big)\psi(P,x,x_0,t_r,t_{0,r}) = y(P)\psi(P,x,x_0,t_r,t_{0,r}), \\ &\psi_1(P,x_0,x_0,t_{0,r},t_{0,r}) = 1, \end{aligned} \tag{3.1}$$

where $V^{(n)} = (\lambda^n V)_+ = (V_{ij}^{(n)})_{3\times 3}$ and $V_{ij}^{(n)} = V_{ij}(a^{(n)},b^{(n)})$,

$$a^{(n)} = \sum_{j=0}^{n} a_{j-1}\lambda^{n-j}, \qquad b^{(n)} = \sum_{j=0}^{n} b_{j-1}\lambda^{n-j}$$

with a_j, b_j determined by (2.11). The compatibility conditions of the first three expressions in (3.1) yield that

$$U_{t_r} - \widetilde{V}_x^{(r)} + \big[U,\widetilde{V}^{(r)}\big] = 0, \tag{3.2}$$

$$-V_x^{(n)} + \big[U,V^{(n)}\big] = 0, \tag{3.3}$$

$$-V_{t_r}^{(n)} + \big[\widetilde{V}^{(r)},V^{(n)}\big] = 0. \tag{3.4}$$

Through a direct calculation we can show that $yI - V^{(n)}$ satisfies equations (3.3) and (3.4). So $\mathcal{F}_m(\lambda,y) = \det(yI - V^{(n)})$ is an independent constant of the variables x and t_r, from which we can define a trigonal curve $\mathcal{K}_{m-1} : \mathcal{F}_m(\lambda,y) = 0$ with the expansion

$$\det\big(yI - V^{(n)}\big) = y^3 + yS_m(\lambda) - T_m(\lambda) = 0, \tag{3.5}$$

where

$$S_m = \sum_{1\le i<j\le 3} \begin{vmatrix} V_{ii}^{(n)} & V_{ij}^{(n)} \\ V_{ji}^{(n)} & V_{jj}^{(n)} \end{vmatrix}, \qquad T_m = \begin{vmatrix} V_{11}^{(n)} & V_{12}^{(n)} & V_{13}^{(n)} \\ V_{21}^{(n)} & V_{22}^{(n)} & V_{23}^{(n)} \\ V_{31}^{(n)} & V_{32}^{(n)} & V_{33}^{(n)} \end{vmatrix}.$$

Immediately, from (2.10) if we choose $\beta_0 = 1$, α_0 an arbitrary constant or $\beta_0 = 0$, $\alpha_0 = 1$, we shall know that the corresponding values of m in (3.5) are $3n+2$ or $3n+1$, respectively. For the convenience, the compactification of the curve $\mathcal{K}_{m-1}$ is denoted by the same symbol $\mathcal{K}_{m-1}$. Thus $\mathcal{K}_{m-1}$ becomes a three-sheeted Riemann surface of arithmetic genus $m-1$ when it is nonsingular or smooth.

Next we shall introduce the meromorphic function $\phi_1(P,x,t_r)$, which is closely related to $\psi(P,x,x_0,t_r,t_{0,r})$, by

$$\phi_1(P,x,t_r) = \frac{\partial_x \psi_1(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})}, \quad P \in \mathcal{K}_{m-1}, x \in \mathbb{C}, \tag{3.6}$$

which implies from (3.1) that

$$\begin{aligned} \phi_1(P,x,t_r) &= \frac{\varepsilon(m)F_m(\lambda,x,t_r)}{y^2 V_{23}^{(n)}(\lambda,x,t_r) - yC_m(\lambda,x,t_r) + D_m(\lambda,x,t_r)} \\ &= \frac{y^2 V_{13}^{(n)}(\lambda,x,t_r) - yA_m(\lambda,x,t_r) + B_m(\lambda,x,t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x,t_r)} \\ &= \frac{yV_{23}^{(n)}(\lambda,x,t_r) + C_m(\lambda,x,t_r)}{yV_{13}^{(n)}(\lambda,x,t_r) + A_m(\lambda,x,t_r)}, \end{aligned} \tag{3.7}$$

where $P = (\lambda, y) \in \mathcal{K}_{m-1}$, $(x,t_r) \in \mathbb{C}^2$,

$$\begin{aligned} A_m &= V_{12}^{(n)}V_{23}^{(n)} - V_{13}^{(n)}V_{22}^{(n)}, \\ B_m &= V_{13}^{(n)}\big(V_{11}^{(n)}V_{33}^{(n)} - V_{13}^{(n)}V_{31}^{(n)}\big) + V_{12}^{(n)}\big(V_{11}^{(n)}V_{23}^{(n)} - V_{13}^{(n)}V_{21}^{(n)}\big), \\ C_m &= V_{13}^{(n)}V_{21}^{(n)} - V_{11}^{(n)}V_{23}^{(n)}, \\ D_m &= V_{23}^{(n)}\big(V_{22}^{(n)}V_{33}^{(n)} - V_{23}^{(n)}V_{32}^{(n)}\big) + V_{21}^{(n)}\big(V_{13}^{(n)}V_{22}^{(n)} - V_{12}^{(n)}V_{23}^{(n)}\big), \end{aligned} \tag{3.8}$$

$$\begin{aligned} E_{m-1} &= -\varepsilon(m)\big[V_{13}^{(n)}\big(V_{13}^{(n)}V_{32}^{(n)} - V_{12}^{(n)}V_{33}^{(n)}\big) + V_{12}^{(n)}\big(V_{13}^{(n)}V_{22}^{(n)} - V_{12}^{(n)}V_{23}^{(n)}\big)\big], \\ F_m &= \varepsilon(m)\big[V_{23}^{(n)}\big(V_{23}^{(n)}V_{31}^{(n)} - V_{21}^{(n)}V_{33}^{(n)}\big) + V_{21}^{(n)}\big(V_{11}^{(n)}V_{23}^{(n)} - V_{13}^{(n)}V_{21}^{(n)}\big)\big], \end{aligned} \tag{3.9}$$

and

$$\varepsilon(m) = \begin{cases} -1 & \text{if } m = 3n+2, \\ 1 & \text{if } m = 3n+1, \end{cases}$$

which is introduced to ensure that E_{m-1}, F_m are both monic polynomials. It is easy to see that there exist various interrelationships between polynomials A_m, B_m, C_m, D_m, E_{m-1}, F_m

and S_m, T_m, some of which are summarized as follows:

$$\begin{aligned}
&\varepsilon(m)V_{13}^{(n)}F_m = V_{23}^{(n)}D_m - S_m\big(V_{23}^{(n)}\big)^2 - C_m^2,\\
&\varepsilon(m)A_mF_m = T_m\big(V_{23}^{(n)}\big)^2 + C_mD_m,\\
&\varepsilon(m)V_{23}^{(n)}E_{m-1} = S_m\big(V_{13}^{(n)}\big)^2 - V_{13}^{(n)}B_m + A_m^2,\\
&-\varepsilon(m)C_mE_{m-1} = T_m\big(V_{13}^{(n)}\big)^2 + A_mB_m,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&V_{23}^{(n)}B_m + V_{13}^{(n)}D_m - V_{13}^{(n)}V_{23}^{(n)}S_m + A_mC_m = 0,\\
&V_{13}^{(n)}V_{23}^{(n)}T_m + V_{23}^{(n)}A_mS_m + V_{13}^{(n)}C_mS_m - B_mC_m - A_mD_m = 0,\\
&V_{23}^{(n)}A_mT_m + V_{13}^{(n)}C_mT_m - E_{m-1}F_m - B_mD_m = 0,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\varepsilon(m)E_{m-1,x} = 2S_mV_{13}^{(n)} - 3B_m,\\
&V_{23}^{(n)}F_{m,x} = -3V_{22}^{(n)}F_m + \varepsilon(m)\big(V_{21}^{(n)} - uV_{23}^{(n)}\big)\big(2V_{23}^{(n)}S_m - 3D_m\big).
\end{aligned} \tag{3.12}$$

For displaying the properties of $\phi_1(P,x,t_r)$ exactly, we introduce the holomorphic map $*$, changing sheets, as

$$\begin{aligned}
&*: \begin{cases} \mathcal{K}_{m-1} \to \mathcal{K}_{m-1},\\ P = (\lambda, y_i(\lambda)) \to P^* = (\lambda, y_{i+1(\mathrm{mod}\,3)}(\lambda)), \quad i = 0,1,2,\end{cases}\\
&P^{**} := \big(P^*\big)^*, \qquad \text{etc.},
\end{aligned}$$

where $y_i(\lambda)$, $i = 0,1,2$, denote the three branches of $y(P)$ satisfying $\mathcal{F}_m(\lambda,y) = 0$. Then it is easy to show the properties of $\phi_1(P,x,t_r)$ immediately:

$$\begin{aligned}
&\phi_{1,xx}(P,x,t_r) + 3\phi_1(P,x,t_r)\phi_{1,x}(P,x,t_r) + \phi_1^3(P,x,t_r) - 2u(x,t_r)\phi_1(P,x,t_r)\\
&\quad = u_x(x,t_r) + v(x,t_r) + \lambda,
\end{aligned} \tag{3.13}$$

$$\begin{aligned}
\phi_{1,t_r}(P,x,t_r) = \partial_x\big[&\widetilde{V}_{11}^{(r)}(\lambda,x,t_r) + \widetilde{V}_{12}^{(r)}(\lambda,x,t_r)\phi_1(P,x,t_r)\\
&+ \widetilde{V}_{13}^{(r)}(\lambda,x,t_r)\big(\phi_{1,x}(P,x,t_r) + \phi_1^2(P,x,t_r) - u(x,t_r)\big)\big],
\end{aligned} \tag{3.14}$$

$$\phi_1(P,x,t_r)\phi_1\big(P^*,x,t_r\big)\phi_1\big(P^{**},x,t_r\big) = \frac{F_m(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}, \tag{3.15}$$

$$\phi_1(P,x,t_r) + \phi_1\big(P^*,x,t_r\big) + \phi_1\big(P^{**},x,t_r\big) = \frac{E_{m-1,x}(\lambda,x,t_r)}{E_{m-1}(\lambda,x,t_r)}, \tag{3.16}$$

$$\begin{aligned}
&y(P)\phi_1(P,x,t_r) + y\big(P^*\big)\phi_1\big(P^*,x,t_r\big) + y\big(P^{**}\big)\phi_1\big(P^{**},x,t_r\big)\\
&\quad = \frac{3T_m(\lambda)V_{32}^{(n)}(\lambda,x,t_r) + 2S_m(\lambda)A_m(\lambda,x,t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x,t_r)},
\end{aligned} \tag{3.17}$$

$$\begin{aligned}
&\frac{1}{\phi_1(P,x,t_r)} + \frac{1}{\phi_1(P^*,x,t_r)} + \frac{1}{\phi_1(P^{**},x,t_r)}\\
&\quad = \frac{-3V_{22}^{(n)}(\lambda,x,t_r)}{V_{21}^{(n)}(\lambda,x,t_r) - u(x,t_r)V_{23}^{(n)}(\lambda,x,t_r)}\\
&\qquad - \frac{V_{23}^{(n)}(\lambda,x,t_r)}{V_{21}^{(n)}(\lambda,x,t_r) - u(x,t_r)V_{23}^{(n)}(\lambda,x,t_r)}\frac{F_{m,x}(\lambda,x,t_r)}{F_m(\lambda,x,t_r)}.
\end{aligned} \tag{3.18}$$

After tedious calculations, we have the following lemma.

Lemma 3.1 *Assume* (3.1), (3.2), *and let* $(\lambda, x, x_0, t_r, t_{0,r}) \in \mathbb{C}^5$. *Then*

$$\begin{aligned} E_{m-1,t_r}(\lambda,x,t_r) &= E_{m-1,x}\left(\widetilde{V}_{12}^{(r)} - \frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}V_{12}^{(n)}\right) + 3E_{m-1}\left(\widetilde{V}_{11}^{(r)} - \frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}V_{11}^{(n)}\right), \\ F_{m,t_r}(\lambda,x,t_r) &= F_{m,x}\left(\widetilde{V}_{23}^{(r)} - \frac{\widetilde{V}_{21}^{(r)} - u\widetilde{V}_{23}^{(r)}}{V_{21}^{(n)} - uV_{23}^{(n)}}V_{23}^{(n)}\right) \\ &\quad + 3F_m\left(\widetilde{V}_{22}^{(r)} - \frac{\widetilde{V}_{21}^{(r)} - u\widetilde{V}_{23}^{(r)}}{V_{21}^{(n)} - uV_{23}^{(n)}}V_{22}^{(n)}\right). \end{aligned} \tag{3.19}$$

Moreover, by institute of (3.2), (3.6), (3.16), and (3.19), we arrive at the properties of $\psi_1(P,x,x_0,t_r,t_{0,r})$ immediately.

Lemma 3.2 *Assume* (3.1), (3.6), $P = (\lambda, y(P)) \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$, *and let* $(\lambda, x, x_0, t_r, t_{0,r}) \in \mathbb{C}^5$. *Then*

$$\begin{aligned} \frac{\psi_{1,t_r}(P,x,x_0,t_r,t_{0,r})}{\psi_1(P,x,x_0,t_r,t_{0,r})} &= \widetilde{V}_{13}^{(r)}(\lambda,x,t_r)\left[\phi_{1,x}(P,x,t_r) + \phi_1^2(P,x,t_r) - u(x,t_r)\right] \\ &\quad + \widetilde{V}_{12}^{(r)}(\lambda,x,t_r)\phi_1(P,x,t_r) + \widetilde{V}_{11}^{(r)}(\lambda,x,t_r), \end{aligned} \tag{3.20}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_1\left(P^*,x,x_0,t_r,t_{0,r}\right)\psi_1\left(P^{**},x,x_0,t_r,t_{0,r}\right) = \frac{E_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}, \tag{3.21}$$

$$\psi_{1,x}(P,x,x_0,t_r,t_{0,r})\psi_{1,x}\left(P^*,x,x_0,t_r,t_{0,r}\right)\psi_{1,x}\left(P^{**},x,x_0,t_r,t_{0,r}\right) = \frac{F_m(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}, \tag{3.22}$$

$$\begin{aligned} &\psi_1(P,x,x_0,t_r,t_{0,r}) \\ &\quad = \exp\left(\int_{x_0}^{x}\phi_1\left(P,x',t_r\right)dx'\right. \\ &\qquad + \int_{t_{0,r}}^{t_r}\left[\widetilde{V}_{13}^{(r)}\left(\lambda,x_0,t'\right)\left(\frac{y(P) - V_{11}^{(n)}(\lambda,x_0,t')}{V_{13}^{(n)}(\lambda,x_0,t')} - \frac{V_{12}^{(n)}(\lambda,x_0,t')}{V_{13}^{(n)}(\lambda,x_0,t')}\phi_1\left(P,x_0,t'\right)\right)\right. \\ &\qquad \left.\left. + \widetilde{V}_{12}^{(r)}\left(\lambda,x_0,t'\right)\phi_1\left(P,x_0,t'\right) + \widetilde{V}_{11}^{(r)}\left(\lambda,x_0,t'\right)\right]dt'\right), \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\psi_1(P,x,x_0,t_r,t_{0,r}) \\ &\quad = \left[\frac{E_{m-1}(\lambda,x,t_r)}{E_{m-1}(\lambda,x_0,t_{0,r})}\right]^{1/3} \\ &\qquad \times \exp\left(\int_{x_0}^{x}\frac{y(P)^2V_{13}^{(n)}(\lambda,x',t_r) - y(P)A_m(\lambda,x',t_r) + \frac{2}{3}S_m(\lambda)V_{13}^{(n)}(\lambda,x',t_r)}{-\varepsilon(m)E_{m-1}(\lambda,x',t_r)}dx'\right. \\ &\qquad + \int_{t_{0,r}}^{t_r}\left[\frac{y(P)^2V_{13}^{(n)}(\lambda,x_0,t') - y(P)A_m(\lambda,x_0,t') + \frac{2}{3}S_m(\lambda)V_{13}^{(n)}(\lambda,x_0,t')}{-\varepsilon(m)E_{m-1}(\lambda,x_0,t')}\right. \\ &\qquad \times\left(\tilde{V}_{12}^{(r)}\left(\lambda,x_0,t'\right) - \frac{\tilde{V}_{13}^{(r)}(\lambda,x_0,t')}{V_{13}^{(n)}(\lambda,x_0,t')}V_{12}^{(n)}\left(\lambda,x_0,t'\right)\right) \\ &\qquad \left.\left. + y(P)\frac{\tilde{V}_{13}^{(r)}(\lambda,x_0,t')}{V_{13}^{(n)}(\lambda,x_0,t')}\right]dt'\right). \end{aligned} \tag{3.24}$$

By inspection of (3.9), one shall know that E_{m-1} and F_m are both monic polynomials with respect to λ of degree $m-1$ and m, respectively. Hence we may decompose them into

$$E_{m-1}(\lambda,x,t_r)=\prod_{j=1}^{m-1}\bigl(\lambda-\mu_j(x,t_r)\bigr),\tag{3.25}$$

$$F_m(\lambda,x,t_r)=\prod_{l=0}^{m-1}\bigl(\lambda-\nu_l(x,t_r)\bigr).\tag{3.26}$$

Define

$$\begin{aligned}&\hat{\mu}_j(x,t_r)=\bigl(\mu_j(x,t_r),y\bigl(\hat{\mu}_j(x,t_r)\bigr)\bigr)=\left(\mu_j(x,t_r),-\frac{A_m(\mu_j(x,t_r),x,t_r)}{V_{13}^{(n)}(\mu_j(x,t_r),x,t_r)}\right)\in\mathcal{K}_{m-1},\\&\quad 1\le j\le m-1,(x,t_r)\in\mathbb{C}^2,\end{aligned}\tag{3.27}$$

$$\begin{aligned}&\hat{\nu}_l(x,t_r)=\bigl(\nu_l(x,t_r),y\bigl(\hat{\nu}_l(x,t_r)\bigr)\bigr)=\left(\nu_l(x,t_r),-\frac{C_m(\nu_l(x,t_r),x,t_r)}{V_{23}^{(n)}(\nu_l(x,t_r),x,t_r)}\right)\in\mathcal{K}_{m-1},\\&\quad 0\le l\le m-1,(x,t_r)\in\mathbb{C}^2.\end{aligned}\tag{3.28}$$

The dynamics of the zeros $\mu_j(x,t_r)$ and $\nu_l(x,t_r)$ of $E_{m-1}(\lambda,x,t_r)$ and $F_m(\lambda,x,t_r)$ are then described in terms of Dubrovin-type equations as follows.

Lemma 3.3 (i) *Suppose that the zeros $\mu_j(x,t_r)_{j=1,\ldots,m-1}$ of $E_{m-1}(P,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Then $\mu_j(x,t_r)_{j=1,\ldots,m-1}$ satisfy the system of differential equations*

$$\begin{aligned}&\mu_{j,x}(x,t_r)=\frac{\varepsilon(m)V_{13}^{(n)}(\mu_j(x,t_r),x,t_r)[3y^2(\hat{\mu}_j(x,t_r))+S_m(\mu_j(x,t_r))]}{\prod_{k=1,k\neq j}^{m-1}(\mu_j(x,t_r)-\mu_k(x,t_r))},\\&\quad 1\le j\le m-1,\end{aligned}\tag{3.29}$$

$$\begin{aligned}\mu_{j,t_r}(x,t_r)={}&\bigl[V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr)\widetilde{V}_{12}^{(r)}\bigl(\mu_j(x,t_r),x,t_r\bigr)\\&-\widetilde{V}_{13}^{(r)}\bigl(\mu_j(x,t_r),x,t_r\bigr)V_{12}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr)\bigr]\\&\times\frac{\varepsilon(m)[3y^2(\hat{\mu}_j(x,t_r))+S_m(\mu_j(x,t_r))]}{\prod_{k=1,k\neq j}^{m-1}(\mu_j(x,t_r)-\mu_k(x,t_r))},\quad 1\le j\le m-1.\end{aligned}\tag{3.30}$$

(ii) *Suppose that the zeros $\nu_l(x,t_r)_{l=0,\ldots,m-1}$ of $F_m(P,x,t_r)$ remain distinct for $(x,t_r)\in\Omega_\nu$, where $\Omega_\nu\subseteq\mathbb{C}^2$ is open and connected. Then $\nu_l(x,t_r)_{l=0,\ldots,m-1}$ satisfy the system of differential equations*

$$\begin{aligned}&\nu_{l,x}(x,t_r)\\&\quad=\frac{\varepsilon(m)[V_{21}^{(n)}(\nu_l(x,t_r),x,t_r)-uV_{23}^{(n)}(\nu_l(x,t_r),x,t_r)][3y^2(\hat{\nu}_l(x,t_r))+S_m(\nu_l(x,t_r))]}{\prod_{k=0,k\neq l}^{m-1}(\nu_l(x,t_r)-\nu_k(x,t_r))},\\&\quad 0\le l\le m-1,\end{aligned}\tag{3.31}$$

$$\begin{aligned}\nu_{l,t_r}(x,t_r) &= \bigl[\bigl(V_{21}^{(n)}\bigl(\nu_l(x,t_r),x,t_r\bigr) - uV_{23}^{(n)}\bigl(\nu_l(x,t_r),x,t_r\bigr)\bigr)\widetilde{V}_{23}^{(r)}\bigl(\nu_l(x,t_r),x,t_r\bigr) \\ &\quad - \bigl(\widetilde{V}_{21}^{(r)}\bigl(\nu_l(x,t_r),x,t_r\bigr) - u\widetilde{V}_{23}^{(r)}\bigl(\nu_l(x,t_r),x,t_r\bigr)\bigr)V_{23}^{(n)}\bigl(\nu_l(x,t_r),x,t_r\bigr)\bigr] \\ &\quad \times \frac{\varepsilon(m)[3y^2(\hat{\nu}_l(x,t_r)) + S_m(\nu_l(x,t_r))]}{\prod_{\substack{k=0\\k\neq l}}^{m-1}(\nu_l(x,t_r)-\nu_k(x,t_r))}, \quad 0 \le l \le m-1.\end{aligned} \tag{3.32}$$

Proof Using (3.10), we have ($\lambda = \mu_j(x,t_r)$)

$$\begin{aligned}&S_m\bigl(\mu_j(x,t_r)\bigr)\bigl(V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr)\bigr)^2 - B_m\bigl(\mu_j(x,t_r),x,t_r\bigr)V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr) \\ &\quad + A_m^2\bigl(\mu_j(x,t_r),x,t_r\bigr) = 0,\end{aligned} \tag{3.33}$$

that is,

$$\begin{aligned}B_m\bigl(\mu_j(x,t_r),x,t_r\bigr) &= S_m\bigl(\mu_j(x,t_r)\bigr)V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr) + \frac{A_m^2(\mu_j(x,t_r),x,t_r)}{V_{13}^{(n)}(\mu_j(x,t_r),x,t_r)} \\ &= \bigl[S_m\bigl(\mu_j(x,t_r)\bigr) + y^2\bigl(\hat{\mu}_j(x,t_r)\bigr)\bigr]V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr).\end{aligned}$$

After substituting B_m into (3.12), we get

$$\begin{aligned}&\varepsilon(m)E_{m-1,x}\bigl(\mu_j(x,t_r),x,t_r\bigr) \\ &\quad = -V_{13}^{(n)}\bigl(\mu_j(x,t_r),x,t_r\bigr)\bigl[3y^2\bigl(\hat{\mu}_j(x,t_r)\bigr) + S_m\bigl(\mu_j(x,t_r)\bigr)\bigr].\end{aligned} \tag{3.34}$$

On the other hand, derivatives of the expression in (3.25) with respect to x and t_r respectively, are

$$E_{m-1,x}\bigl(\mu_j(x,t_r),x,t_r\bigr) = -\mu_{j,x}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}\bigl(\mu_j(x,t_r)-\mu_k(x,t_r)\bigr), \tag{3.35}$$

$$E_{m-1,t_r}\bigl(\mu_j(x,t_r),x,t_r\bigr) = -\mu_{j,t_r}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}\bigl(\mu_j(x,t_r)-\mu_k(x,t_r)\bigr). \tag{3.36}$$

Comparing (3.34) and (3.35), we can obtain (3.29). From (3.19), one can know

$$\begin{aligned}&E_{m-1,t_r}\bigl(\mu_j(x,t_r),x,t_r\bigr) \\ &\quad = E_{m-1,x}\bigl(\mu_j(x,t_r),x,t_r\bigr)\frac{V_{13}^{(n)}\widetilde{V}_{12}^{(r)} - \widetilde{V}_{13}^{(r)}V_{12}^{(n)}}{V_{13}^{(n)}} \\ &\quad = -\mu_{j,x}(x,t_r)\prod_{\substack{k=1\\k\neq j}}^{m-1}\bigl(\mu_j(x,t_r)-\mu_k(x,t_r)\bigr)\frac{V_{13}^{(n)}\widetilde{V}_{12}^{(r)} - \widetilde{V}_{13}^{(r)}V_{12}^{(n)}}{V_{13}^{(n)}} \\ &\quad = -\varepsilon(m)\bigl[3y^2\bigl(\hat{\mu}_j(x,t_r)\bigr) + S_m\bigl(\mu_j(x,t_r)\bigr)\bigr]\bigl(V_{13}^{(n)}\widetilde{V}_{12}^{(r)} - \widetilde{V}_{13}^{(r)}V_{12}^{(n)}\bigr),\end{aligned} \tag{3.37}$$

then we have (3.30). Similarly, we can prove (3.31) and (3.32). □

4 Algebro-geometric solutions to the second-order Benjamin-Ono hierarchy

In our final and principal section, we obtain Riemann theta function representations for the Baker-Akhiezer function and the meromorphic function; especially, the theta function representations for general algebro-geometric solutions u, v of the second-order Benjamin-Ono hierarchy. For the convenience, we assume that the curve $\mathcal{K}_{m-1}$ is non-singular.

For investigating the asymptotic expansion of $\phi_1(P,x,t_r)$ near P_∞, we choose the local coordinate $\zeta = \lambda^{-\frac{1}{3}}$, then we get the following lemma.

Lemma 4.1 *Let $(x,t_r) \in \mathbb{C}^2$, near $P_\infty \in \mathcal{K}_{m-1}$, we have*

$$\phi_1(P,x,t_r) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j(x,t_r)\zeta^j \quad \text{as } P \to P_\infty, \tag{4.1}$$

where

$$\begin{aligned}
&\kappa_0 = 1, \qquad \kappa_1 = 0, \qquad \kappa_2 = \frac{2}{3}u, \qquad \kappa_3 = \frac{1}{3}(v - u_x),\\
&\kappa_4 = \frac{1}{9}u_{xx} - \frac{1}{3}v_x, \qquad \kappa_5 = \frac{2}{9}(v_{xx} - uu_x - uv),\\
&\kappa_j = -\frac{1}{3}\left[\kappa_{j-2,xx} + 3\sum_{i=2}^{j-1}\kappa_{j-1-i}\kappa_{i,x} + \sum_{i=2}^{j-1}\kappa_i\kappa_{j-i} + \sum_{i=2}^{j-1}\sum_{l=0}^{j-i}\kappa_i\kappa_l\kappa_{j-i-l} - 2u\kappa_{j-2}\right] \quad (j \geq 4).
\end{aligned} \tag{4.2}$$

Proof In terms of the local coordinate $\zeta = \lambda^{-\frac{1}{3}}$, (3.13) reads

$$\phi_{1,xx} + 3\phi_1\phi_{1,x} + \phi_1^3 - 2u\phi_1 = u_x + v + \zeta^{-3}. \tag{4.3}$$

Then, by inserting the power series ansatz of $\phi_1(P,x,t_r)$ in ζ as follows:

$$\phi_1(P,x,t_r) \underset{\zeta\to 0}{=} \frac{1}{\zeta}\sum_{j=0}^{\infty}\kappa_j(x,t_r)\zeta^j \tag{4.4}$$

into (4.3)

$$\begin{aligned}
&\zeta^{-1}\sum_{j=0}^{\infty}\kappa_{j,xx}\zeta^j + 3\zeta^{-2}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\kappa_j\kappa_{i,x}\zeta^{(j+i)} + \zeta^{-3}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\kappa_j\kappa_i\kappa_l\zeta^{(j+i+l)} - 2u\zeta^{-1}\sum_{j=0}^{\infty}\kappa_j\zeta^j\\
&\quad = u_x + v + \zeta^{-3},
\end{aligned} \tag{4.5}$$

and comparing the same powers of ζ in (4.5), we arrive at (4.2). □

One infers, from (3.7), (3.25), (3.26), and (4.1), that the divisor $(\phi_1(P,x,t_r))$ of $\phi_1(P,x,t_r)$ is given by

$$\big(\phi_1(P,x,t_r)\big) = \mathcal{D}_{\hat{\nu}_0(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)}(P) - \mathcal{D}_{P_\infty,\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-1}(x,t_r)}(P). \tag{4.6}$$

That is, $\hat{\nu}_0(x,t_r),\ldots,\hat{\nu}_{m-1}(x,t_r)$ are the m zeros of $\phi_1(P,x,t_r)$ and P_∞, $\hat{\mu}_1(x,t_r),\ldots,\hat{\mu}_{m-1}(x,t_r)$ are its m poles.

A straightforward calculation reveals that the asymptotic behaviors of $y(P)$ and $S_m(\lambda)$ near P_∞ are

$$y(P) \underset{\zeta\to 0}{=} \begin{cases} \zeta^{-3n-2}[1+\alpha_0\zeta+\beta_1\zeta^3+\alpha_1\zeta^4+O(\zeta^6)] & \text{as } P\to P_\infty, m=3n+2, \\ \zeta^{-3n-1}[1+\beta_1\zeta^2+\alpha_1\zeta^3+O(\zeta^5)] & \text{as } P\to P_\infty, m=3n+1, \end{cases} \tag{4.7}$$

$$S_m(\lambda) \underset{\zeta\to 0}{=} \begin{cases} -3\zeta^{-6n-3}[\alpha_0+(\alpha_1+\beta_1\alpha_0)\zeta^3+O(\zeta^6)] & \text{as } P\to P_\infty, m=3n+2, \\ -3\zeta^{-6n}[\beta_1+O(\zeta^3)] & \text{as } P\to P_\infty, m=3n+1. \end{cases} \tag{4.8}$$

Next we will introduce the three kinds of holomorphic differentials and show some properties of them. The holomorphic differentials $\eta_l(P)$ on $\mathcal{K}_{m-1}$ are defined by

$$\eta_l(P) = \frac{1}{3y(P)^2+S_m} \begin{cases} \lambda^{l-1}\,d\lambda, & 1\le l\le m-n-1, \\ y(P)\lambda^{l+n-m}\,d\lambda, & m-n\le l\le m-1. \end{cases} \tag{4.9}$$

To construct the theta function and normalize the holomorphic differentials, we choose a homology basis $\{\mathbb{a}_j,\mathbb{b}_j\}_{j=1}^{m-1}$ on $\mathcal{K}_{m-1}$ so that they satisfy

$$\mathbb{a}_j\circ\mathbb{b}_k=\delta_{j,k}, \qquad \mathbb{a}_j\circ\mathbb{a}_k=0, \qquad \mathbb{b}_j\circ\mathbb{b}_k=0, \quad j,k=1,\dots,m-1.$$

Introducing an invertible matrix $E=(E_{j,k})_{(m-1)\times(m-1)}$ and $\underline{e}(k)=(e_1(k),\dots,e_{m-1}(k))$, where

$$E_{j,k}=\int_{\mathbb{a}_k}\eta_j, \qquad e_j(k)=\left(E^{-1}\right)_{j,k},$$

and the normalized holomorphic differentials ω_j for $j=1,\dots,m-1$,

$$\begin{aligned} &\omega_j=\sum_{l=1}^{m-1}e_j(l)\eta_l, \qquad \int_{\mathbb{a}_k}\omega_j=\delta_{j,k}, \\ &\int_{\mathbb{b}_k}\omega_j=\tau_{j,k} \quad (\tau_{j,k}=\tau_{k,j}), j,k=1,\dots,m-1. \end{aligned} \tag{4.10}$$

Let $\omega^{(2)}_{P_\infty,2}(P)$ denote the normalized second Abel differential defined by

$$\omega^{(2)}_{P_\infty,2}(P) = -\sum_{j=1}^{m-1}z_j\eta_j(P) - \frac{1}{3y(P)^2+S_m} \begin{cases} \lambda^{2n}\,d\lambda, & m=3n+1, \\ y(P)\lambda^n\,d\lambda, & m=3n+2, \end{cases} \tag{4.11}$$

which is holomorphic on $\mathcal{K}_{m-1}\setminus\{P_\infty\}$ with a pole of order 2 at P_∞, and the constants $\{z_j\}_{j=1,\dots,m-1}$ are determined by the normalization condition

$$\int_{\mathbb{a}_j}\omega^{(2)}_{P_\infty,2}(P)=0, \quad j=1,\dots,m-1.$$

The $\mathbb{b}$-periods of the differential $\omega^{(2)}_{P_\infty,2}$ are denoted by

$$\underline{U}^{(2)}_2=\left(U^{(2)}_{2,1},\dots,U^{(2)}_{2,m-1}\right), \qquad U^{(2)}_{2,j}=\frac{1}{2\pi i}\int_{\mathbb{b}_j}\omega^{(2)}_{P_\infty,2}(P), \quad j=1,\dots,m-1. \tag{4.12}$$

On the other hand, $\omega^{(2)}_{P_\infty,3}(P)$ denotes the normalized third Abel differential which is holomorphic on $\mathcal{K}_{m-1}\setminus\{P_\infty\}$ with a pole of order 3 at P_∞

$$\omega^{(2)}_{P_\infty,3}(P)\underset{\zeta\to 0}{=}\big(\zeta^{-3}+O(1)\big)\,d\zeta \quad \text{as } P\to P_\infty, \tag{4.13}$$

and the $\mathbb{b}$-periods of it are defined by

$$\underline{U}^{(2)}_3=\big(U^{(2)}_{3,1},\ldots,U^{(2)}_{3,m-1}\big),\qquad U^{(2)}_{3,j}=\frac{1}{2\pi i}\int_{\mathbb{b}_j}\omega^{(2)}_{P_\infty,3},\quad j=1,\ldots,m-1.$$

Furthermore, the normalized third Abel differential $\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P)$ is holomorphic on $\mathcal{K}_{m-1}\setminus\{P_\infty,\hat{\nu}_0(x)\}$ with simple poles at P_∞ and $\hat{\nu}_0(x)$ with residues ± 1, respectively, that is,

$$\begin{aligned}
&\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P)\underset{\zeta\to 0}{=}\big(\zeta^{-1}+O(1)\big)\,d\zeta \quad \text{as } P\to P_\infty,\\
&\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P)\underset{\zeta\to 0}{=}\big(-\zeta^{-1}+O(1)\big)\,d\zeta \quad \text{as } P\to \hat{\nu}_0(x).
\end{aligned} \tag{4.14}$$

Then

$$\begin{aligned}
&\int_{P_0}^{P}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P)=\ln\zeta+e^{(3)}(P_0)+O(\zeta) \quad \text{as } P\to P_\infty,\\
&\int_{P_0}^{P}\omega^{(3)}_{P_\infty,\hat{\nu}_0(x)}(P)=-\ln\zeta+e^{(3)}(P_0)+O(\zeta) \quad \text{as } P\to \hat{\nu}_0(x)
\end{aligned} \tag{4.15}$$

with $e^{(3)}(P_0)$ being an integration constant.

A straightforward Laurent expansion of (4.9), (4.10), and (4.11) near P_∞ yields the following results.

Lemma 4.2 *Near P_∞ in the local coordinate $\zeta=\lambda^{-\frac{1}{3}}$, the differentials $\underline{\omega}$ and $\omega^{(2)}_{P_\infty,2}$ have the Laurent series*

$$\underline{\omega}=(\omega_1,\ldots,\omega_{m-1})\underset{\zeta\to 0}{=}\big(\underline{\rho}_0+\underline{\rho}_1\zeta+\underline{\rho}_2\zeta^3+O(\zeta^4)\big)\,d\zeta, \tag{4.16}$$

with

$$\begin{aligned}
&\underline{\rho}_0=\begin{cases}-\underline{e}(m-n-1), & m=3n+2,\\ -\underline{e}(m-1), & m=3n+1,\end{cases}\\
&\underline{\rho}_1=\begin{cases}-\underline{e}(m-1)+\alpha_0\underline{e}(m-n-1), & m=3n+2,\\ -\underline{e}(m-n-1), & m=3n+1,\end{cases}\\
&\underline{\rho}_2=\begin{cases}(2\beta_1-\alpha_0^3)\underline{e}(m-n-1)+\alpha_0^2\underline{e}(m-1)-\underline{e}(m-n-2), & m=3n+2,\\ \alpha_1\underline{e}(m-1)+\beta_1\underline{e}(m-n-1)-\underline{e}(m-2), & m=3n+1,\end{cases}
\end{aligned} \tag{4.17}$$

$$\omega^{(2)}_{P_\infty,2}(P)\underset{\zeta\to 0}{=}\begin{cases}(\zeta^{-2}+z_{m-n-1}-\alpha_0^2+(-\beta_1+\alpha_0^3-\alpha_0 z_{m-n-1}+z_{m-1})\zeta+O(\zeta^2))\,d\zeta,\\ \quad m=3n+2,\\ (\zeta^{-2}+z_{m-1}-\beta_1+(z_{m-n-1}-2\alpha_1)\zeta+O(\zeta^2))\,d\zeta,\\ \quad m=3n+1.\end{cases}$$

From Lemma 4.2 we infer

$$\int_{P_0}^{P} \omega_{P_\infty,2}^{(2)}(P) \underset{\zeta\to 0}{=} -\zeta^{-1} + e_2^{(2)}(P_0) - q_1\zeta + q_2\zeta^2 + O(\zeta^3) \quad \text{as } P \to P_\infty, \tag{4.18}$$

where $e_2^{(2)}(P_0)$ is an appropriate constant, and

$$q_1 = \begin{cases} -z_{m-n-1} + \alpha_0^2, & m = 3n+2, \\ -z_{m-1} + \beta_1, & m = 3n+1, \end{cases}$$
$$q_2 = \begin{cases} \frac{1}{2}(-\beta_1 + \alpha_0^3 - \alpha_0 z_{m-n-1} + z_{m-1}), & m = 3n+2, \\ \frac{1}{2} z_{m-n-1} - \alpha_1, & m = 3n+1. \end{cases} \tag{4.19}$$

Let $\theta(\underline{\lambda})$ denote the Riemann theta function [20–22] associated with $\mathcal{K}_{m-1}$ and the appropriately fixed homology basis $\{\mathbb{a}_j, \mathbb{b}_j\}_{j=1}^{m-1}$. Next we choose a convenient base point $P_0 \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$. For brevity, define the function $\underline{\lambda} : \mathcal{K}_{m-1} \times \sigma^{m-1}\mathcal{K}_{m-1} \to \mathbb{C}$ by

$$\underline{\lambda}(P, \underline{Q}) = \underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}), \quad P \in \mathcal{K}_{m-1},$$
$$\underline{Q} = (Q_1, \ldots, Q_{m-1}) \in \sigma^{m-1}\mathcal{K}_{m-1},$$

where $\underline{\Xi}_{P_0}$ is the vector of Riemann constants, and the Abel maps $\underline{A}_{P_0}(P)$ and $\underline{\alpha}_{P_0}(P)$ are defined by (period lattice $L_{m-1} = \{\underline{z} \in \mathbb{C}^{m-1} | \underline{z} = \underline{N} + \tau\underline{M}, \underline{N}, \underline{M} \in \mathbb{Z}^{m-1}\}$)

$$\underline{A}_{P_0} : \mathcal{K}_{m-1} \to \mathcal{J}(\mathcal{K}_{m-1}) = \mathbb{C}^{m-1}/L_{m-1},$$
$$P \mapsto \underline{A}_{P_0}(P) = \big(A_{P_0,1}(P), \ldots, A_{P_0,m-1}(P)\big) = \left(\int_{P_0}^{P} \omega_1, \ldots, \int_{P_0}^{P} \omega_{m-1}\right) (\operatorname{mod} L_{m-1}),$$

and

$$\underline{\alpha}_{P_0} : \operatorname{Div}(\mathcal{K}_{m-1}) \to \mathcal{J}(\mathcal{K}_{m-1}),$$
$$\mathcal{D} \mapsto \underline{\alpha}_{P_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_{m-1}} \mathcal{D}(P)\underline{A}_{P_0}(P).$$

In view of these preparations, we give the theta function representation of our fundamental object $\phi_1(P, x, t_r)$.

Theorem 4.3 *Let $P = (\lambda, y) \in \mathcal{K}_{m-1} \setminus \{P_\infty\}$, and let $(x, t_r), (x_0, t_{0,r}) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{C}^2$ is open and connected. Suppose also that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ is nonspecial for $(x, t_r) \in \Omega_\mu$. Then*

$$\phi_1(P, x, t_r) = \frac{\theta(\underline{\lambda}(P, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{\lambda}(P_\infty, \underline{\hat{\mu}}(x, t_r)))}{\theta(\underline{\lambda}(P_\infty, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{\lambda}(P, \underline{\hat{\mu}}(x, t_r)))} \exp\left(e^{(3)}(P_0) - \int_{P_0}^{P} \omega_{P_\infty, \hat{\nu}_0(x,t_r)}^{(3)}\right). \tag{4.20}$$

Proof Let Φ denote the right-hand side of (4.20). From (4.15) it follows that

$$\exp\left(e^{(3)}(P_0) - \int_{P_0}^{P} \omega_{P_\infty, \hat{\nu}_0(x,t_r)}^{(3)}\right) \underset{\zeta\to 0}{=} \zeta^{-1} + O(1). \tag{4.21}$$

Using (4.6) we immediately know that ϕ_1 has simple poles at $\underline{\hat{\mu}}(x,t_r)$ and P_∞, and simple zeros at $\hat{\nu}_0(x,t_r)$, $\underline{\hat{\nu}}(x,t_r)$. By (4.20) and the Riemann vanishing theorem, we see that Φ has the same properties. Using the Riemann-Roch theorem [21, 22], we conclude that the holomorphic function $\frac{\Phi}{\phi_1} = \gamma$, where γ is a constant. Using (4.21) and Lemma 4.1, we have

$$\frac{\Phi}{\phi_1} \underset{\zeta\to 0}{=} \frac{(1+O(\zeta))(\zeta^{-1}+O(1))}{\zeta^{-1}+O(\zeta)} \underset{\zeta\to 0}{=} 1+O(\zeta) \quad \text{as } P\to P_\infty, \tag{4.22}$$

from which we conclude $\gamma = 1$. □

Let $\omega^{(2)}_{P_\infty,s}$, $s = 3r+2$ (or $3r+1$), $r \in \mathbb{N}_0$, be the normalized differential of the second kind holomorphic on $\mathcal{K}_{m-1} \setminus \{P_\infty\}$, with a pole of order s at P_∞,

$$\omega^{(2)}_{P_\infty,s}(P) \underset{\zeta\to 0}{=} \left(\zeta^{-s}+O(1)\right)d\zeta \quad \text{as } P\to P_\infty.$$

Then we define the normalized differentials as

$$\begin{aligned}\widetilde{\Omega}^{(2)}_{P_\infty,s+1} &= \sum_{l=0}^{r}\tilde{\beta}_{r-l}(3l+2)\widetilde{\omega}^{(2)}_{P_\infty,3l+3} + \sum_{l=0}^{r}\tilde{\alpha}_{r-l}(3l+1)\widetilde{\omega}^{(2)}_{P_\infty,3l+2},\\ &s = 3r+2 \text{ (or } 3r+1), r\in\mathbb{N}_0,\end{aligned} \tag{4.23}$$

where

$$(\tilde{\alpha}_0,\tilde{\beta}_0) = \begin{cases}(\tilde{\alpha}_0,1), & s=3r+2,\\ (1,0), & s=3r+1,\end{cases} \qquad \tilde{\alpha}_0\in\mathbb{C}.$$

In addition, we define the vector of $\mathbb{b}$-periods of them as

$$\begin{aligned}&\underline{\widetilde{U}}^{(2)}_{s+1} = \left(\widetilde{U}^{(2)}_{s+1,1},\ldots,\widetilde{U}^{(2)}_{s+1,m-1}\right), \qquad \widetilde{U}^{(2)}_{s+1,j} = \frac{1}{2\pi i}\int_{\mathbb{b}_j}\widetilde{\Omega}^{(2)}_{P_\infty,s+1},\\ &j=1,\ldots,m-1, s=3r+2 \text{ (or } 3r+1), r\in\mathbb{N}_0.\end{aligned} \tag{4.24}$$

Motivated by the second integration in (3.23), one defines the function $I_s(P,x,t_r)$, meromorphic on $\mathcal{K}_{m-1}\times\mathbb{C}^2$, by

$$\begin{aligned}I_s(P,x,t_r) = {}& \widetilde{V}^{(r)}_{11}(\lambda,x,t_r) + \widetilde{V}^{(r)}_{12}(\lambda,x,t_r)\phi_1(P,x,t_r) + \widetilde{V}^{(r)}_{13}(\lambda,x,t_r)\big(\phi_{1,x}(P,x,t_r)\\ &+\phi_1^2(P,x,t_r) - u(x,t_r)\big).\end{aligned} \tag{4.25}$$

Denote by $\bar{I}_s(P,x,t_r)$ the associated homogeneous one replacing $\widetilde{V}^{(r)}_{1j}$ by $\bar{\widetilde{V}}^{(r)}_{1j}$, where

$$\bar{\widetilde{V}}^{(r)}_{1j} = \begin{cases}\widetilde{V}^{(r)}_{1j}|_{\tilde{\alpha}_0=1,\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_0=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0}, & s=3r+1,\\ \widetilde{V}^{(r)}_{1j}|_{\tilde{\beta}_0=1,\tilde{\alpha}_0=\tilde{\alpha}_1=\cdots=\tilde{\alpha}_r=\tilde{\beta}_1=\cdots=\tilde{\beta}_r=0}, & s=3r+2,\end{cases} \qquad j=1,2,3.$$

Lemma 4.4 *Let $s=3r+2$ (or $3r+1$), $r\in\mathbb{N}_0$, $(x,t_r)\in\mathbb{C}^2$, and $\lambda=\zeta^{-3}$ be the local coordinate near P_∞. Then*

$$\bar{I}_s(P,x,t_r) \underset{\zeta\to 0}{=} -\zeta^{-s}+O(\zeta) \quad \text{as } P\to P_\infty. \tag{4.26}$$

Proof For the sake of convenience, we introduce the notation $\widetilde{V}_{1j}^{(r,s)} = \widetilde{V}_{1j}^{(r)}$, $j = 1, 2, 3$. From (2.12) and (4.25), one easily gets

$$\begin{aligned}
\bar{I}_s(P, x, t_r) &= \bar{\widetilde{V}}_{11}^{(r,s)}(\lambda, x, t_r) + \bar{\widetilde{V}}_{12}^{(r,s)}(\lambda, x, t_r)\phi_1(P, x, t_r) \\
&\quad + \bar{\widetilde{V}}_{13}^{(r,s)}(\lambda, x, t_r)\big(\phi_{1,x}(P, x, t_r) + \phi_1^2(P, x, t_r) - u\big) \\
&= \frac{1}{6}\bar{\tilde{b}}_{xx}^{(r,s)}(\lambda, x, t_r) - \frac{1}{3}u\bar{\tilde{b}}^{(r,s)}(\lambda, x, t_r) - \bar{\tilde{a}}_x^{(r,s)}(\lambda, x, t_r) \\
&\quad - \left[\bar{\tilde{a}}^{(r,s)}(\lambda, x, t_r) - \frac{1}{2}\bar{\tilde{b}}_x^{(r,s)}(\lambda, x, t_r)\right]\phi_1(P, x, t_r) \\
&\quad + \bar{\tilde{b}}^{(r,s)}\big[\phi_{1,x}(P, x, t_r) + \phi_1^2(P, x, t_r) - u(x, t_r)\big].
\end{aligned}$$

From (4.1), we can see

$$\begin{aligned}
&\bar{I}_1 = \phi_3(P, x, t_r) = \zeta^{-1} + O(\zeta), \\
&\bar{I}_2 = -\frac{1}{3}u(x, t_r) + \phi_{1,x}(P, x, t_r) - \phi_1^2(P, x, t_r) - u(x, t_r) = \zeta^{-2} + O(\zeta).
\end{aligned}$$

So (4.26) is correct for $s = 1$ and $s = 2$. Then one may rewrite (4.26) as

$$\bar{I}_s(P, x, t_r) \underset{\zeta\to 0}{=} \zeta^{-s} + \sum_{j=1}^{\infty}\delta_j(x, t_r)\zeta^j \quad \text{as } P \to P_\infty \tag{4.27}$$

for some coefficients $\{\delta_j(x, t_r)\}_{j\in\mathbb{N}}$. From (3.20) and (4.25), we can see

$$\begin{aligned}
&\partial_x\bar{I}_s(P, x, t_r) \\
&\quad = \partial_x\big(\bar{\widetilde{V}}_{12}^{(r,s)}(\lambda, x, t_r)\phi_1(P, x, t_r) + \bar{\widetilde{V}}_{13}^{(r,s)}(\lambda, x, t_r)\big(\phi_{1,x}(P, x, t_r) + \phi_1^2(P, x, t_r) - u\big) \\
&\qquad + \bar{\widetilde{V}}_{11}^{(r,s)}(\lambda, x, t_r)\big) \\
&\quad = \phi_{1,t_r}(P, x, t_r),
\end{aligned}$$

that is,

$$\partial_x\left(-\zeta^{-s} + \sum_{j=1}^{\infty}\delta_j(x, t_r)\zeta^j\right) = \left(\zeta^{-1} + \sum_{j=1}^{\infty}\kappa_j(x, t_r)\zeta^{j-1}\right)_{t_r} = \left(\sum_{j=1}^{\infty}\kappa_{j+1}(x, t_r)\zeta^j\right)_{t_r}. \tag{4.28}$$

Using (3.2), (4.2), and comparing coefficients of ζ in (4.28), we should obtain

$$\begin{aligned}
&\delta_{j,x}(x, t_r) = \kappa_{j+1,t_r}(x, t_r), \quad j = 1, 2, \ldots \\
&\delta_{1,x}(x, t_r) = \kappa_{2,t_r}(x, t_r) = \frac{2}{3}u_{t_r}(x, t_r) = -\bar{\tilde{b}}_{r,x}^{(r,s)}(x, t_r), \\
&\delta_{2,x}(x, t_r) = \kappa_{3,t_r}(x, t_r) = \frac{1}{3}\big(-u(x, t_r) + v(x, t_r)\big)_{t_r} = \frac{1}{2}\bar{\tilde{b}}_{r,xx}^{(r,s)}(x, t_r) - \bar{\tilde{a}}_{r,x}^{(r,s)}(x, t_r), \\
&\delta_{3,x}(x, t_r) = \kappa_{4,t_r}(x, t_r) = \left(\frac{1}{9}u_{xx}(x, t_r) - \frac{1}{3}v_x(x, t_r)\right)_{t_r} = -\frac{1}{6}\bar{\tilde{b}}_{r,xxx}^{(r,s)}(x, t_r) + \bar{\tilde{a}}_{r,xx}^{(r,s)}(x, t_r).
\end{aligned} \tag{4.29}$$

That is,

$$\begin{aligned}
&\delta_1(x,t_r) = \gamma_1(t_r) - \bar{\bar{b}}_r^{(r,s)}(x,t_r), \\
&\delta_2(x,t_r) = \gamma_2(t_r) + \frac{1}{2}\bar{\bar{b}}_{r,x}^{(r,s)}(x,t_r) - \bar{\bar{a}}_r^{(r,s)}(x,t_r), \\
&\delta_3(x,t_r) = \gamma_3(t_r) - \frac{1}{6}\bar{\bar{b}}_{r,xx}^{(r,s)}(x,t_r) + \bar{\bar{a}}_{r,x}^{(r,s)}(x,t_r),
\end{aligned} \tag{4.30}$$

with $\gamma_1(t_r)$, $\gamma_2(t_r)$, $\gamma_3(t_r)$ being integration constants. From the definition of $\bar{I}_s$, the power series for $\phi_1(P,x,t_r)$ and the coefficients of $\bar{\bar{a}}(\zeta,x,t_r)$, $\bar{\bar{b}}(\zeta,x,t_r)$, we deduce that $\gamma_1(t_r) = \gamma_2(t_r) = \gamma_3(t_r) = 0$. Hence one concludes

$$\begin{aligned}
\bar{I}_s(P,x,t_r) = {} & \zeta^{-s} - \bar{\bar{b}}_r^{(r,s)}\zeta + \left(\frac{1}{2}\bar{\bar{b}}_{r,x}^{(r,s)} - \bar{\bar{a}}_r^{(r,s)}\right)\zeta^2 + \left(-\frac{1}{6}\bar{\bar{b}}_{r,xx}^{(r,s)} + \bar{\bar{a}}_{r,x}^{(r,s)}\right)\zeta^3 \\
& + O\bigl(\zeta^4\bigr) \quad \text{as } P \to P_\infty.
\end{aligned} \tag{4.31}$$

On the other hand, we will get

$$\begin{aligned}
\bar{I}_{s+3}(P,x,t_r) = {} & \zeta^{-3}\bar{I}_s + \left(\bar{\bar{a}}_r^{(r+1,s+3)} - \frac{1}{2}\bar{\bar{b}}_{r,x}^{(r+1,s+3)}\right)\phi_1 + \bar{\bar{b}}_r^{(r+1,s+3)}\bigl(\phi_{1,x} + \phi_1^2 - u\bigr) \\
& + \frac{1}{6}\bar{\bar{b}}_{r,xx}^{(r+1,s+3)} - \frac{1}{3}u\bar{\bar{b}}_r^{(r+1,s+3)} - \bar{\bar{a}}_{r,x}^{(r+1,s+3)} \\
= {} & \zeta^{-s-3} + O(\zeta).
\end{aligned} \tag{4.32}$$

$\square$

By (3.1) one knows that

$$\begin{aligned}
I_s(P,x,t_r) = {} & \sum_{l=0}^{r} \tilde{\beta}_{r-l}\bar{I}_{3l+2}(P,x,t_r) \\
& + \sum_{l=0}^{r} \tilde{\alpha}_{r-l}\bar{I}_{3l+1}(P,x,t_r), \quad s = 3r+2 \text{ (or } s = 3r+1).
\end{aligned} \tag{4.33}$$

Thus

$$\begin{aligned}
\int_{t_{0,r}}^{t_r} I_s(P,x,\tau)\,d\tau \underset{\zeta\to 0}{=} {} & (t_r - t_{0,r}) \sum_{l=0}^{r}\left(\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+2}} + \tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+1}}\right) \\
& + O(\zeta) \quad \text{as } P \to P_\infty.
\end{aligned} \tag{4.34}$$

Furthermore, integrating (4.23) yields

$$\begin{aligned}
& \int_{P_0}^{P} \widetilde{\Omega}_{P_\infty,s+1}^{(2)} \\
& \quad = \sum_{l=0}^{r} \tilde{\beta}_{r-l}(3l+2)\int_{\zeta_0}^{\zeta} \widetilde{\omega}_{P_\infty,3l+3}^{(2)} + \sum_{l=0}^{r} \tilde{\alpha}_{r-l}(3l+1)\int_{\zeta_0}^{\zeta} \widetilde{\omega}_{P_\infty,3l+2}^{(2)}
\end{aligned}$$

$$\underset{\zeta\to 0}{=}\sum_{l=0}^{r}\tilde{\beta}_{r-l}(3l+2)\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{3l+3}}\,d\zeta+\sum_{l=0}^{r}\tilde{\alpha}_{r-l}(3l+1)\int_{\zeta_0}^{\zeta}\frac{1}{\zeta^{3l+2}}\,d\zeta+O(\zeta)$$

$$\underset{\zeta\to 0}{=}-\sum_{l=0}^{r}\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+2}}-\sum_{l=0}^{r}\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+1}}+e_{s+1}^{(2)}(P_0)+O(\zeta)\quad\text{as } P\to P_\infty, \tag{4.35}$$

where $e_{s+1}^{(2)}(P_0)$ is a constant. Combing (4.34) and (4.35) indicates

$$\int_{t_{0,r}}^{t_r} I_s(P,x,\tau)\,d\tau \underset{\zeta\to 0}{=} (t_r-t_{0,r})\left(e_{s+1}^{(2)}(P_0)-\int_{P_0}^{P}\widetilde{\Omega}_{P_\infty,s+1}^{(2)}\right)+O(\zeta)\quad\text{as } P\to P_\infty. \tag{4.36}$$

Given these preparations, the theta function representation of $\psi_1(P,x,x_0,t_r,t_{0,r})$ reads as follows.

Theorem 4.5 *Let $P=(\lambda,y)\in\mathcal{K}_{m-1}\setminus\{P_\infty\}$ and let $(x,t_r),(x_0,t_{0,r})\in\Omega_\mu$, where $\Omega_\mu\subseteq\mathbb{C}^2$ is open and connected. Suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ is nonspecial for $(x,t_r)\in\Omega_\mu$. Then*

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) &= \frac{\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))}\\ &\quad\times\exp\left((x-x_0)\left(e_2^{(2)}(P_0)-\int_{P_0}^{P}\omega_{P_\infty,2}^{(2)}\right)\right.\\ &\quad\left.+(t_r-t_{0,r})\left(e_{s+1}^{(2)}(P_0)-\int_{P_0}^{P}\widetilde{\Omega}_{P_\infty,s+1}^{(2)}\right)\right).\end{aligned} \tag{4.37}$$

Proof Let $\psi_1(P,x,x_0,t_r,t_{0,r})$ be defined as in (3.23) and denote the right-hand side of (4.37) by $\Psi(P,x,x_0,t_r,t_{0,r})$. In order to prove that $\psi_1=\Psi$, one uses (3.7), (3.12), (3.29), (3.30) and

$$V_{12}^{(n)}\phi_1+V_{13}^{(n)}\left(\phi_{1,x}+\phi_1^2-u\right)+V_{11}^{(n)}=y,$$

to compute

$$\begin{aligned}\phi_1(P,x,t_r) &= \frac{y^2V_{13}^{(n)}-yA_m+B_m}{-\varepsilon(m)E_{m-1}}\\ &= \frac{y^2V_{13}^{(n)}-yA_m+\frac{2}{3}V_{13}^{(n)}S_m-\frac{1}{3}\varepsilon(m)E_{m-1,x}}{-\varepsilon(m)E_{m-1}}\\ &= \frac{2}{3}V_{13}^{(n)}\frac{3y^2+S_m}{-\varepsilon(m)E_{m-1}}+\frac{1}{3}\partial_x\ln E_{m-1}+\frac{V_{13}^{(n)}y(y+\frac{A_m}{V_{13}^{(n)}})}{\varepsilon(m)E_{m-1}}\\ &\underset{\lambda\to\mu_j(x,t_r)}{=}-\frac{\mu_{j,x}}{\lambda-\mu_j}+O(1)\underset{\lambda\to\mu_j(x,t_r)}{=}\partial_x\ln\left(\lambda-\mu_j(x,t_r)\right)+O(1),\\ I_s(P,x,t_r) &= \widetilde{V}_{12}^{(r)}\phi_1+\widetilde{V}_{13}^{(r)}\left(\phi_{1,x}+\phi_1^2-u\right)+\widetilde{V}_{11}^{(r)}\\ &= \left(\widetilde{V}_{12}^{(r)}-\widetilde{V}_{13}^{(r)}\frac{V_2^{(n)}}{V_{13}^{(n)}}\right)\phi_1+\widetilde{V}_{11}^{(r)}-\widetilde{V}_{13}^{(r)}\frac{V_{11}^{(n)}}{V_{13}^{(n)}}+y\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\\ &\quad+\widetilde{V}_{11}^{(r)}-\widetilde{V}_{13}^{(r)}\frac{V_{11}^{(n)}}{V_{13}^{(n)}}+y\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}\end{aligned}$$

$$= \frac{1}{3}\frac{E_{m-1,t_r}}{E_{m-1}} + \left(\widetilde{V}_{12}^{(r)} - \widetilde{V}_{13}^{(r)}\frac{V_{12}^{(n)}}{V_{13}^{(n)}}\right)\frac{y^2 V_{13}^{(n)} - yA_m + \frac{2}{3}S_m V_{13}^{(n)}}{-\varepsilon(m)E_{m-1}} + y\frac{\widetilde{V}_{13}^{(r)}}{V_{13}^{(n)}}$$
$$\underset{\lambda\to\mu_j(x,t_r)}{=} -\frac{\mu_{j,t_r}}{\lambda-\mu_j} + O(1)$$
$$\underset{\lambda\to\mu_j(x,t_r)}{=} \partial_{t_r}\ln\big(\lambda-\mu_j(x,t_r)\big) + O(1) \quad \text{as } P\to\hat{\mu}_j(x,t_r).$$

Hence

$$\begin{aligned}
&\psi_1(P,x,x_0,t_r,t_{0,r})\\
&\quad= \frac{\lambda-\mu_j(x,t_r)}{\lambda-\mu_j(x_0,t_r)}\frac{\lambda-\mu_j(x_0,t_r)}{\lambda-\mu_j(x_0,t_{0,r})}O(1)\\
&\quad= \begin{cases} (\lambda-\mu_j(x,t_r))O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r}),\\ O(1) & \text{for } P \text{ near } \hat{\mu}_j(x,t_r)=\hat{\mu}_j(x_0,t_{0,r}),\\ (\lambda-\mu_j(x_0,t_{0,r}))^{-1}O(1) & \text{for } P \text{ near } \hat{\mu}_j(x_0,t_{0,r})\neq\hat{\mu}_j(x,t_r),\end{cases}
\end{aligned} \tag{4.38}$$

where $O(1)\neq 0$ in (4.38). Consequently, all zeros and poles of ψ_1 and Ψ on $\mathcal{K}_{m-1}\setminus\{P_\infty\}$ are simple and coincide. It remains to identify the essential singularity of ψ_1 and Ψ at P_∞. By (4.35) we see that the singularities in the exponential terms of ψ_1 and Ψ coincide. The uniqueness result for Baker-Akhiezer functions completes the proof that $\psi_1=\Psi$ on Ω_μ. □

The straightening out of the second-order Benjiamin-Ono flows by the Abel map is showed in our next result.

Theorem 4.6 *Let* $(x,t_r),(x_0,t_{0,r})\in\mathbb{C}^2$. *Then*

$$\begin{aligned}
&\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\widetilde{U}}_{s+1}^{(2)}(t_r-t_{0,r}),\\
&\underline{\alpha}_{P_0}(\mathcal{D}_{\hat{\nu}_0(x,t_r)\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\hat{\nu}_0(x_0,t_{0,r})\underline{\hat{\nu}}(x_0,t_{0,r})}) + \underline{U}_2^{(2)}(x-x_0) + \underline{\widetilde{U}}_{s+1}^{(2)}(t_r-t_{0,r}).
\end{aligned} \tag{4.39}$$

Our main result, the theta function representation of the algebro-geometric solutions of the second-order Benjamin-Ono hierarchy, now quickly follows.

Theorem 4.7 *Let* $(x,t_r)\in\Omega_\mu$, *where* $\Omega_\mu\subseteq\mathbb{C}^2$ *is open and connected. Suppose also that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, *or equivalently*, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ *is nonspecial for* $(x,t_r)\in\Omega_\mu$. *Then*

$$\begin{aligned}
&u(x,t_r) = -\frac{3}{2}\partial_x^2\ln\Big(\theta\big(\underline{\lambda}\big(P_\infty,\underline{\hat{\mu}}(x,t_r)\big)\big)\Big) + \frac{3}{2}q_1,\\
&v(x,t_r) = -3\partial_x\partial_{\underline{U}_3^{(2)}}\ln\Big(\theta\big(\underline{\lambda}\big(P_\infty,\underline{\hat{\mu}}(x,t_r)\big)\big)\Big) - 3q_2,
\end{aligned} \tag{4.40}$$

with q_1 *and* q_2 *defined in* (4.19), *and* $\partial_{\underline{U}_3^{(2)}}$ *denotes the directional derivative* $\partial_{\underline{U}_3^{(2)}} = \sum_{j=1}^{m-1}U_{3,j}^{(2)}\frac{\partial}{\partial\lambda_j}$.

Proof Using Theorem 4.5, one can write ψ_1 near P_∞ in the coordinate ζ as

$$\begin{aligned}
&\psi_1(P,x,x_0,t_r,t_{0,r}) \\
&\quad \underset{\zeta\to 0}{=} \bigl(1+\sigma_1(x,t_r)\zeta+\sigma_2(x,t_r)\zeta^2+O(\zeta^3)\bigr)\exp\Biggl[(x-x_0)\bigl(\zeta^{-1}+q_1\zeta \\
&\qquad -q_2\zeta^2+O(\zeta^3)\bigr)+(t_r-t_{0,r})\sum_{l=0}^{r}\biggl(\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+2}}+\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+1}}\biggr)+O(\zeta)\Biggr],
\end{aligned}\tag{4.41}$$

where the terms $\sigma_1(x,t_r)$ and $\sigma_2(x,t_r)$ in (4.41) come from the Taylor expansion about P_∞ of the ratios of the theta functions in (4.37). That is,

$$\begin{aligned}
&\frac{\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x,t_r)))} \\
&\quad \underset{\zeta\to 0}{=} \frac{\theta(\underline{\Xi}_{P_0}-\underline{A}_{P_0}(P)+\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))}{\theta(\underline{\Xi}_{P_0}-\underline{A}_{P_0}(P_\infty)+\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))} \\
&\quad \underset{\zeta\to 0}{=} \frac{\theta(\ldots,\Xi_{P_0,j}-A_{P_0,j}(P_\infty)+\alpha_{P_0,j}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)})-\rho_{0,j}\zeta-\frac{1}{2}\rho_{1,j}\zeta^2-\frac{1}{4}\rho_{2,j}\zeta^4+O(\zeta^5),\ldots)}{\theta(\underline{\Xi}_{P_0}-\underline{A}_{P_0}(P_\infty)+\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))} \\
&\quad \underset{\zeta\to 0}{=} \frac{\theta_0-\sum_{j=1}^{m-1}\frac{\partial\theta_0}{\partial\lambda_j}\rho_{0,j}\zeta-\frac{1}{2}\sum_{j=1}^{m-1}(\frac{\partial\theta_0}{\partial\lambda_j}\rho_{1,j}-\sum_{k=1}^{m-1}\frac{\partial^2\theta_0}{\partial\lambda_j\,\partial\lambda_k}\rho_{0,j}\rho_{0,k})\zeta^2+O(\zeta^3)}{\theta(\underline{\Xi}_{P_0}-\underline{A}_{P_0}(P_\infty)+\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))} \\
&\quad \underset{\zeta\to 0}{=} 1-\partial_x\ln\theta_0\zeta+\biggl(\frac{1}{2}\partial_x^2\ln\theta_0+\frac{1}{2}(\partial_x\ln\theta_0)^2-\partial_{\underline{U}_3^{(2)}}\ln\theta_0\biggr)\zeta^2+O(\zeta^3), \\
&\quad P\to P_\infty,
\end{aligned}\tag{4.42}$$

where $\theta_0=\theta(\underline{\Xi}_{P_0}-\underline{A}_{P_0}(P_\infty)+\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))$. Similarly, we can have

$$\frac{\theta(\underline{\lambda}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{\lambda}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r})))}\underset{\zeta\to 0}{=} O(1),\quad P\to P_\infty.\tag{4.43}$$

So, we give the Taylor expansion about ψ_1 as follows:

$$\begin{aligned}
&\psi_1(P,x,x_0,t_r,t_{0,r}) \\
&\quad \underset{\zeta\to 0}{=} \biggl(1-\partial_x\ln\theta_0\zeta+\biggl(\frac{1}{2}\partial_x^2\ln\theta_0+\frac{1}{2}(\partial_x\ln\theta_0)^2-\partial_{\underline{U}_3^{(2)}}\ln\theta_0\biggr)\zeta^2+O(\zeta^3)\biggr)O(1) \\
&\qquad \times\exp\bigl[(x-x_0)\bigl(\zeta^{-1}+q_1\zeta-q_2\zeta^2+O(\zeta^3)\bigr)\bigr] \\
&\qquad \times\Biggl[(t_r-t_{0,r})\sum_{l=0}^{r}\biggl(\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+2}}+\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+1}}\biggr)+O(\zeta)\Biggr],\quad P\to P_\infty.
\end{aligned}\tag{4.44}$$

Then it is clear that

$$\begin{aligned}
&\sigma_{1,x}(x,t_r)=-\partial_x^2\ln\theta_0, \\
&\frac{1}{2}\sigma_{1,xx}(x,t_r)-\sigma_1(x,t_r)\sigma_{1,x}(x,t_r)+\sigma_{2,x}(x,t_r)=-\partial_x\,\partial_{\underline{U}_3^{(2)}}\ln\theta_0.
\end{aligned}\tag{4.45}$$

If we set

$$\psi_1 \underset{\zeta\to 0}{=} \left(1+\sigma_1(x,t_r)\zeta+\sigma_2(x,t_r)\zeta^2+O\left(\zeta^3\right)\right)\exp(\Delta), \quad P\to P_\infty$$

with $\Delta=(x-x_0)(\zeta^{-1}+q_1\zeta-q_2\zeta^2+O(\zeta^3))+(t_r-t_{0,r})\sum_{l=0}^{r}(\tilde{\beta}_{r-l}\frac{1}{\zeta^{3l+2}}+\tilde{\alpha}_{r-l}\frac{1}{\zeta^{3l+1}})+O(\zeta)$, then we can show

$$\begin{aligned}
\psi_{1,x} &\underset{\zeta\to 0}{=} \left(\sigma_{1,x}\zeta+\sigma_{2,x}\zeta^2+O\left(\zeta^3\right)\right)\exp(\Delta)+\left(\zeta^{-1}+q_1\zeta-q_2\zeta^2+O\left(\zeta^3\right)\right)\psi_1,\\
&\underset{\zeta\to 0}{=} \zeta^{-1}\psi_1+O(\zeta)\psi_1,\\
\psi_{1,xx} &\underset{\zeta\to 0}{=} \left(\sigma_{1,x}+(\sigma_{2,x}+\sigma_{1,xx})\zeta+(\sigma_{2,xx}+q_1\sigma_{1,x})\zeta^2+O\left(\zeta^3\right)\right)\exp(\Delta)\\
&\qquad +\left(\zeta^{-1}+q_1\zeta-q_2\zeta^2+O\left(\zeta^3\right)\right)\psi_{1,x},\\
\psi_{1,xxx} &\underset{\zeta\to 0}{=} \left(3\sigma_{1,xx}+2\sigma_{2,x}+2\sigma_{1,x}\zeta^{-1}O(\zeta)\right)\exp(\Delta)+\left(\zeta^{-2}+2q_1-2q_2\zeta+O\left(\zeta^2\right)\right)\psi_{1,x},\\
&\underset{\zeta\to 0}{=} 3(\sigma_{1,xx}+\sigma_{2,x}-\sigma_1\sigma_{1,x}-q_2)\psi_1+3(\sigma_{1,x}+q_1)\psi_{1,x}+\zeta^{-3}\psi_1+O(\zeta)\psi_1,\\
&P\to P_\infty.
\end{aligned} \tag{4.46}$$

On the other hand, we know that

$$\psi_{1,xxx}=\left(u_x(x,t_r)+v(x,t_r)+\lambda\right)\psi_1+2u(x,t_r)\psi_{1,x}.$$

Hence

$$\begin{aligned}
u(x,t_r) &= \frac{3}{2}(\sigma_{1,x}+q_1),\\
v(x,t_r) &= 3(\sigma_{1,xx}+\sigma_{2,x}-\sigma_1\sigma_{1,x}-q_2)-u_x(x,t_r).
\end{aligned} \tag{4.47}$$

That is just (4.40). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, read and approved the final manuscript.

Author details
[1]Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, 450002, China. [2]Department of Economics, Zhengzhou Vocational College of Economics and Trade, Zhengzhou, 450000, China.

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