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Uniformly asymptotic stability of almost periodic solutions for a competitive system with impulsive perturbations

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Abstract
Impulsive differential models play an important role in modeling population systems. In this article, we consider an almost periodic competitive model subject to impulsive perturbations and establish sufficient conditions for the uniformly asymptotic stability of a unique positive almost periodic solution for the system. The example and its numerical simulations are carried out to illustrate the feasibility of our main results.

Keywords: impulsive competitive system; positive almost periodic solutions; uniformly asymptotic stability; Lyapunov function

1 Introduction

In [1], Gopalsamy introduced the following autonomous two-species competitive system

$$\begin{cases} x_1'(t) = x_1(t)[a_1 - b_1x_1(t) - c_1x_2(t) - d_1x_1^2(t)], \\ x_2'(t) = x_2(t)[a_2 - b_2x_2(t) - c_2x_1(t) - d_2x_2^2(t)], \end{cases} \tag{1.1}$$

where $x_1(t)$, $x_2(t)$ can be interpreted as the density of two competing species at time t, respectively. a_1, a_2 stand for the intrinsic growth rates of two species, b_1, d_1, b_2, d_2 represent the effects of intra-specific competition, and c_1, c_2 are the effects of inter-specific competition. Notice that the coefficients, in the real world, are not unchanged constants owing to the variation of environment, and the effect of a varying environment is significant for evolutionary theory as the selective forces on systems in such a fluctuating environment differ from those in a stable environment. So it is realistic to consider a corresponding non-autonomous version with the form

$$\begin{cases} x_1'(t) = x_1(t)[a_1(t) - b_1(t)x_1(t) - c_1(t)x_2(t) - d_1(t)x_1^2(t)], \\ x_2'(t) = x_2(t)[a_2(t) - b_2(t)x_2(t) - c_2(t)x_1(t) - d_2(t)x_2^2(t)]. \end{cases} \tag{1.2}$$

Here, all the coefficients $a_i(t)$, $b_i(t)$, $c_i(t)$, $d_i(t)$ ($i = 1, 2$) are subject to fluctuation in time. Furthermore, it is known that the assumption of almost periodicity of the coefficients is a way of incorporating the time-dependent variability of the environment, and especially, if the various components of the environment are with incommensurable periods, then it is reasonable to consider the environment to be almost periodic, which leads to the almost periodicity of the coefficients of system (1.2). On the other hand, species live in a real fluctuating medium, and human exploitation activities might result in the duration of abrupt changes. Such changes can be well approximated as impulses, and these processes tend to be reasonably modeled by impulsive differential equations.

Motivated by the above facts, we establish the following almost periodic competitive system with impulsive perturbations:

$$\begin{cases} \left.\begin{aligned} x_1'(t) &= x_1(t)[a_1(t) - b_1(t)x_1(t) - c_1(t)x_2(t) - d_1(t)x_1^2(t)], \\ x_2'(t) &= x_2(t)[a_2(t) - b_2(t)x_2(t) - c_2(t)x_1(t) - d_2(t)x_2^2(t)], \end{aligned}\right\} & t \neq \tau_k, \\ \left.\begin{aligned} x_1(\tau_k^+) &= (1 + \gamma_{1k})x_1(\tau_k), \\ x_2(\tau_k^+) &= (1 + \gamma_{2k})x_2(\tau_k), \end{aligned}\right\} & t = \tau_k, k \in \mathbb{N}. \end{cases} \tag{1.3}$$

Here, $x_1(0^+) = x_1(0) > 0$, $x_2(0^+) = x_2(0) > 0$, $\mathbb{N}$ is the set of positive integers, the coefficients $a_i(t)$, $b_i(t)$, $c_i(t)$, $d_i(t)$ are all continuous almost periodic functions which are bounded above and below by positive constants, $\gamma_{1k} > -1$ and $\gamma_{2k} > -1$ are constants and $0 < \tau_1 < \tau_2 < \cdots < \tau_k < \tau_{k+1} < \cdots$ are impulse points with $\lim_{k\to+\infty} \tau_k = +\infty$. The jump conditions reflect the possibility of impulsive effects on two species. From biological viewpoints, when $\gamma_{ik} > 0$, the perturbations may stand for stocking, while $\gamma_{ik} < 0$ the perturbations mean harvesting.

In the research of population ecology, competitive systems are very important to describe the interactions in the multi-species population dynamics. Many competitive systems have been studied recently by many authors and there is quite extensive literature concerned with the dynamics such as stability of equilibrium [2], persistence [3], permanence or partial extinction [4–7], positive periodic solution [8–11], positive almost periodic solution [12–15] *etc.* However, there are not many papers considering the stability of positive almost periodic solutions for impulsive competitive systems [14–16]. In this article, we make an attempt to discuss such an issue by considering system (1.3). The rest of this paper is arranged as follows. In Section 2, we present some notations, definitions and lemmas. In Section 3, we give the main result on the uniformly asymptotic stability of a unique positive almost periodic solution for system (1.3). In Section 4, an example together with its numerical simulations is presented to verify the validity of the proposed criteria.

2 Preliminaries

In this section, we give some notations, definitions, lemmas which are useful for establishing our main result (*i.e.*, Theorem 3.1).

Denote by $\mathbb{R}^+$, $\mathbb{R}$ and $\mathbb{Z}$ the sets of nonnegative real numbers, real numbers and integers, respectively. $\mathbb{R}^2$ and $\mathbb{R}^n$ denote the cone of a two-dimensional and n-dimensional real Euclidean space, respectively.

Definition 2.1 (see [17]) A sequence $\{\tau_k\}$ is called almost periodic if for any $\epsilon > 0$ there exists a relatively dense set of its ϵ-periods, *i.e.*, there exists such a positive integer $N = N(\epsilon)$ that, for any arbitrary $k \in \mathbb{Z}$, there is at least an integer p in the segment $[k, k + N]$, for which $|\tau_{k+p} - \tau_k| < \epsilon$ holds.

Definition 2.2 (see [17]) The set of sequences $\{\tau_k^j = \tau_{k+j} - \tau_k\}$, $k, j \in \mathbb{Z}$, is said to be uniformly almost periodic if for arbitrary $\epsilon > 0$, there exists a relatively dense set of ϵ-almost periodic common for any sequences.

Let $PC(\mathbb{R},\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R}$: u is continuous for $t \in \mathbb{R}$, $t \neq \tau_k$, continuous from the left for $t \in \mathbb{R}$ and discontinuities of the first kind occur at the point $\tau_k \in \mathbb{R}$, $k \in \mathbb{N}\}$.

Definition 2.3 (see [17]) The function $\varphi \in PC(\mathbb{R},\mathbb{R})$ is said to be almost periodic if the following conditions hold:

(1) The set of sequences $\{\tau_k^j\}$, $k, j \in \mathbb{Z}$ is uniformly almost periodic.
(2) For any $\epsilon > 0$, there exists a positive number $\delta = \delta(\epsilon)$ such that if the points t' and t'' belong to the same interval of continuity and $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \epsilon$.
(3) For any $\epsilon > 0$, there exists a relative dense set T of ϵ-almost periods such that if $\tau \in T$, then $|\varphi(t + \tau) - \varphi(t)| < \epsilon$ for all $t \in \mathbb{R}$, satisfying the condition $|t - \tau_k| > \epsilon$, $k \in \mathbb{Z}$.

Consider the following non-impulsive system which corresponds to system (1.3)

$$\begin{cases} y_1'(t) = y_1(t)[a_1(t) - B_1(t)y_1(t) - C_1(t)y_2(t) - D_1(t)y_1^2(t)], \\ y_2'(t) = y_2(t)[a_2(t) - B_2(t)y_2(t) - C_2(t)y_1(t) - D_2(t)y_2^2(t)], \end{cases} \tag{2.1}$$

where

$$\begin{aligned} &B_i(t) = b_i(t) \prod_{0<\tau_k<t} (1 + \gamma_{ik}), \qquad C_i(t) = c_i(t) \prod_{0<\tau_k<t} (1 + \gamma_{jk}), \\ &D_i(t) = d_i(t) \prod_{0<\tau_k<t} (1 + \gamma_{ik})^2, \quad i, j = 1, 2, i \neq j. \end{aligned} \tag{2.2}$$

The following Lemma 2.1 is obvious.

Lemma 2.1 *Any solution* $(y_1(t), y_2(t))$ *of system* (2.1) *satisfies* $y_i(t) > 0$ *for all* $t \geq 0$.

Lemma 2.2 *For systems* (1.3) *and* (2.1), *we have the following conclusions.*

(1) *If* $(y_1(t), y_2(t))$ *is a solution of system* (2.1), *then* $(x_1(t), x_2(t)) = (\prod_{0<\tau_k<t}(1 + \gamma_{1k})y_1(t), \prod_{0<\tau_k<t}(1 + \gamma_{2k})y_2(t))$ *is a solution of system* (1.3).
(2) *If* $(x_1(t), x_2(t))$ *is a solution of system* (1.3), *then* $(y_1(t), y_2(t)) = (\prod_{0<\tau_k<t}(1 + \gamma_{1k})^{-1}x_1(t), \prod_{0<\tau_k<t}(1 + \gamma_{2k})^{-1}x_2(t))$ *is a solution of system* (2.1).

Proof (1) Assume that $(y_1(t), y_2(t))$ is a solution of system (2.1). It is easy to see that $x_i(t) = \prod_{0<\tau_k<t}(1 + \gamma_{ik})y_i(t)$ are continuous on the interval $(\tau_k, \tau_{k+1}]$, then for any $t \neq \tau_k$, $k \in \mathbb{N}$, one has

$$\begin{aligned} &x_1'(t) - x_1(t)\big[a_1(t) - b_1(t)x_1(t) - c_1(t)x_2(t) - d_1(t)x_1^2(t)\big] \\ &\quad = \prod_{0<\tau_k<t} (1 + \gamma_{1k}) \Bigg\{ y_1'(t) - y_1(t) \Bigg[a_1(t) - b_1(t) \prod_{0<\tau_k<t} (1 + \gamma_{1k}) y_1(t) \\ &\qquad - c_1(t) \prod_{0<\tau_k<t} (1 + \gamma_{2k}) y_2(t) - d_1(t) \prod_{0<\tau_k<t} (1 + \gamma_{1k})^2 y_1^2(t) \Bigg] \Bigg\} \\ &\quad = \prod_{0<\tau_k<t} (1 + \gamma_{1k}) \big\{ y_1'(t) - y_1(t)\big[a_1(t) - B_1(t)y_1(t) - C_1(t)y_2(t) - D_1(t)y_1^2(t)\big]\big\} \\ &\quad = 0 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&x_2'(t) - x_2(t)\bigl[a_2(t) - b_2(t)x_2(t) - c_2(t)x_1(t) - d_2(t)x_2^2(t)\bigr] \\
&\quad = \prod_{0<\tau_k<t} (1+\gamma_{2k})\biggl\{y_2'(t) - y_2(t)\biggl[a_2(t) - b_2(t)\prod_{0<\tau_k<t}(1+\gamma_{2k})y_2(t) \\
&\qquad - c_2(t)\prod_{0<\tau_k<t}(1+\gamma_{1k})y_1(t) - d_2(t)\prod_{0<\tau_k<t}(1+\gamma_{2k})^2 y_2^2(t)\biggr]\biggr\} \\
&\quad = \prod_{0<\tau_k<t}(1+\gamma_{2k})\bigl\{y_2'(t) - y_2(t)\bigl[a_2(t) - B_2(t)y_2(t) - C_2(t)y_1(t) - D_2(t)y_2^2(t)\bigr]\bigr\} \\
&\quad = 0.
\end{aligned} \tag{2.4}$$

On the other hand, for every $t = \tau_k$, $k \in \mathbb{N}$, we get

$$\begin{aligned}
x_1\bigl(\tau_k^+\bigr) &= \lim_{t\to\tau_k^+}\prod_{0<\tau_k<t}(1+\gamma_{1k})y_1(t) = \prod_{0<\tau_j\le\tau_k}(1+\gamma_{1j})y_1(\tau_k) \\
&= (1+\gamma_{1k})\prod_{0<\tau_j<\tau_k}(1+\gamma_{1j})y_1(\tau_k) = (1+\gamma_{1k})x_1(\tau_k)
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
x_2\bigl(\tau_k^+\bigr) &= \lim_{t\to\tau_k^+}\prod_{0<\tau_k<t}(1+\gamma_{2k})y_2(t) = \prod_{0<\tau_j\le\tau_k}(1+\gamma_{2j})y_2(\tau_k) \\
&= (1+\gamma_{2k})\prod_{0<\tau_j<\tau_k}(1+\gamma_{2j})y_2(\tau_k) = (1+\gamma_{2k})x_2(\tau_k).
\end{aligned} \tag{2.6}$$

Thus $(x_1(t), x_2(t))$ is a solution of system (1.3).

(2) Since $y_1(t)$ and $y_2(t)$ are continuous on each interval $(\tau_k, \tau_{k+1}]$. From system (2.1), one can easily check the continuity of $y_i(t)$ at the impulse points $t = \tau_k$, $k \in \mathbb{N}$. Recalling system (1.3), we have

$$\begin{aligned}
y_1\bigl(\tau_k^+\bigr) &= \prod_{0<\tau_j\le\tau_k}(1+\gamma_{1j})^{-1}x_1\bigl(\tau_k^+\bigr) = \prod_{0<\tau_j<\tau_k}(1+\gamma_{1j})^{-1}x_1(\tau_k) = y_1(\tau_k), \\
y_2\bigl(\tau_k^+\bigr) &= \prod_{0<\tau_j\le\tau_k}(1+\gamma_{2j})^{-1}x_2\bigl(\tau_k^+\bigr) = \prod_{0<\tau_j<\tau_k}(1+\gamma_{2j})^{-1}x_2(\tau_k) = y_2(\tau_k).
\end{aligned} \tag{2.7}$$

Also, by the basic theory of impulsive differential equations in [18, 19], we know that $x_i(\tau_k^-) = x_i(\tau_k)$. So we get

$$\begin{aligned}
y_1\bigl(\tau_k^-\bigr) &= \prod_{0<\tau_j<\tau_k}(1+\gamma_{1j})^{-1}x_1\bigl(\tau_k^-\bigr) = \prod_{0<\tau_j<\tau_k}(1+\gamma_{1j})^{-1}x_1(\tau_k) = y_1(\tau_k), \\
y_2\bigl(\tau_k^-\bigr) &= \prod_{0<\tau_j<\tau_k}(1+\gamma_{2j})^{-1}x_2\bigl(\tau_k^-\bigr) = \prod_{0<\tau_j<\tau_k}(1+\gamma_{2j})^{-1}x_2(\tau_k) = y_2(\tau_k).
\end{aligned} \tag{2.8}$$

Equations (2.7) and (2.8) imply that $y_1(t)$ and $y_2(t)$ are continuous on $\mathbb{R}^+$. It is easy to see that $(y_1(t), y_2(t))$ is a solution of system (2.1). The proof of Lemma 2.2 is complete. □

Consider the following differential equation:

$$X' = f(t, X), \quad t \in \mathbb{R}^+, \tag{2.9}$$

where $f(t,X) \in C(\mathbb{R}^+ \times \mathbb{D}, \mathbb{R}^n)$ and $\mathbb{D}$ is an open set in $\mathbb{R}^n$, $f(t,X)$ is almost periodic in t uniformly with respect to $X \in \mathbb{D}$. The following associate product system of system (2.9) can be expressed as

$$X' = f(t, X), \qquad Y' = f(t, Y). \tag{2.10}$$

Lemma 2.3 (see Theorem 6.3 in [20]) *Suppose that there exists a Lyapunov function $V(t,X,Y)$ defined on $[0,+\infty) \times \mathbb{D} \times \mathbb{D}$, which satisfies the following conditions*:

(1) $a(\|X - Y\|) \le V(t,X,Y) \le b(\|X - Y\|)$, *where $a(\kappa)$ and $b(\kappa)$ are continuous, increasing and positive definite functions*.

(2) $|V(t,X_1,Y_1) - V(t,X_2,Y_2)| \le K\{\|X_1 - X_2\| + \|Y_1 - Y_2\|\}$, *where $K > 0$ is a constant*.

(3) $V'_{(2.10)}(t,X,Y) \le -\mu V(t,X,Y)$, *where $\mu > 0$ is a constant*.

Moreover, suppose that system (2.9) *has a solution that remains in a compact set $\mathbb{S} \subset \mathbb{D}$ for all $t \ge 0$. Then system* (2.9) *has a unique almost periodic solution in $\mathbb{S}$, which is uniformly asymptotically stable in $\mathbb{D}$.*

Lemma 2.4 (see [21]) (1) *If $a > 0$, $b > 0$ and $x'(t) \ge x(t)(a - bx(t))$, when $t \ge 0$ and $x(0) > 0$, we have* $\liminf_{t\to+\infty} x(t) \ge a/b$. (2) *If $a > 0$, $b > 0$ and $x'(t) \le x(t)(a - bx(t))$, when $t \ge 0$ and $x(0) > 0$, we have* $\limsup_{t\to+\infty} x(t) \le a/b$.

For convenience, given an almost periodic function $g(t)$ defined on $\mathbb{R}^+$, let g^L and g^U be defined as $g^L = \inf_{t\in\mathbb{R}^+} g(t)$, $g^U = \sup_{t\in\mathbb{R}^+} g(t)$.

Lemma 2.5 *Assume that the following two conditions*

(A1) *there exist positive constants α_i, β_i such that* $\alpha_i \le \prod_{0<\tau_k<t}(1 + \gamma_{ik}) \le \beta_i$, $i = 1, 2$,

(A2) $a_1^L - c_1^U \beta_2 M_2 > 0$ *and* $a_2^L - c_2^U \beta_1 M_1 > 0$

hold, then any solution $(y_1(t), y_2(t))$ of system (2.1) *satisfies*

$$m_i \le \liminf_{t\to+\infty} y_i(t) \le \limsup_{t\to+\infty} y_i(t) \le M_i, \tag{2.11}$$

where $m_i = (a_i^L - c_i^U \beta_j M_j)/(b_i^U \beta_i + d_i^U \beta_i^2 M_i)$, $M_i = a_i^U/(b_i^L \alpha_i)$, $1 \le i,j \le 2$; $i \ne j$.

Proof Let $(y_1(t), y_2(t))$ be any solution of system (2.1). It follows from system (2.1) and (A1) that we have

$$y_i'(t) \le y_i(t)\big[a_i(t) - B_i(t)y_i(t)\big] \le y_i(t)\big[a_i^U - b_i^L \alpha_i y_i(t)\big].$$

Using (2) in Lemma 2.4, one has

$$\limsup_{t\to+\infty} y_i(t) \le a_i^U/\big(b_i^L \alpha_i\big) \stackrel{\text{def}}{=} M_i. \tag{2.12}$$

Hence, for any small constant $\varepsilon > 0$, there exists $T_0 > 0$ such that for $t \ge T_0$,

$$y_i(t) \le M_i + \varepsilon, \quad i = 1, 2.$$

Together with system (2.1), we can derive that

$$y_i'(t) \ge y_i(t)\big[a_i^L - c_i^U \beta_j(M_j + \varepsilon) - \big(b_i^U \beta_i + d_i^U \beta_i^2 (M_i + \varepsilon)\big) y_i(t)\big], \quad i,j = 1,2, i \ne j.$$

Thus from (A2), (1) in Lemma 2.4 and $\varepsilon > 0$ is arbitrarily small, one has

$$\liminf_{t \to +\infty} y_i(t) \ge \big(a_i^L - c_i^U \beta_j M_j\big) / \big(b_i^U \beta_i + d_i^U \beta_i^2 M_i\big) \stackrel{\text{def}}{=} m_i, \quad i,j = 1,2, i \ne j. \tag{2.13}$$

The proof of Lemma 2.5 is complete. □

By (2.12) and (2.13), we denote by Θ the set of all solutions $(y_1(t), y_2(t))$ of system (2.1) satisfying $m_i \le y_i(t) \le M_i$, that is,

$$\Theta = \big\{\big(y_1(t), y_2(t)\big) | m_i \le y_i(t) \le M_i, i = 1,2\big\}. \tag{2.14}$$

Lemma 2.6 *Assume that* (A1) *and* (A2) *are satisfied. Suppose further that*

(A3) *the set of sequences* $\{\tau_k^j = \tau_{k+j} - \tau_k\}$, $k, j \in \mathbb{Z}$ *is uniformly almost periodic,*

(A4) $\prod_{0<\tau_k<t}(1 + \gamma_{ik})$ *is an almost periodic function.*

Then $\Theta \ne \phi$.

Proof The almost periodicity of $\{a_i(t)\}$, $\{B_i(t)\}$, $\{C_i(t)\}$, $\{D_i(t)\}$ implies that there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that

$$\begin{aligned} &a_i(t + t_n) \to a_i(t), \qquad B_i(t + t_n) \to B_i(t), \\ &C_i(t + t_n) \to C_i(t), \qquad D_i(t + t_n) \to D_i(t), \quad i = 1,2, \end{aligned} \tag{2.15}$$

as $n \to +\infty$ for $t \in \mathbb{R}^+$. It follows from (2.11) that, for any small enough $\varepsilon > 0$, there exists $T_1 > 0$ such that for $t \ge T_1$,

$$m_i - \varepsilon \le y_i(t) \le M_i + \varepsilon.$$

It is obvious that the sequence $\{y_i(t + t_n)\}$ is uniformly bounded and equicontinuous on each bounded subset of $\mathbb{R}^+$, $i = 1,2$. By Ascoli's theorem, we obtain that $\{y_i(t + t_n)\}$ exists a subsequence, also denoted by $\{y_i(t + t_n)\}$, converging on each bounded subset of $\mathbb{R}^+$ as $n \to +\infty$. Therefore, there is a continuous function $\bar{y}_i(t)$ satisfying

$$y_i(t + t_n) \to \bar{y}_i(t) \quad \text{for } t \in \mathbb{R}^+ \text{ as } n \to +\infty. \tag{2.16}$$

For any $T_2 \in \mathbb{R}^+$ such that $t_n + T_2 \ge T_1$ for large enough n. Let $t \ge 0$, we have

$$\begin{cases} y_1(t + t_n + T_2) = y_1(t_n + T_2) + \int_{T_2}^{t+T_2} y_1(s + t_n)[a_1(s + t_n) - B_1(s + t_n) y_1(s + t_n) \\ \qquad - C_1(s + t_n) y_2(s + t_n) - D_1(s + t_n) y_1^2(s + t_n)]\, ds, \\ y_2(t + t_n + T_2) = y_2(t_n + T_2) + \int_{T_2}^{t+T_2} y_2(s + t_n)[a_2(s + t_n) - B_2(s + t_n) y_2(s + t_n) \\ \qquad - C_2(s + t_n) y_1(s + t_n) - D_2(s + t_n) y_2^2(s + t_n)]\, ds. \end{cases} \tag{2.17}$$

Using Lebesgue' dominated convergence theorem, and letting $n \to +\infty$ in (2.17), one has

$$\begin{cases} \bar{y}_1(t+T_2) = \bar{y}_1(T_2) + \int_{T_2}^{t+T_2} \bar{y}_1(s)[a_1(s) - B_1(s)\bar{y}_1(s) \\ \qquad - C_1(s)\bar{y}_2(s) - D_1(s)\bar{y}_1^2(s)]\,ds, \\ \bar{y}_2(t+T_2) = \bar{y}_2(T_2) + \int_{T_2}^{t+T_2} \bar{y}_2(s)[a_2(s) - B_2(s)\bar{y}_2(s) \\ \qquad - C_2(s)\bar{y}_1(s) - D_2(s)\bar{y}_2^2(s)]\,ds. \end{cases} \tag{2.18}$$

Since $T_2 \in \mathbb{R}^+$ is arbitrary, $(\bar{y}_1(t), \bar{y}_2(t))$ is a solution of system (2.1) on $\mathbb{R}^+$. We easily obtain that $m_i - \varepsilon \le \bar{y}_i(t) \le M_i + \varepsilon$ for $t \in \mathbb{R}^+$, $i = 1, 2$. Furthermore, since ε is arbitrarily small, we get that $m_i \le \bar{y}_i(t) \le M_i$, $i = 1, 2$, for $t \in \mathbb{R}^+$. The proof of Lemma 2.6 is complete. □

3 The main result

In this section, we give our main result and establish the uniformly asymptotic stability of a unique positive almost periodic solution for system (1.3).

Theorem 3.1 *Assume that* (A1)-(A4) *hold. Furthermore, assume that*

(A5) *there exist positive constants* θ_1, θ_2, σ, *where* $\sigma = \min\{s_1, s_2\}$, *and*

$$\begin{aligned} s_1 &= \theta_1 b_1^L \alpha_1 m_1 + 2\theta_1 d_1^L \alpha_1^2 m_1^2 - \theta_2 c_2^U \beta_1 M_1, \\ s_2 &= \theta_2 b_2^L \alpha_2 m_2 + 2\theta_2 d_2^L \alpha_2^2 m_2^2 - \theta_1 c_1^U \beta_2 M_2. \end{aligned} \tag{3.1}$$

Then system (1.3) *has a unique uniformly asymptotically stable positive almost periodic solution.*

Proof Let us make the change of variables

$$z_1(t) = \ln y_1(t), \qquad z_2(t) = \ln y_2(t),$$

then system (2.1) can be rewritten as

$$\begin{cases} z_1'(t) = a_1(t) - B_1(t)\exp\{z_1(t)\} - C_1(t)\exp\{z_2(t)\} - D_1(t)\exp\{2z_1(t)\}, \\ z_2'(t) = a_2(t) - B_2(t)\exp\{z_2(t)\} - C_2(t)\exp\{z_1(t)\} - D_2(t)\exp\{2z_2(t)\}. \end{cases} \tag{3.2}$$

Obviously, the existence of a unique almost periodic solution of system (2.1) is equivalent to that of system (3.2). By Lemma 2.6, there is a bounded solution $(z_1(t), z_2(t))$ of system (3.2) satisfying

$$\ln m_1 \le z_1(t) \le \ln M_1, \qquad \ln m_2 \le z_2(t) \le \ln M_2.$$

Define the norm $\|(z_1(t), z_2(t))\| = |z_1(t)| + |z_2(t)|$, where $(z_1(t), z_2(t)) \in \mathbb{R}^2$. Consider the associate product system of system (3.2)

$$\begin{cases} z_1'(t) = a_1(t) - B_1(t)\exp\{z_1(t)\} - C_1(t)\exp\{z_2(t)\} - D_1(t)\exp\{2z_1(t)\}, \\ z_2'(t) = a_2(t) - B_2(t)\exp\{z_2(t)\} - C_2(t)\exp\{z_1(t)\} - D_2(t)\exp\{2z_2(t)\}, \\ w_1'(t) = a_1(t) - B_1(t)\exp\{w_1(t)\} - C_1(t)\exp\{w_2(t)\} - D_1(t)\exp\{2w_1(t)\}, \\ w_2'(t) = a_2(t) - B_2(t)\exp\{w_2(t)\} - C_2(t)\exp\{w_1(t)\} - D_2(t)\exp\{2w_2(t)\}. \end{cases} \tag{3.3}$$

Here, $Z(t) = (z_1(t), z_2(t))$ and $W(t) = (w_1(t), w_2(t))$ are any two solutions of system (3.2) defined on $\mathbb{S}$, and $\mathbb{S} = \{(z_1(t), z_2(t)) \in \mathbb{R}^2 | \ln m_i \le z_i(t) \le \ln M_i, i = 1, 2, t \in \mathbb{R}^+\}$.

Next, let us consider a Lyapunov function defined on $\mathbb{R}^+ \times \mathbb{S} \times \mathbb{S}$ as follows:

$$V\big(t, Z(t), W(t)\big) = \theta_1 \big|z_1(t) - w_1(t)\big| + \theta_2 \big|z_2(t) - w_2(t)\big|. \tag{3.4}$$

It is obvious that

$$\min\{\theta_1, \theta_2\} \big\|Z(t) - W(t)\big\| \le V\big(t, Z(t), W(t)\big) \le \max\{\theta_1, \theta_2\} \big\|Z(t) - W(t)\big\|. \tag{3.5}$$

Let $a(\kappa) = \min\{\theta_1, \theta_2\}\kappa$, $b(\kappa) = \max\{\theta_1, \theta_2\}\kappa$, then condition (1) in Lemma 2.3 is satisfied.

In addition, for any $(t, Z, W), (t, \tilde{Z}, \tilde{W}) \in \mathbb{R}^+ \times \mathbb{S} \times \mathbb{S}$, one has

$$\begin{aligned}
&\big|V\big(t, Z(t), W(t)\big) - V\big(t, \tilde{Z}(t), \tilde{W}(t)\big)\big| \\
&\quad = \big|\theta_1 \big|z_1(t) - w_1(t)\big| + \theta_2 \big|z_2(t) - w_2(t)\big| - \theta_1 \big|\tilde{z}_1(t) - \tilde{w}_1(t)\big| - \theta_2 \big|\tilde{z}_2(t) - \tilde{w}_2(t)\big|\big| \\
&\quad \le \theta_1 \big|z_1(t) - \tilde{z}_1(t)\big| + \theta_1 \big|w_1(t) - \tilde{w}_1(t)\big| + \theta_2 \big|z_2(t) - \tilde{z}_2(t)\big| + \theta_2 \big|w_2(t) - \tilde{w}_2(t)\big| \\
&\quad \le \max\{\theta_1, \theta_2\} \big\{\big\|Z(t) - \tilde{Z}(t)\big\| + \big\|W(t) - \tilde{W}(t)\big\|\big\} \\
&\quad = \lambda \big\{\big\|Z(t) - \tilde{Z}(t)\big\| + \big\|W(t) - \tilde{W}(t)\big\|\big\},
\end{aligned} \tag{3.6}$$

where $\tilde{Z}(t) = (\tilde{z}_1(t), \tilde{z}_2(t))$, $\tilde{W}(t) = (\tilde{w}_1(t), \tilde{w}_2(t))$, $\lambda = \max\{\theta_1, \theta_2\}$. Hence, condition (2) in Lemma 2.3 is satisfied.

Finally, calculating the right derivative $D^+V(t)$ of $V(t)$ along the solutions of system (3.3), one has

$$\begin{aligned}
D^+_{(3.3)} V(t) &= \theta_1 \operatorname{sgn}\big(z_1(t) - w_1(t)\big)\big(z_1'(t) - w_1'(t)\big) \\
&\quad + \theta_2 \operatorname{sgn}\big(z_2(t) - w_2(t)\big)\big(z_2'(t) - w_2'(t)\big) \\
&= \theta_1 \operatorname{sgn}\big(z_1(t) - w_1(t)\big)\big[-B_1(t)\big(\exp\big\{z_1(t)\big\} - \exp\big\{w_1(t)\big\}\big) \\
&\quad - C_1(t)\big(\exp\big\{z_2(t)\big\} - \exp\big\{w_2(t)\big\}\big) - D_1(t)\big(\exp\big\{2z_1(t)\big\} - \exp\big\{2w_1(t)\big\}\big)\big] \\
&\quad + \theta_2 \operatorname{sgn}\big(z_2(t) - w_2(t)\big)\big[-B_2(t)\big(\exp\big\{z_2(t)\big\} - \exp\big\{w_2(t)\big\}\big) \\
&\quad - C_2(t)\big(\exp\big\{z_1(t)\big\} - \exp\big\{w_1(t)\big\}\big) \\
&\quad - D_2(t)\big(\exp\big\{2z_2(t)\big\} - \exp\big\{2w_2(t)\big\}\big)\big].
\end{aligned} \tag{3.7}$$

By the mean-value theorem, we have

$$\begin{aligned}
&\exp\big\{z_i(t)\big\} - \exp\big\{w_i(t)\big\} = \xi_i(t)\big(z_i(t) - w_i(t)\big), \\
&\exp\big\{2z_i(t)\big\} - \exp\big\{2w_i(t)\big\} = 2\eta_i^2(t)\big(z_i(t) - w_i(t)\big),
\end{aligned} \tag{3.8}$$

$i = 1, 2$, where $\xi_i(t)$ and $\eta_i(t)$ lie between $\exp\{z_i(t)\}$ and $\exp\{w_i(t)\}$, respectively. Substituting (3.8) into (3.7), one has

$$\begin{aligned}
D^+_{(3.3)} V(t) &= \theta_1 \operatorname{sgn}\big(z_1(t) - w_1(t)\big)\big[-B_1(t)\xi_1(t)\big(z_1(t) - w_1(t)\big) \\
&\quad - C_1(t)\xi_2(t)\big(z_2(t) - w_2(t)\big) - 2D_1(t)\eta_1^2(t)\big(z_1(t) - w_1(t)\big)\big]
\end{aligned}$$

$$\begin{aligned}
&+ \theta_2 \operatorname{sgn}\bigl(z_2(t) - w_2(t)\bigr)\bigl[-B_2(t)\xi_2(t)\bigl(z_2(t) - w_2(t)\bigr) \\
&- C_2(t)\xi_1(t)\bigl(z_1(t) - w_1(t)\bigr) - 2D_2(t)\eta_2^2(t)\bigl(z_2(t) - w_2(t)\bigr)\bigr] \\
\le{}& -\theta_1 B_1(t)\xi_1(t)\bigl|z_1(t) - w_1(t)\bigr| + \theta_1 C_1(t)\xi_2(t)\bigl|z_2(t) - w_2(t)\bigr| \\
&- 2\theta_1 D_1(t)\eta_1^2(t)\bigl|z_1(t) - w_1(t)\bigr| \\
&- \theta_2 B_2(t)\xi_2(t)\bigl|z_2(t) - w_2(t)\bigr| + \theta_2 C_2(t)\xi_1(t)\bigl|z_1(t) - w_1(t)\bigr| \\
&- 2\theta_2 D_2(t)\eta_2^2(t)\bigl|z_2(t) - w_2(t)\bigr| \\
\le{}& -\bigl(\theta_1 b_1^L \alpha_1 m_1 + 2\theta_1 d_1^L \alpha_1^2 m_1^2 - \theta_2 c_2^U \beta_1 M_1\bigr)\bigl|z_1(t) - w_1(t)\bigr| \\
&- \bigl(\theta_2 b_2^L \alpha_2 m_2 + 2\theta_2 d_2^L \alpha_2^2 m_2^2 - \theta_1 c_1^U \beta_2 M_2\bigr)\bigl|z_2(t) - w_2(t)\bigr| \\
={}& -s_1\bigl|z_1(t) - w_1(t)\bigr| - s_2\bigl|z_2(t) - w_2(t)\bigr| \\
\le{}& -\sigma \min\{1/\theta_1, 1/\theta_2\} V(t) \\
={}& -\mu V(t),
\end{aligned} \tag{3.9}$$

where $\sigma = \min\{s_1, s_2\}$ and $\mu = \sigma \min\{1/\theta_1, 1/\theta_2\}$. It follows from condition (A5) in Theorem 3.1 that we have $\mu > 0$, that is, condition (3) in Lemma 2.3 is also satisfied. Therefore, it follows from Lemma 2.3 that system (3.2) has a unique almost periodic solution $(z_1^*(t), z_2^*(t))$ which is uniformly asymptotically stable in $\mathbb{S}$. That is, system (2.1) has a unique uniformly asymptotically stable positive almost periodic solution $(y_1^*(t), y_2^*(t)) = (\exp\{z_1^*(t)\}, \exp\{z_2^*(t)\})$.

Finally, we will prove that system (1.3) has a unique uniformly asymptotically stable positive almost periodic solution. It follows from Lemma 2.2 that

$$\bigl(x_1^*(t), x_2^*(t)\bigr) = \biggl(\prod_{0<\tau_k<t} (1 + \gamma_{1k}) y_1^*(t), \prod_{0<\tau_k<t} (1 + \gamma_{2k}) y_2^*(t)\biggr) \tag{3.10}$$

is a solution of system (1.3). By conditions (A3) and (A4), we can prove that $x_i^*(t) = \prod_{0<\tau_k<t}(1 + \gamma_{ik}) y_i^*(t)$ is an almost periodic function based on the proofs of Lemma 31 and Theorem 79 in [17]. Thus $(x_1^*(t), x_2^*(t))$ is a unique uniformly asymptotically stable positive almost periodic solution of system (1.3). The proof of Theorem 3.1 is complete. □

4 An example and numerical simulations

In this section, to illustrate the feasibility of our analytical results, we give the following example.

Example 4.1 Consider the competitive system with impulsive perturbations

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
x_1'(t) = x_1(t)[1.18 + 0.02\sin(\sqrt{3}t) - (0.95 + 0.02\sin(\sqrt{2}t))x_1(t) \\
\qquad - (0.00025 + 0.00002\sin(\sqrt{3}t))x_2(t) \\
\qquad - (0.54 + 0.01\sin(\sqrt{2}t))x_1^2(t)], \\
x_2'(t) = x_2(t)[1.05 - 0.01\cos(\sqrt{2}t) - (0.82 + 0.01\cos(\sqrt{2}t))x_2(t) \\
\qquad - (0.00015 + 0.00001\cos(\sqrt{2}t))x_1(t) \\
\qquad - (0.45 + 0.02\cos(\sqrt{3}t))x_2^2(t)],
\end{array}\right\} \quad t \neq \tau_k, \\
\left.\begin{array}{l}
x_1(\tau_k^+) = (1 + \gamma_{1k})x_1(\tau_k), \\
x_2(\tau_k^+) = (1 + \gamma_{2k})x_2(\tau_k),
\end{array}\right\} \quad t = \tau_k, k \in \mathbb{N}.
\end{array}\right. \tag{4.1}$$

Let $\gamma_{ik} = \exp\{(-1)^{k+1}\frac{1}{k^2}\} - 1$, $\tau_k = k$, $k \in \mathbb{N}$, we obtain that $1 < \prod_{0<\tau_k<t}(1 + \gamma_{ik}) < e$, $i = 1, 2$. So we can choose $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = e$. A computation shows that

$$\begin{aligned} &M_1 \approx 1.2903, \qquad M_2 \approx 1.3086, \qquad m_1 \approx 0.1471, \qquad m_2 \approx 0.1528, \\ &a_1^L - c_1^U \beta_2 M_2 \approx 1.1590 > 0, \qquad a_2^L - c_2^U \beta_1 M_1 \approx 1.0394 > 0. \end{aligned} \tag{4.2}$$

Obviously, (A1) and (A2) in Lemma 2.5 are satisfied; moreover, letting $\theta_1 = \theta_2 = 1$, one has

$$\begin{aligned} &s_1 = \theta_1 b_1^L \alpha_1 m_1 + 2\theta_1 d_1^L \alpha_1^2 m_1^2 - \theta_2 c_2^U \beta_1 M_1 \approx 0.1592 > 0, \\ &s_2 = \theta_2 b_2^L \alpha_2 m_2 + 2\theta_2 d_2^L \alpha_2^2 m_2^2 - \theta_1 c_1^U \beta_2 M_2 \approx 0.1429 > 0, \end{aligned} \tag{4.3}$$

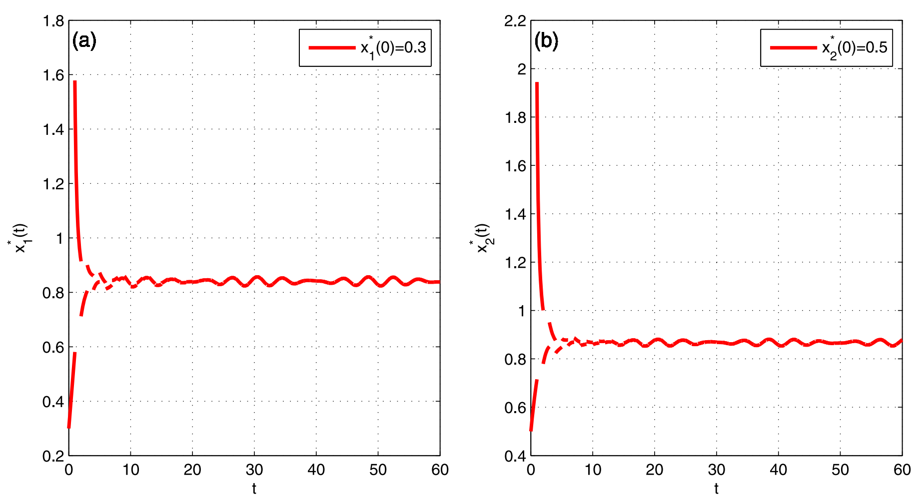

Figure 1 Positive almost periodic solution of system (4.1). (a) Time-series of $x_1^*(t)$ with initial value $x_1^*(0) = 0.3$ for $t \in [0, 60]$. **(b)** Time-series of $x_2^*(t)$ with initial value $x_2^*(0) = 0.5$ for $t \in [0, 60]$.

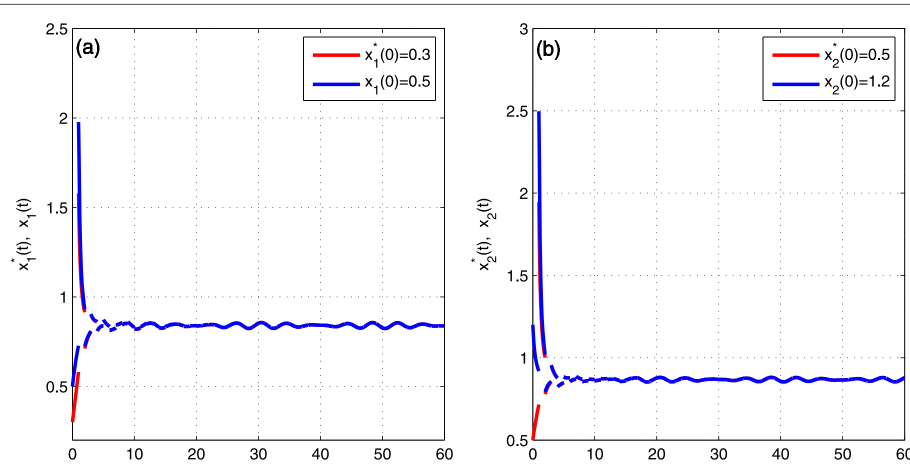

Figure 2 Uniformly asymptotic stability. (a) Time-series of $x_1^*(t)$ with $x_1^*(0) = 0.3$ and $x_1(t)$ with $x_1(0) = 0.5$ for $t \in [0, 60]$, respectively. **(b)** Time-series of $x_2^*(t)$ with $x_2^*(0) = 0.5$ and $x_2(t)$ with $x_2(0) = 1.2$ for $t \in [0, 60]$, respectively.

that is, $\sigma = \min\{s_1, s_2\} \approx 0.1429 > 0$, so (A3)-(A5) in Theorem 3.1 are satisfied. Thus, system (4.1) has a unique uniformly asymptotically stable positive almost periodic solution. From Figure 1, we can easily see that system (4.1) with initial value $(0.3, 0.5)$ has a positive almost periodic solution denoted by $(x_1^*(t), x_2^*(t))$. Figure 2 shows that a positive solution with initial value $(0.5, 1.2)$, denoted by $(x_1(t), x_2(t))$, tends to the above positive almost periodic solution $(x_1^*(t), x_2^*(t))$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, RT, WL *et al.* contributed to each part of this work equally and read and approved the final version of the manuscript.

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