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Asymptotic stability of positive periodic solution for semilinear evolution equations

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Abstract

The aim of this paper is to study the asymptotic stability of positive periodic solution for semilinear evolution equation in an ordered Banach space E: $u'(t) + Au(t) = f(t, u(t))$, $t \in \mathbb{R}^+ = [0, +\infty)$, where $A : D(A) \subset E \to E$ is a closed linear operator, and $f : \mathbb{R}^+ \times E \to E$ is a continuous mapping which is ω-periodic in t. Under order conditions on the nonlinearity f, the asymptotic stability results of positive ω-periodic mild solution are obtained on $\mathbb{R}^+$ by using operator semigroup theory and a monotone iterative technique.

MSC: 35B10; 35B40

Keywords: positive ω-periodic mild solution; asymptotic stability; monotone iterative technique

1 Introduction

The problems concerning periodic solutions of partial differential equations are an important area of investigation in recent years. Especially, the existence of periodic solutions for the evolution equations has been considered by several authors; see [1–14] and the references therein. In [1], Xiang and Ahmad proved an existence result of the periodic solution to the delay evolution equations in Banach spaces under the assumption that the corresponding initial value problem has an *a priori* estimate. In [2, 3], Liu derived periodic solutions from bounded solutions or ultimate bounded solutions for finite or infinite delay evolution equations in Banach spaces. In [4], Liang *et al.* proved that if the solutions of the corresponding initial value problem are ultimately bounded, then the delay impulsive evolution equation has a periodic solution. In all these works, the key assumption of *a priori* boundedness of solutions is employed. In [5], Li studied the existence and uniqueness of positive periodic solutions for semilinear evolution equations in ordered Banach spaces by using a monotone iterative technique. In [6], under the spectral separation conditions of a selfadjoint operator, Li studied the existence and uniqueness of periodic solutions for semilinear evolution equations in Hilbert spaces by using the method of fixed point theorems.

Recently, Li in [7] investigated the existence and asymptotic stability of time ω-periodic solutions for the delay parabolic boundary value problem (DPBVP),

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + A(x, D)u(x, t) \\ \quad = g(x, t, u(x, t), u(x, t - \tau_1), \dots, u(x, t - \tau_n)), & (x, t) \in \Omega \times \mathbb{R}, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

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where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$,

$$A(x, D) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + a_0(x)$$

is a uniformly elliptic differential operator of divergence form in $\overline{\Omega}$ with the coefficients $a_{ij} \in C^{1+\mu}(\overline{\Omega})$ $(i, j = 1, 2, \ldots, N)$ and $a_0 \in C^\mu(\overline{\Omega})$ for some $\mu \in (0, 1)$. That is, $[a_{ij}(x)]_{N \times N}$ is a positive define symmetric matrix for every $x \in \overline{\Omega}$, and $\tau_1, \tau_2, \ldots, \tau_n$ are positive constants which denote the time delays. Let $a_0(x) \geq 0$ on $\overline{\Omega}$, $g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function which is ω-periodic in t. Assume we work under the following assumptions:

(A$_1$) There exist positive constants $\beta_0, \ldots, \beta_n$ such that

$$\left| g(x, t, \eta_0, \ldots, \eta_n) - g(x, t, \zeta_0, \ldots, \zeta_n) \right| \leq \sum_{i=0}^{n} \beta_i |\eta_i - \zeta_i|,$$

for any $(x, t, \eta_0, \ldots, \eta_n), (x, t, \zeta_0, \ldots, \zeta_n) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^{n+1}$.

(A$_2$) $\sum_{i=0}^{n} e^{\lambda_1 \tau_i} \beta_i < \lambda_1$.

The authors obtained the existence and asymptotic stability of time ω-periodic solutions for the DPBVP (1).

If we have the case without delays, in $\mathbb{R}^+$, the DPBVP (1) degenerates into the following problem:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + A(x, D)u(x, t) = g(x, t, u(x, t)), & (x, t) \in \Omega \times \mathbb{R}^+, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

In this case, the assumptions (A$_1$) and (A$_2$) degenerate into the following.

(A$_3$) There exists a constant $\beta \in (0, \lambda_1)$ such that

$$\left| g(x, t, \eta) - g(x, t, \zeta) \right| \leq \beta |\eta - \zeta|,$$

for any $(x, t, \eta), (x, t, \zeta) \in \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}$.

Sometimes the condition (A$_3$) is not easy to verify in applications. To make the work better applicable, in this paper, we obtain the following result.

Theorem A *Let $g(x, t, 0) \geq 0$ and $g(x, t, 0) \not\equiv 0$. Assume that the following conditions are satisfied.*

(C$_1$) *For any $R \geq 0$, there exists a constant $M = M(R) > 0$ such that*

$$g(x, t, \xi_2) - g(x, t, \xi_1) \geq -M(\xi_2 - \xi_1), \quad t \in \mathbb{R}^+,$$

for $\xi_1, \xi_2 \in \mathbb{R}$ with $0 \leq \xi_1 \leq \xi_2$, $|\xi_i| \leq R$ $(i = 1, 2)$.

(C$_2$) *There exists a constant $L < \lambda_1$ such that*

$$g(x, t, \xi_2) - g(x, t, \xi_1) \leq L(\xi_2 - \xi_1), \quad t \in \mathbb{R}^+,$$

for $\xi_1, \xi_2 \in \mathbb{R}$ with $0 \leq \xi_1 \leq \xi_2$.

Then the problem (2) *has a unique positive time ω-periodic solution and it exponentially attracts every solution of the corresponding initial value problem in $L^2(\Omega)$.*

Our discussion will be made in the framework of ordered Banach spaces. Let E be an ordered Banach space with norm $\|\cdot\|$, whose positive cone K is normal with normal constant $N = 1$, $A : D(A) \subset E \to E$ be a closed linear operator, $-A$ generate a C_0-semigroup $T(t)$ $(t \geq 0)$ in E, and let $f : \mathbb{R}^+ \times E \to E$ be a continuous mapping which is ω-periodic in t. It is well known (see [15]) that for a C_0-semigroup $T(t)$ $(t \geq 0)$ that there exist $C > 0$ and $\delta \in \mathbb{R}$ such that

$$\|T(t)\| \leq Ce^{\delta t}, \quad t \geq 0.$$

Let $\nu_0 = \inf\{\delta \in R, \text{there exists } C > 0 \text{ such that } \|T(t)\| \leq Ce^{\delta t}, t \geq 0\}$. Then ν_0 is called the growth exponent of the semigroup $T(t)$ $(t \geq 0)$. Furthermore, ν_0 can also be obtained by the following formula:

$$\nu_0 = \limsup_{t \to +\infty} \frac{\ln\|T(t)\|}{t}.$$

More generally, we consider the existence and asymptotic stability of time ω-periodic solution for the abstract evolution equation in E

$$u'(t) + Au(t) = f\big(t, u(t)\big), \quad t \in \mathbb{R}^+. \tag{3}$$

For the abstract evolution equation (3), we obtain the following results.

Theorem 1 *Let E be an ordered Banach space, whose positive cone K is normal. Assume that $-A$ generates a positive C_0-semigroup $T(t)$ $(t \geq 0)$ in E, $f : \mathbb{R}^+ \times E \to E$ is a continuous mapping which is ω-periodic in t, and $f(t, \theta) \geq \theta, f(t, \theta) \not\equiv \theta$ for $t \in R^+$, where θ is the zero element in E. Assume $f(t, u)$ satisfies the following conditions.*

(H$_1$) *For any $R \geq 0$, there exists a constant $M = M(R) > 0$ such that*

$$f(t, y) - f(t, x) \geq -M(y - x), \quad t \in \mathbb{R}^+,$$

for $x, y \in E$ with $\theta \leq x \leq y$, $\|x\| \leq R$, $\|y\| \leq R$.

(H$_2$) *There exists a constant $L < -\nu_0$ such that*

$$f(t, y) - f(t, x) \leq L(y - x), \quad t \in \mathbb{R}^+,$$

for $x, y \in E$ with $\theta \leq x \leq y$.

Then the positive ω-periodic mild solution of Eq. (3) *is globally asymptotically stable.*

If C_0-semigroup $T(t)$ is continuous in uniform operator topology for every $t > 0$ in E, it is well known (see [16]) that ν_0 can also be determined by $\sigma(A)$ and

$$\nu_0 = -\inf\big\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\big\},$$

where $\sigma(A)$ is the spectrum of A. We know (see [15]) that compact semigroup is continuous in uniform operator topology for $t > 0$. Let K be a regeneration cone, $T(t)$ ($t \geq 0$) be a compact and positive C_0-semigroup. By the characteristic of positive semigroups (see [17]) and the Krein-Rutmann theorem, A has the first eigenvalue $\lambda_1 > 0$ and

$$\lambda_1 = \inf\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}.$$

That is, $\nu_0 = -\lambda_1$. Hence by Theorem 1, we have the following.

Corollary 1 *Let E be an ordered Banach space, whose positive cone K is a normal regeneration cone. Assume that $-A$ generates a compact and positive C_0-semigroup $T(t)$ ($t \geq 0$) in E, $f : \mathbb{R}^+ \times E \to E$ is a continuous mapping which is ω-periodic in t and $f(t,\theta) \geq \theta$, $f(t,\theta) \not\equiv \theta$ for $t \in \mathbb{R}^+$. If $f(t,u)$ satisfies the assumptions* (H_1) *and*

(H_2)* *there exists a constant $L < \lambda_1$ such that*

$$f(t,y) - f(t,x) \leq L(y-x), \quad t \in \mathbb{R}^+,$$

for $x, y \in E$ with $\theta \leq x \leq y$.

Then the positive ω-periodic mild solution of Eq. (3) is globally asymptotically stable.

Remark 1 Under the assumptions of Theorem 1 or Corollary 1, the existence and uniqueness of positive ω-periodic mild solutions for Eq. (3) were obtained by Li in [5]. So, in this paper, we mainly focus on the asymptotic stability of the positive ω-periodic mild solutions.

We apply the above abstract results to the problem (2). Let $E = L^2(\Omega)$, $K = \{u \in L^2(\Omega) : u(x) \geq 0, \text{a.e. } x \in \Omega\}$. Then K is a normal regeneration cone in E. Define an operator $A : D(A) \subset E \to E$ by

$$D(A) = H^2(\Omega) \cap H_0^1(\Omega), \qquad Au = A(x,D)u.$$

It is well known (see [7]) that $-A$ generates a compact C_0-semigroup in E which is also positive. Define a mapping $f : \mathbb{R}^+ \times E \to E$ by

$$f(t,u) = g(\cdot, t, u(\cdot)), \quad u \in L^2(\Omega), t \in \mathbb{R}^+. \tag{4}$$

It is clear that $f : \mathbb{R}^+ \times E \to E$ is continuous and it is ω-periodic in t. Thus, the problem (2) is rewritten into the form of abstract evolution equation (3). When the conditions (C_1) and (C_2) of Theorem A are satisfied, the mapping $f : \mathbb{R}^+ \times E \to E$ defined by (4) satisfies the conditions (H_1) and (H_2)*. Hence, by Corollary 1, we obtain the conclusion of Theorem A.

The abstract result of Theorem 1 will be proved in Section 3. In Section 2, some preliminary conclusions are given.

2 Preliminaries

Let E be an ordered Banach space, whose positive cone K is normal, $A : D(A) \to E$ be a closed linear operator in E. Denote by $C([0,\omega], E)$ the continuous function space from

$[0, \omega]$ to E. Let $C_\omega(\mathbb{R}^+, E)$ be the Banach space $\{u \in C(\mathbb{R}^+, E) : u(t) = u(t + \omega), t \in \mathbb{R}^+\}$ endowed with the maximum norm $\|u\|_C = \max_{t \in [0, \omega]} \|u(t)\|$. We first consider the existence of the initial value problem (IVP) of the evolution equation in E

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t > 0, \\ u(0) = x_0. \end{cases} \tag{5}$$

For IVP (5), we obtain the following existence result.

Lemma 1 *Let E be an ordered Banach space, whose positive cone K is normal. Assume that $-A$ generates a positive C_0-semigroup $T(t)$ ($t \geq 0$) in E, $f : \mathbb{R}^+ \times K \to E$ is continuous and $f(t, \theta) \geq \theta, f(t, \theta) \not\equiv \theta$ for $t \in \mathbb{R}^+$. If $x_0 \in K$ and $f(t, u)$ satisfies the conditions* (H$_1$) *and* (H$_2$), *then IVP* (5) *has a unique mild solution.*

Proof Let $h_0(t) = f(t, \theta)$. Then $h_0 \in C(\mathbb{R}^+, E)$, $h_0(t) \geq \theta$ and $h_0(t) \not\equiv \theta$. We first consider the initial value problem of linear evolution equation (LIVP)

$$\begin{cases} u'(t) + (A - LI)u(t) = h_0(t), & t > 0, \\ u(0) = x_0. \end{cases} \tag{6}$$

It is well known (see [15]) that the LIVP (6) has a unique mild solution $u_0 \in C(\mathbb{R}^+, E)$ expressed by

$$u_0(t) = S_1(t)x_0 + \int_0^t S_1(t - s)h_0(s)\,ds, \quad t \geq 0,$$

where $S_1(t) = e^{Lt}T(t)$ ($t \geq 0$) is a positive C_0-semigroup generated by $-(A - LI)$, whose norm satisfies $\|S_1(t)\| \leq Ce^{(L+v_0)t} \leq C$ for $t \geq 0$. Hence, we have

$$\|u_0(t)\| \leq C\|x_0\| + C\int_0^t e^{(L+v_0)(t-s)}\,ds \cdot \|h_0\|_C \leq C\|x_0\| + \frac{C\|h_0\|_C}{-(L + v_0)} \triangleq \overline{R}.$$

Let $v_0 \equiv \theta$, $w_0 = u_0$. By $h_0(t) \geq \theta$, $h_0(t) \not\equiv \theta$, $x_0 \in K$ and the positive property of semigroup $S_1(t)$ ($t \geq 0$), we see that $v_0 \leq w_0$. Let $M = M(\overline{R}) > 0$ be the constant in assumption (H$_1$). We consider the following IVP of the evolution equation:

$$\begin{cases} u'(t) + (A + MI)u(t) = f(t, u(t)) + Mu(t), & t > 0, \\ u(0) = x_0. \end{cases} \tag{7}$$

Without loss of generality, we assume $M > -L$ (otherwise, replacing M by $M + |L|$, the assumption (H$_1$) still holds). Then the operator $-(A + MI)$ generates a positive C_0-semigroup $S_2(t) = e^{-Mt}T(t)$ ($t \geq 0$), whose norm satisfies $\|S_2(t)\| \leq Ce^{-(M-v_0)t} \leq C$ for $t \geq 0$.

Define the operator Q by

$$(Qu)(t) = S_2(t)x_0 + \int_0^t S_2(t - s)\big[f(s, u(s)) + Mu(s)\big]\,ds, \quad t \geq 0. \tag{8}$$

It is clear that the mild solution of IVP (7) is equivalent to the fixed point of operator Q.

Let $D = [v_0, w_0]$. It follows from assumption (H$_1$) that $Q : D \to C(\mathbb{R}^+, E)$ is a continuously increasing operator. Let

$$v_n = Qv_{n-1}, \qquad w_n = Qw_{n-1}, \quad n = 1, 2, \ldots. \tag{9}$$

Then

$$\theta = v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq w_n \leq \cdots \leq w_1 \leq w_0. \tag{10}$$

Therefore, for any $t \geq 0$, we have

$$\theta \leq w_n(t) - v_n(t)$$
$$= \int_0^t S_2(t-s)\big[f\big(s, w_{n-1}(s)\big) - f\big(s, v_{n-1}(s)\big) + M\big(w_{n-1}(s) - v_{n-1}(s)\big)\big]\, ds$$
$$\leq (M + L) \int_0^t S_2(t-s)\big(w_{n-1}(s) - v_{n-1}(s)\big)\, ds.$$

By the normality of cone K in E, we have

$$\big\| w_n(t) - v_n(t) \big\| \leq C(M + L) \int_0^t \big\| w_{n-1}(s) - v_{n-1}(s) \big\|\, ds.$$

Continuing such a procedure, we have

$$\big\| w_n(t) - v_n(t) \big\| \leq \frac{[C(M + L)]^n}{n!} t^n \cdot \| w_0 \|_C \to 0 \quad (n \to \infty).$$

This implies that there is a unique $u^* \in C(\mathbb{R}^+, E)$ such that

$$u^*(t) = \lim_{n\to\infty} w_n(t) = \lim_{n\to\infty} v_n(t), \quad t \geq 0.$$

Combining this with (9), since the convergence is uniform in each compact interval and the operator Q is continuous, we obtain $u^* = Qu^*$. Therefore, $u(t; x_0) := u^*(t)$ is the unique mild solution of IVP (7) on $\mathbb{R}^+$. This proof is completed. □

To prove our main result, we also need the following lemma.

Lemma 2 *Let $y_1, y_2 \in E$ with $\theta \leq y_1 \leq y_2$. Then $u(t; y_1) \leq u(t; y_2)$.*

Proof Consider the following two initial value problems:

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t > 0, \\ u(0) = y_1 \end{cases} \tag{11}$$

and

$$\begin{cases} u'(t) + Au(t) = f(t, u(t)), & t > 0, \\ u(0) = y_2. \end{cases} \tag{12}$$

Let $x_0 = y_1 + y_2$. Then the solution of LIVP (6) is the w_0 corresponding to IVP (11) and IVP (12). Let $v_0 = \theta$. A similar argument as in Lemma 1 shows that

$$v_{i,n} = Qv_{i,n-1}, \qquad w_{i,n} = Qw_{i,n-1}, \quad i = 1, 2, n = 1, 2, \ldots$$

and

$$u(t; y_1) = \lim_{n \to \infty} v_{1,n}(t) = \lim_{n \to \infty} w_{1,n}(t), \qquad u(t; y_2) = \lim_{n \to \infty} v_{2,n}(t) = \lim_{n \to \infty} w_{2,n}(t), \quad t \geq 0.$$

For any $t \geq 0$, noticing that $v_{1,0}(t) = v_{2,0}(t) \equiv \theta$ and $\theta \leq y_1 \leq y_2$, it follows from the positivity of the operators $S_2(t)$ and

$$v_{1,1}(t) = (Qv_{1,0})(t) = S_2(t)y_1 + \int_0^t S_2(t-s)h_0(s)\,ds,$$

$$v_{2,1}(t) = (Qv_{2,0})(t) = S_2(t)y_2 + \int_0^t S_2(t-s)h_0(s)\,ds$$

that $v_{1,1}(t) \leq v_{2,1}(t)$. Inductively, when $v_{1,n-1}(t) \leq v_{2,n-1}(t)$, it follows from

$$v_{1,n}(t) = (Qv_{1,n-1})(t) = S_2(t)y_1 + \int_0^t S_2(t-s)\big[f\big(s, v_{1,n-1}(s)\big) + Mv_{1,n-1}(s)\big]\,ds,$$

$$v_{2,n}(t) = (Qv_{2,n-1})(t) = S_2(t)y_2 + \int_0^t S_2(t-s)\big[f\big(s, v_{2,n-1}(s)\big) + Mv_{2,n-1}(s)\big]\,ds$$

that

$$v_{1,n}(t) \leq v_{2,n}(t), \quad t \geq 0. \tag{13}$$

Taking the limits on both sides of inequality (13) as $n \to \infty$, we obtain

$$u(t; y_1) \leq u(t; y_2).$$

This proof is completed. □

For the existence and uniqueness of ω-periodic mild solutions of Eq. (3), we have the following result.

Lemma 3 (see [5]) *Let E be an ordered Banach space, whose positive cone K is normal. Assume that $-A$ generates a positive C_0-semigroup $T(t)$ ($t \geq 0$) in E, $f : \mathbb{R}^+ \times K \to E$ is a continuous mapping which is ω-periodic in t and $f(t, \theta) \geq \theta$, $f(t, \theta) \not\equiv \theta$ for $t \geq 0$. If $f(t, u)$ satisfies the conditions* (H$_1$) *and* (H$_2$), *then Eq. (3) has a unique positive ω-periodic mild solution on $\mathbb{R}^+$.*

3 The proof of Theorem 1

Proof of Theorem 1 Define an equivalent norm $|\cdot|_E$ in E by

$$|x|_E = \sup_{t \geq 0} \big\| e^{(M-v_0)t} S_2(t)x \big\|.$$

Then $\|x\| \le |x|_E \le C\|x\|$ and

$$
\begin{aligned}
\left|S_2(t)x\right|_E &= \sup_{s \ge 0} \left\| e^{(M-v_0)s} S_2(s) \cdot S_2(t)x \right\| = \sup_{s \ge 0} \left\| e^{(M-v_0)s} S_2(t+s)x \right\| \\
&= e^{-(M-v_0)t} \cdot \sup_{s \ge 0} \left\| e^{(M-v_0)(t+s)} S_2(t+s)x \right\| \\
&\le e^{-(M-v_0)t} \cdot |x|_E,
\end{aligned}
$$

which implies that $|S_2(t)|_E \le e^{-(M-v_0)t}$.

From Lemma 3, Eq. (3) has a unique positive ω-periodic mild solution $\widetilde{u}$ on $\mathbb{R}^+$. By Lemma 1, IVP (5) has a unique positive mild solution $u = u(t; x_0) \in C(\mathbb{R}^+, K)$. Let $y_0 = \widetilde{u}(0)$. Then $\widetilde{u}(t) = u(t; y_0)$. Setting $x_1 := y_0 + x_0$, then $x_1 \ge y_0$, $x_1 \ge x_0$. By Lemma 2, we see that $\theta \le \widetilde{u}(t) \le u(t; x_1)$, $\theta \le u(t; x_0) \le u(t; x_1)$. Setting $u_1(t) = u(t; x_1)$, $u^*(t) = u(t; x_0)$, by the semigroup representation of the solutions, we have

$$
\begin{aligned}
\theta \le u_1(t) - \widetilde{u}(t) &= (Qu_1)(t) - (Q\widetilde{u})(t) \\
&= S_2(t)x_1 + \int_0^t S_2(t-s)\big(f\big(s, u_1(s)\big) + Mu_1(s)\big)\, ds \\
&\quad - S_2(t)y_0 - \int_0^t S_2(t-s)\big(f\big(s, \widetilde{u}(s)\big) + M\widetilde{u}(s)\big)\, ds \\
&\le S_2(t)(x_1 - y_0) + \int_0^t S_2(t-s)\big[f\big(s, u_1(s)\big) - f\big(s, \widetilde{u}(s)\big) + M\big(u_1(s) - \widetilde{u}(s)\big)\big]\, ds \\
&\le S_2(t)(x_1 - y_0) + (M+L)\int_0^t S_2(t-s)\big(u_1(s) - \widetilde{u}(s)\big)\, ds.
\end{aligned}
$$

By the normality of cone K in E, we have

$$
\begin{aligned}
\left|u_1(t) - \widetilde{u}(t)\right|_E &\le \left|S_2(t)(x_1 - y_0)\right|_E + (M+L)\left|\int_0^t S_2(t-s)\big(u_1(s) - \widetilde{u}(s)\big)\, ds\right|_E \\
&\le e^{-(M-v_0)t}|x_1 - y_0|_E + (M+L)\int_0^t e^{-(M-v_0)(t-s)}\left|u_1(s) - \widetilde{u}(s)\right|_E\, ds,
\end{aligned}
$$

that is,

$$
e^{(M-v_0)t}\left|u_1(t) - \widetilde{u}(t)\right|_E \le |x_1 - y_0|_E + (M+L)\int_0^t e^{(M-v_0)s}\left|u_1(s) - \widetilde{u}(s)\right|_E\, ds.
$$

By the Gronwall-Bellman inequality, we have

$$
\left|u_1(t) - \widetilde{u}(t)\right|_E \le |x_1 - y_0|_E \cdot e^{(L+v_0)t} \to 0 \quad (t \to \infty).
$$

Similarly, $|u_1(t) - u^*(t)|_E \to 0$ $(t \to \infty)$. Therefore, we obtain

$$
\left|\widetilde{u}(t) - u^*(t)\right|_E \le \left|\widetilde{u}(t) - u_1(t)\right|_E + \left|u_1(t) - u^*(t)\right|_E \to 0 \quad (t \to \infty).
$$

This proof is completed. □

4 Application

To illustrate our results, we consider the semilinear partial differential equations in $\mathbb{R}^2$ of the form

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \frac{\partial u(x,t)}{\partial x} = g(x,t,u(x,t)), & (x,t) \in \mathbb{R}^2, \\ u(x + 2\pi, t) = u(x, t + 2\pi) = u(x,t), & (x,t) \in \mathbb{R}^2, \end{cases} \tag{14}$$

where $g \in C^1(\mathbb{R}^3)$ is 2π-periodic both in x and t.

Let $E = C_{2\pi}(\mathbb{R})$. Define an operator A in E by

$$D(A) = C_{2\pi}^1(\mathbb{R}), \qquad Au = \frac{\partial u(x,t)}{\partial x}.$$

By [5], $-A$ generates a contraction C_0-semigroup $T(t)$ ($t \geq 0$) in E, which is also a positive C_0-semigroup. By the contraction property of $T(t)$ ($t \geq 0$), we know that $\nu_0 \leq 0$.

Let $f(t, u(t))(\cdot) = g(\cdot, t, u(\cdot, t))$. Then $f : \mathbb{R} \times E \to E$ is continuous and is 2π-periodic in t. From Theorem 1, we can obtain the following.

Theorem 2 *Let $g \in C^1(\mathbb{R}^3)$ which is 2π-periodic both in x and t, and $g(x,t,0) \geq 0$, $g(x,t,0) \not\equiv 0$. Assume that the following conditions are satisfied:*

(P₁) *For any $R \geq 0$, there exists a constant $M = M(R) > 0$ such that*

$$g(x,t,\xi_2) - g(x,t,\xi_1) \geq -M(\xi_2 - \xi_1), \quad t \in \mathbb{R}^+,$$

for $\xi_1, \xi_2 \in \mathbb{R}$ with $0 \leq \xi_1 \leq \xi_2$, $|\xi_i| \leq R$ ($i = 1, 2$).

(P₂) *There exists a constant $L < -\nu_0$ such that*

$$g(x,t,\xi_2) - g(x,t,\xi_1) \leq L(\xi_2 - \xi_1), \quad t \in \mathbb{R}^+,$$

for $\xi_1, \xi_2 \in \mathbb{R}$ with $0 \leq \xi_1 \leq \xi_2$.

Then the problem (14) *has a unique double 2π-periodic mild solution in $C_{2\pi}(\mathbb{R}, E)$ which is globally asymptotic stable.*

Remark 2 It is clear that if $\sup g_u(x,t,u) < 0$, then the assumptions (P₁) and (P₂) hold automatically.

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