Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Complex oscillation of a second-order linear differential equation with entire coefficients of $[p,q]-\varphi$ order

Xia Shen[1], Jin Tu[2] and Hong Yan Xu[3*]

*Correspondence: xhyhhh@126.com
[3]Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, 333403, China
Full list of author information is available at the end of the article

Abstract

In this paper, the authors investigate the interaction between the growth, zeros of solutions with the coefficients of second-order linear differential equations in terms of $[p,q]-\varphi$ order and obtain some results in general form.

MSC: 30D35; 34A20

Keywords: linear differential equations; $[p,q]-\varphi$ order; $[p,q]-\varphi$ exponent of convergence of zero sequence

1 Introduction and notations

In this paper, we shall assume that readers are familiar with the standard notations of Nevanlinna value distribution theory (see [1–3]). The theory of complex linear equations has been developed since 1960s. Many authors have investigated the second-order linear differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function or a meromorphic function of finite order or finite iterated order, and have obtained many results about the interaction between the solutions and the coefficient of (1.1) (see [4–7]). What about the case when $A(z)$ is an entire function of $[p,q]$-order or more general growth? In the following, we will introduce some notations about $[p,q]$-order, where p and q are two positive integers and satisfy $p \geq q \geq 1$ throughout this paper (see [8–11]). Firstly, for $r \in [0,+\infty)$, we define $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in \mathbb{N}$, and for all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $i \in \mathbb{N}$. Especially, we have $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Secondly, we denote the linear measure and the logarithmic measure of a set $E \subset (1,+\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E \frac{dt}{t}$.

Definition 1.1 ([10]) If $f(z)$ is a meromorphic function, the $[p,q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \varlimsup_{r\to\infty} \frac{\log_p T(r,f)}{\log_q r}. \tag{1.2}$$

Especially, if $f(z)$ is an entire function, then the $[p,q]$-order of $f(z)$ is defined by (see [8, 9, 11, 12])

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q r}. \tag{1.3}$$

Remark 1.1 We use $\sigma_{[1,1]}(f) = \sigma(f)$ and $\sigma_{[p,1]}(f) = \sigma_p(f)$ to denote the order and the iterated order of a function $f(z)$.

Definition 1.2 ([10, 13]) The growth index (or the finiteness degree) of the iterated order of a meromorphic function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{if } f \text{ is rational,} \\ \min\{n \in \mathbb{N} : \sigma_n(f) < \infty\} & \text{if } f \text{ is transcendental and } \sigma_n(f) < \infty \text{ for some } n \in \mathbb{N}, \\ \infty & \text{if with } \sigma_n(f) = \infty \text{ for all } n \in \mathbb{N}. \end{cases}$$

Remark 1.2 By Definition 1.2, we can similarly give the definition of the growth index of the iterated exponent of convergence of the zero-sequence of a meromorphic function $f(z)$ by $i_\lambda(f,0)$.

Definition 1.3 ([10, 11]) The $[p,q]$ exponent of convergence of the (distinct) zero-sequence of a meromorphic function $f(z)$ is respectively defined by

$$\lambda_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f})}{\log_q r}, \tag{1.4}$$

$$\overline{\lambda}_{[p,q]}(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q r} = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q r}. \tag{1.5}$$

Definition 1.4 ([10]) The $[p,q]$ exponent of convergence of the (distinct) pole-sequence of a meromorphic function $f(z)$ is respectively defined by

$$\lambda_{[p,q]}\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,f)}{\log_q r}, \tag{1.6}$$

$$\overline{\lambda}_{[p,q]}\left(\frac{1}{f}\right) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r,f)}{\log_q r}. \tag{1.7}$$

Remark 1.3 We use $\lambda_{[1,1]}(f) = \lambda(f)$, $\lambda_{[p,1]}(f) = \lambda_p(f)$ and $\lambda_{[1,1]}(\frac{1}{f}) = \lambda(\frac{1}{f})$, $\lambda_{[p,1]}(\frac{1}{f}) = \lambda_p(\frac{1}{f})$ to denote the (iterated) exponent of convergence of the zero-sequence and pole-sequence of a meromorphic function $f(z)$.

Recently, some authors have investigated the exponent of convergence of the zero-sequence and pole-sequence of the solutions of second-order linear differential equations (see [13–15]) and have obtained the following results.

Theorem A ([5]) *Let A be a transcendental meromorphic function of order $\sigma(A)$, where $0 < \sigma(A) \leq \infty$, and assume that $\overline{\lambda}(A) < \sigma(A)$. Then, if $f \not\equiv 0$ is a meromorphic solution of*

(1.1), *we have*

$$\sigma(A) \leq \max\left\{\overline{\lambda}(f), \overline{\lambda}\left(\frac{1}{f}\right)\right\}.$$

Theorem B ([13]) *Let $A(z)$ be an entire function with $i(A) = p \in \mathbb{N}_+$. Let f_1, f_2 be two linearly independent solutions of* (1.1) *and denote $F = f_1 f_2$. Then $i_\lambda(F,0) \leq p+1$ and*

$$\lambda_{p+1}(F,0) = \sigma_{p+1}(F) = \max\left\{\lambda_{p+1}(f_1,0), \lambda_{p+1}(f_2,0)\right\} \leq \sigma_p(A).$$

If $i_\lambda(F,0) \leq p$, then $i_\lambda(f,0) = p+1$ holds for all solutions of type $f = c_1 f_1 + c_2 f_2$, where $c_1 c_2 \neq 0$.

Theorem C ([13]) *Let $A(z)$ be an entire function with $0 < i(A) = p < \infty$, let f be any nontrivial solution of* (1.1), *and assume $\overline{\lambda}_p(A,0) < \sigma_p(A) \neq 0$. Then $\lambda_{p+1}(f,0) \leq \sigma_p(A) \leq \lambda_p(f,0)$.*

Theorem D ([13]) *Let $A(z)$ be an entire function with $i(A) = p$ and $\sigma_p(A) = \sigma < \infty$. Let f_1 and f_2 be two linearly independent solutions of* (1.1) *such that* $\max\{\lambda_p(f_1,0), \lambda_p(f_2,0)\} < \sigma$. *Let $\Pi(z) \not\equiv 0$ be any entire function for which either $i(\Pi) < p$ or $i(\Pi) = p$ and $\sigma_p(\Pi) < \sigma$. Then any two linearly independent solutions g_1 and g_2 of the differential equation $y'' + (A(z) + \Pi(z))y = 0$ satisfy* $\max\{\lambda_p(g_1), \lambda_p(g_2)\} \geq \sigma$.

Theorem E ([14]) *Let A be a meromorphic function with $i(A) = p \in \mathbb{N}_+$, and assume that $\overline{\lambda}_p(A) < \sigma_p(A)$. Then, if f is a nonzero meromorphic solution of* (1.1), *we have*

$$\sigma_p(A) \leq \max\left\{\overline{\lambda}_p(f), \overline{\lambda}_p\left(\frac{1}{f}\right)\right\}.$$

In the special case where either $\delta(\infty, f) > 0$ or the poles of f are of uniformly bounded multiplicities, we can conclude that

$$\max\left\{\lambda_{p+1}(f), \lambda_{p+1}\left(\frac{1}{f}\right)\right\} \leq \sigma_p(f) \leq \left\{\overline{\lambda}_p(f), \overline{\lambda}_p\left(\frac{1}{f}\right)\right\}.$$

In [16], Chyzhykov and his co-authors introduced the definition of φ-order of $f(z)$, where $f(z)$ is a meromorphic function in the unit disc and used it to investigate the interaction between the analytic coefficients and solutions of

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0$$

in the unit disc, where the definition of φ-order of $f(z)$ is given as follows.

Definition 1.5 ([16]) Let $\varphi : [0,1) \to (0, +\infty)$ be a non-decreasing unbounded function, the φ-order of a meromorphic function $f(z)$ in the unit disc is defined by

$$\sigma(f, \varphi) = \overline{\lim_{r \to 1^-}} \frac{\log^+ T(r,f)}{\log \varphi(r)}. \tag{1.8}$$

On the basis of Definition 1.5, it is natural for us to give the $[p,q] - \varphi$ order of a meromorphic function $f(z)$ in the complex plane.

Definition 1.6 Let $\varphi : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function, the $[p, q] - \varphi$ order and $[p, q] - \varphi$ lower order of a meromorphic function $f(z)$ are respectively defined by

$$\sigma_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)}, \tag{1.9}$$

$$\mu_{[p,q]}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)}. \tag{1.10}$$

Similar to Definition 1.6, we can also define the $[p, q] - \varphi$ exponent of convergence of the (distinct) zero-sequence of a meromorphic function $f(z)$.

Definition 1.7 The $[p, q] - \varphi$ exponent of convergence of the (distinct) zero-sequence of a meromorphic function $f(z)$ is respectively defined by

$$\lambda_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r, \frac{1}{f})}{\log_q \varphi(r)}, \tag{1.11}$$

$$\overline{\lambda}_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r, \frac{1}{f})}{\log_q \varphi(r)}. \tag{1.12}$$

Proposition 1.1 *If $f_1(z)$, $f_2(z)$ are meromorphic functions satisfying $\sigma_{[p,q]}(f_1, \varphi) = a$, $\sigma_{[p,q]}(f_2, \varphi) = b$, then*

(i) $\sigma_{[p,q]}(f_1 + f_2, \varphi) \le \max\{a, b\}$, $\sigma_{[p,q]}(f_1 \cdot f_2, \varphi) \le \max\{a, b\}$;

(ii) *If $a \ne b$, $\sigma_{[p,q]}(f_1 + f_2, \varphi) = \max\{a, b\}$, $\sigma_{[p,q]}(f_1 \cdot f_2, \varphi) = \max\{a, b\}$.*

In this paper, we add two conditions on $\varphi(r)$ as follows: $\varphi(r) : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function and satisfies (i) $\lim_{r\to\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, (ii) $\lim_{r\to\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$. Throughout this paper, we assume that $\varphi(r)$ always satisfies the above two conditions without special instruction.

Proposition 1.2 *Let $\varphi(r)$ satisfy the above two conditions* (i)-(ii).

(i) *If $f(z)$ is an entire function, then*

$$\sigma_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)},$$

$$\mu_{[p,q]}(f, \varphi) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log_q \varphi(r)} = \underline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)}.$$

(ii) *If $f(z)$ is a meromorphic function, then*

$$\lambda_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r, \frac{1}{f})}{\log_q \varphi(r)} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log_q \varphi(r)},$$

$$\overline{\lambda}_{[p,q]}(f, \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r, \frac{1}{f})}{\log_q \varphi(r)} = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f})}{\log_q \varphi(r)}.$$

Proof (i) By the inequality $T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f)$ $(0 < r < R)$, set $R = \alpha r$ $(\alpha > 1)$, we have

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{\alpha+1}{\alpha-1} T(\alpha r,f). \tag{1.13}$$

By (1.13) and $\lim_{r\to\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$, it is easy to see that conclusion (i) holds.

(ii) Without loss of generality, assume that $f(0) \neq 0$, then $N(r, \frac{1}{f}) = \int_0^r \frac{n(t,\frac{1}{f})}{t}\,dt$. Since

$$N\left(r, \frac{1}{f}\right) - N\left(r_0, \frac{1}{f}\right) = \int_{r_0}^{r} \frac{n(t,\frac{1}{f})}{t}\,dt \leq n\left(r, \frac{1}{f}\right) \log \frac{r}{r_0} \quad (0 < r_0 < r), \tag{1.14}$$

then by (1.14) and $\lim_{r\to\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f})}{\log_q \varphi(r)} \leq \max\left\{ \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)}, \overline{\lim_{r\to\infty}} \frac{\log_{p+1} r}{\log_q \varphi(r)} \right\} = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)}. \tag{1.15}$$

On the other hand, since $\alpha > 1$, we have

$$N\left(\alpha r, \frac{1}{f}\right) = \int_0^{\alpha r} \frac{n(t,\frac{1}{f})}{t}\,dt \geq \int_r^{\alpha r} \frac{n(t,\frac{1}{f})}{t}\,dt \geq n\left(r, \frac{1}{f}\right) \log \alpha. \tag{1.16}$$

By (1.16) and $\lim_{r\to\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f})}{\log_q \varphi(r)} \geq \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)}. \tag{1.17}$$

By (1.15) and (1.17), it is easy to see that $\lambda_{[p,q]}(f,\varphi) = \overline{\lim}_{r\to\infty} \frac{\log_p n(r,\frac{1}{f})}{\log_q \varphi(r)} = \overline{\lim}_{r\to\infty} \frac{\log_p N(r,\frac{1}{f})}{\log_q \varphi(r)}$. By the same proof above, we can obtain the conclusion $\overline{\lambda}_{[p,q]}(f,\varphi) = \overline{\lim}_{r\to\infty} \frac{\log_p \overline{n}(r,\frac{1}{f})}{\log_q \varphi(r)} = \overline{\lim}_{r\to\infty} \frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q \varphi(r)}$. □

Remark 1.4 If $\varphi(r) = r$, Definitions 1.1 and 1.3 are special cases of Definitions 1.6 and 1.7.

2 Main results

In this paper, our aim is to make use of the concept of $[p,q] - \varphi$ order of entire functions to investigate the growth, zeros of the solutions of equation (1.1).

Theorem 2.1 *Let $A(z)$ be an entire function satisfying $\sigma_{[p,q]}(A,\varphi) > 0$. Then $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A,\varphi)$ holds for all non-trivial solutions of* (1.1).

Theorem 2.2 *Let $A(z)$ be an entire function satisfying $\sigma_{[p,q]}(A,\varphi) > 0$, let f_1, f_2 be two linearly independent solutions of* (1.1) *and denote $F = f_1f_2$. Then* $\max\{\lambda_{[p+1,q]}(f_1,\varphi), \lambda_{[p+1,q]}(f_2,\varphi)\} = \lambda_{[p+1,q]}(F,\varphi) = \sigma_{[p+1,q]}(F,\varphi) \leq \sigma_{[p,q]}(A,\varphi)$. *If $\sigma_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A,\varphi)$, then $\lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A,\varphi)$ holds for all solutions of type $f = c_1f_1 + c_2f_2$, where $c_1c_2 \neq 0$.*

Theorem 2.3 *Let $A(z)$ be an entire function satisfying $\overline{\lambda}_{[p,q]}(A,\varphi) < \sigma_{[p,q]}(A,\varphi)$. Then $\lambda_{[p+1,q]}(f,\varphi) \le \sigma_{[p,q]}(A,\varphi) \le \lambda_{[p,q]}(f,\varphi)$ holds for all non-trivial solutions of* (1.1).

Theorem 2.4 *Let $A(z)$ be an entire function satisfying $\sigma_{[p,q]}(A,\varphi) = \sigma_1 > 0$, let f_1 and f_2 be two linearly independent solutions of* (1.1) *such that* $\max\{\lambda_{[p,q]}(f_1,\varphi), \lambda_{[p,q]}(f_2,\varphi)\} < \sigma_1$. *Let $\Pi(z) \not\equiv 0$ be any entire function satisfying $\sigma_{[p,q]}(\Pi,\varphi) < \sigma_1$. Then any two linearly independent solutions g_1 and g_2 of the differential equation $f'' + (A(z) + \Pi(z))f = 0$ satisfy* $\max\{\lambda_{[p,q]}(g_1,\varphi), \lambda_{[p,q]}(g_2,\varphi)\} \ge \sigma_1$.

3 Some lemmas

Lemma 3.1 ([17–19]) *Let $f(z)$ be a transcendental entire function, and let z be a point with $|z| = r$ at which $|f(z)| = M(r,f)$. Then, for all $|z|$ outside a set E_1 of r of finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j \bigl(1 + o(1)\bigr) \quad (j \in \mathbb{N}), \tag{3.1}$$

where $\nu_f(r)$ is the central index of $f(z)$.

Lemma 3.2 ([7, 19, 20]) *Let $g : [0,+\infty) \longrightarrow \mathbb{R}$ and $h : [0,+\infty) \longrightarrow \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \le h(r)$ outside of an exceptional set E_2 of finite linear measure or finite logarithmic measure. Then, for any $d > 1$, there exists $r_0 > 0$ such that $g(r) \le h(dr)$ for all $r > r_0$.*

Lemma 3.3 ([18, 21]) *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu(r)$ be the maximum term, i.e., $\mu(r) = \max\{|a_n|r^n; n = 0,1,\ldots\}$, and let $\nu_f(r)$ be the central index of f.*

(i) *If $|a_0| \ne 0$, then*

$$\log \mu(r) = \log |a_0| + \int_0^r \frac{\nu_f(t)}{t}\,dt. \tag{3.2}$$

(ii) *For $r < R$, we have*

$$M(r,f) < \mu(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\}. \tag{3.3}$$

Lemma 3.4 *Let $f(z)$ be an entire function satisfying $\sigma_{[p,q]}(f,\varphi) = \sigma_2$ and $\mu_{[p,q]}(f,\varphi) = \mu_1$, and let $\nu_f(r)$ be the central index of f, then*

$$\varlimsup_{r\to\infty} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \sigma_2, \qquad \varliminf_{r\to\infty} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} = \mu_1.$$

Proof Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Without loss of generality, we can assume that $|a_0| \ne 0$. From (3.2), for any $1 < \alpha_1 < \alpha$, we have

$$\log \mu(\alpha_1 r) = \log |a_0| + \int_0^{\alpha_1 r} \frac{\nu_f(t)}{t}\,dt \ge \log |a_0| + \int_r^{\alpha_1 r} \frac{\nu_f(t)}{t}\,dt \ge \log |a_0| + \nu_f(r) \log \alpha_1.$$

By the Cauchy inequality, it is easy to see $\mu(\alpha_1 r) \le M(\alpha_1 r,f)$, hence

$$\nu_f(r) \log \alpha_1 \le \log M(\alpha_1 r,f) + c_3, \tag{3.4}$$

where $c_3 > 0$ is a constant. By Proposition 1.2, (3.4) and $\lim_{r\to\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ $(1 < \alpha_1 < \alpha)$, we have

$$\overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} \le \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(\alpha_1 r, f)}{\log_q \varphi(\alpha_1 r)} \cdot \overline{\lim_{r\to\infty}} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = \sigma_{[p,q]}(f,\varphi), \tag{3.5}$$

$$\underline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)} \le \underline{\lim_{r\to\infty}} \frac{\log_{p+1} M(\alpha_1 r, f)}{\log_q \varphi(\alpha_1 r)} \cdot \lim_{r\to\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = \mu_{[p,q]}(f,\varphi). \tag{3.6}$$

On the other hand, set $R = \alpha_1 r$, by (3.3), we have

$$M(r,f) < \mu(r)\left(\nu_f(\alpha_1 r) + \frac{\alpha_1}{\alpha_1 - 1}\right) = |a_{\nu_f(\alpha_1 r)}| r^{\nu_f(\alpha_1 r)}\left(\nu_f(\alpha_1 r) + \frac{\alpha_1}{\alpha_1 - 1}\right). \tag{3.7}$$

Since $\{|a_n|\}_{n=1}^{\infty}$ is a bounded sequence, by (3.7), we have

$$\log_{p+1} M(r,f) \le \log_p \nu_f(\alpha_1 r)\left[1 + \frac{\log_{p+1} \nu_f(\alpha_1 r)}{\log_p \nu_f(\alpha_1 r)}\right] + \log_{p+1} r + c_4, \tag{3.8}$$

where $c_4 > 0$ is a constant. By Proposition 1.2, (3.8), $\lim_{r\to\infty} \frac{\log_q \varphi(\alpha_1 r)}{\log_q \varphi(r)} = 1$ $(1 < \alpha_1 < \alpha)$ and $\lim_{r\to\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, we have

$$\sigma_{[p,q]}(f,\varphi) = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)} \le \overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(\alpha_1 r)}{\log_q \varphi(\alpha_1 r)} = \overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)}, \tag{3.9}$$

$$\mu_{[p,q]}(f,\varphi) = \underline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log_q \varphi(r)} \le \underline{\lim_{r\to\infty}} \frac{\log_p \nu_f(\alpha_1 r)}{\log_q \varphi(\alpha_1 r)} = \underline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log_q \varphi(r)}. \tag{3.10}$$

By (3.5), (3.6), (3.9) and (3.10), we obtain the conclusion of Lemma 3.4. □

Lemma 3.5 *Let $f_1(z)$ and $f_2(z)$ be entire functions of $[p,q] - \varphi$ order and denote $F = f_1 f_2$. Then*

$$\lambda_{[p,q]}(F,\varphi) = \max\{\lambda_{[p,q]}(f_1,\varphi), \lambda_{[p,q]}(f_2,\varphi)\}.$$

Proof Let $n(r,F)$, $n(r,f_1)$ and $n(r,f_2)$ be unintegrated counting functions for the number of zeros of $F(z), f_1(z)$ and $f_2(z)$. For any $r > 0$, it is easy to see

$$n(r,F) \ge \max\{n(r,f_1), n(r,f_2)\}. \tag{3.11}$$

By Definition 1.7 and (3.11), we have

$$\lambda_{[p,q]}(F,\varphi) \ge \max\{\lambda_{[p,q]}(f_1,\varphi), \lambda_{[p,q]}(f_2,\varphi)\}. \tag{3.12}$$

On the other hand, since the zeros of $F(z)$ must be the zeros of $f_1(z)$ or the zeros of $f_2(z)$, for any $r > 0$, we have

$$n(r,F) \le n(r,f_1) + n(r,f_2) \le 2\max\{n(r,f_1), n(r,f_2)\}. \tag{3.13}$$

By Definition 1.7 and (3.13), we have

$$\lambda_{[p,q]}(F,\varphi) \leq \max\bigl\{\lambda_{[p,q]}(f_1,\varphi),\lambda_{[p,q]}(f_2,\varphi)\bigr\}. \tag{3.14}$$

Therefore, by (3.12) and (3.14), we have $\lambda_{[p,q]}(F,\varphi) = \{\lambda_{[p,q]}(f_1,\varphi),\lambda_{[p,q]}(f_2,\varphi)\}$. □

Lemma 3.6 *Let $f(z)$ be a transcendental meromorphic function satisfying $\sigma_{[p,q]}(f,\varphi) = \sigma_3$, where $\varphi(r)$ only satisfies $\frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, and let k be any positive integer. Then, for any $\varepsilon > 0$, there exists a set E_3 having finite linear measure such that for all $r \notin E_3$, we have*

$$m\left(r,\frac{f^{(k)}}{f}\right) = O\bigl\{\exp_{p-1}\bigl\{(\sigma_3+\varepsilon)\log_q \varphi(r)\bigr\}\bigr\}.$$

Proof Set $k = 1$, since $\sigma_{[p,q]}(f,\varphi) = \sigma_3 < \infty$, for sufficiently large r and for any given $\varepsilon > 0$, we have

$$T(r,f) < \exp_p\bigl\{(\sigma_3+\varepsilon)\log_q \varphi(r)\bigr\}. \tag{3.15}$$

By the lemma of logarithmic derivative, we have

$$m\left(r,\frac{f'}{f}\right) = O\bigl\{\log T(r,f) + \log r\bigr\} \quad (r \notin E_3), \tag{3.16}$$

where $E_3 \subset [0,+\infty)$ is a set of finite linear measure, not necessarily the same at each occurrence. By (3.15), (3.16) and $\frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, we have $m(r,\frac{f'}{f}) = O\{\exp_{p-1}\{(\sigma+\varepsilon)\log_q \varphi(r)\}\}$ $(r \notin E_3)$.

We assume that $m(r,\frac{f^{(k)}}{f}) = O\{\exp_{p-1}\{(\sigma_3+\varepsilon)\log_q \varphi(r)\}\}$ $(r \notin E_3)$ holds for any positive integer k. By $N(r,f^{(k)}) \leq (k+1)N(r,f)$, for all $r \notin E_3$, we have

$$\begin{aligned} T\bigl(r,f^{(k)}\bigr) &= m\bigl(r,f^{(k)}\bigr) + N\bigl(r,f^{(k)}\bigr) \leq m\left(r,\frac{f^{(k)}}{f}\right) + m(r,f) + (k+1)N(r,f) \\ &\leq (k+1)T(r,f) + O\bigl\{\exp_{p-1}\bigl\{(\sigma_3+\varepsilon)\log_q \varphi(r)\bigr\}\bigr\}. \end{aligned} \tag{3.17}$$

By (3.16) and (3.17), for $r \notin E_3$, we have

$$m\left(r,\frac{f^{(k+1)}}{f}\right) \leq m\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right) + m\left(r,\frac{f^{(k)}}{f}\right) = O\bigl\{\exp_{p-1}\bigl\{(\sigma_3+\varepsilon)\log_q \varphi(r)\bigr\}\bigr\}.$$ □

Lemma 3.7 ([19]) *Let $f(z)$ be an entire function of $[p,q]$-order, and $f(z)$ can be represented by the form*

$$f(z) = U(z)e^{V(z)},$$

where $U(z)$ and $V(z)$ are entire functions such that

$$\lambda_{[p,q]}(f) = \lambda_{[p,q]}(U) = \sigma_{[p,q]}(U), \qquad \sigma_{[p,q]}(f) = \max\bigl\{\sigma_{[p,q]}(U),\sigma_{[p,q]}\bigl(e^V\bigr)\bigr\}.$$

If $f(z)$ is an entire function of $[p,q]-\varphi$ order, we have a similar result as follows.

Lemma 3.8 *Let $f(z)$ be an entire function of $[p,q]-\varphi$ order, and $f(z)$ can be represented by the form*

$$f(z)=U(z)e^{V(z)},$$

where $U(z)$ and $V(z)$ are entire functions of $[p,q]-\varphi$ order such that

$$\lambda_{[p,q]}(f,\varphi)=\lambda_{[p,q]}(U,\varphi)=\sigma_{[p,q]}(U,\varphi),$$
$$\sigma_{[p,q]}(f,\varphi)=\max\{\sigma_{[p,q]}(U,\varphi),\sigma_{[p,q]}(e^V,\varphi)\}.$$

4 Proofs of Theorems 2.1-2.4

Proof of Theorem 2.1 Set $\sigma_{[p,q]}(A,\varphi)=\sigma_4>0$. First, we prove that every solution of (1.1) satisfies $\sigma_{[p+1,q]}(f,\varphi)\le\sigma_4$. If $f(z)$ is a polynomial solution of (1.1), it is easy to know that $\sigma_{[p+1,q]}(f,\varphi)=0\le\sigma_4$ holds. If $f(z)$ is a transcendental solution of (1.1), by (1.1) and Lemma 3.1, there exists a set $E_1\subset(1,+\infty)$ having finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_1$ and $|f(z)|=M(r,f)$, we have

$$\left(\frac{\nu_f(r)}{r}\right)^2\bigl(1+o(1)\bigr)\le\exp_{p+1}\left\{\left(\sigma_4+\frac{\varepsilon}{2}\right)\log_q\varphi(r)\right\}.$$

And hence, we have

$$\nu_f(r)\le r\exp_{p+1}\{(\sigma_4+\varepsilon)\log_q\varphi(r)\}\quad(r\notin E_1).\tag{4.1}$$

By (4.1) and Lemma 3.2, there exists some α_1 $(1<\alpha_1<\alpha)$ such that for all $r\ge r_0$, we have

$$\nu_f(r)\le\alpha_1 r\exp_{p+1}\{(\sigma_4+\varepsilon)\log_q\varphi(\alpha_1 r)\}.\tag{4.2}$$

By Lemma 3.4, (4.2) and the two conditions on $\varphi(r)$, we have

$$\sigma_{[p+1,q]}(f,\varphi)=\varlimsup_{r\to\infty}\frac{\log_{p+1}\nu_f(r)}{\log_q\varphi(r)}\le\sigma_4.\tag{4.3}$$

On the other hand, by (1.1), we have

$$m(r,A)=m\left(r,-\frac{f''}{f}\right)=O\{\log rT(r,f)\}.\tag{4.4}$$

By (4.4), we have $\sigma_{[p,q]}(A,\varphi)\le\sigma_{[p+1,q]}(f,\varphi)$. Therefore, we have that $\sigma_{[p+1,q]}(f,\varphi)=\sigma_{[p,q]}(A,\varphi)$ holds for all non-trivial solutions of (1.1). □

Proof of Theorem 2.2 Set $\sigma_{[p,q]}(A,\varphi)=\sigma_5>0$, by Theorem 2.1, we have $\sigma_{[p+1,q]}(f_1,\varphi)=\sigma_{[p+1,q]}(f_2,\varphi)=\sigma_{[p,q]}(A,\varphi)=\sigma_5$. Hence, we have

$$\lambda_{[p+1,q]}(F,\varphi)\le\sigma_{[p+1,q]}(F,\varphi)\le\max\{\sigma_{[p+1,q]}(f_1,\varphi),\sigma_{[p+1,q]}(f_2,\varphi)\}=\sigma_{[p,q]}(A,\varphi).\tag{4.5}$$

By Lemma 3.5 and (4.5), we have

$$\max\{\lambda_{[p+1,q]}(f_1,\varphi),\lambda_{[p+1,q]}(f_2,\varphi)\} = \lambda_{[p+1,q]}(F,\varphi) \le \sigma_{[p+1,q]}(F,\varphi) \le \sigma_{[p,q]}(A,\varphi). \tag{4.6}$$

It remains to show that $\lambda_{[p+1,q]}(F,\varphi) = \sigma_{[p+1,q]}(F,\varphi)$. By (1.1), we have (see [13, pp.76-77]) that all zeros of $F(z)$ are simple and that

$$F^2 = C^2\left(\left(\frac{F'}{F}\right)^2 - 2\left(\frac{F''}{F}\right) - 4A\right)^{-1}, \tag{4.7}$$

where $C \neq 0$ is a constant. Hence,

$$\begin{aligned} 2T(r,F) &= T\left(r,\left(\frac{F'}{F}\right)^2 - 2\left(\frac{F''}{F}\right) - 4A\right) + O(1) \\ &\le O\left(\overline{N}\left(r,\frac{1}{F}\right) + m\left(r,\frac{F'}{F}\right) + m\left(r,\frac{F''}{F}\right) + m(r,A)\right). \end{aligned} \tag{4.8}$$

By Lemma 3.6, for all $r \notin E_3$, we have $m(r,A) = m(r,\frac{f''}{f}) = O\{\exp_p\{(\sigma_5+\varepsilon)\log_q\varphi(r)\}\}$, $m(r,\frac{F'}{F}) = O\{\exp_p\{(\sigma_5+\varepsilon)\log_q\varphi(r)\}\}$ and $m(r,\frac{F''}{F}) = O\{\exp_p\{(\sigma_5+\varepsilon)\log_q\varphi(r)\}\}$. By (4.8), for all $r \notin E_3$, we have

$$T(r,F) = O\left\{\overline{N}\left(r,\frac{1}{F}\right) + \exp_p\{(\sigma_5+\varepsilon)\log_q\varphi(r)\}\right\}. \tag{4.9}$$

Let us assume $\lambda_{[p+1,q]}(F,\varphi) < \beta < \sigma_{[p+1,q]}(F,\varphi)$. Since all zeros of $F(z)$ are simple, we have

$$\overline{N}\left(r,\frac{1}{F}\right) = N\left(r,\frac{1}{F}\right) = O\{\exp_{p+1}\{\beta\log_q\varphi(r)\}\}. \tag{4.10}$$

By (4.9) and (4.10), for all $r \notin E_3$, we have

$$T(r,F) = O\{\exp_{p+1}\{\beta\log_q\varphi(r)\}\}.$$

By Definition 1.6 and Lemma 3.2, we have $\sigma_{[p+1,q]}(F,\varphi) \le \beta < \sigma_{[p+1,q]}(F,\varphi)$, this is a contradiction. Therefore, the first assertion is proved.

If $\sigma_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A,\varphi)$, let us assume that $\lambda_{[p+1,q]}(f,\varphi) < \sigma_{[p,q]}(A,\varphi)$ holds for any solution of type $f = c_1f_1 + c_2f_2$ $(c_1c_2 \neq 0)$. We denote $F = f_1f_2$ and $F_1 = ff_1$, then we have $\lambda_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A,\varphi)$ and $\lambda_{[p+1,q]}(F_1,\varphi) < \sigma_{[p,q]}(A,\varphi)$. Since (4.9) holds for $F(z)$ and $F_1(z)$ and $F_1 = ff_1 = (c_1f_1 + c_2f_2)f_1 = c_1f_1^2 + c_2F$, we have

$$\begin{aligned} T(r,f_1) &= O\big(T(r,F_1) + T(r,F)\big) \\ &= O\left\{\overline{N}\left(r,\frac{1}{F_1}\right) + \overline{N}\left(r,\frac{1}{F}\right) + \exp_p\{(\sigma_5+\varepsilon)\log_q\varphi(r)\}\right\}. \end{aligned} \tag{4.11}$$

By $\lambda_{[p+1,q]}(F,\varphi) < \sigma_{[p,q]}(A,\varphi)$, $\lambda_{[p+1,q]}(F_1,\varphi) < \sigma_{[p,q]}(A,\varphi)$ and (4.10), for some $\beta < \sigma_{[p,q]}(A,\varphi)$, we have

$$T(r,f_1) = O\{\exp_{p+1}\{\beta\log_q\varphi(r)\}\}. \tag{4.12}$$

By Definition 1.6 and (4.12), we have $\sigma_{[p+1,q]}(f_1,\varphi) \leq \beta < \sigma_{[p,q]}(A,\varphi)$, this is a contradiction with Theorem 2.1. Therefore, we have that $\lambda_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A,\varphi)$ holds for all solutions of type $f = c_1f_1 + c_2f_2$, where $c_1c_2 \neq 0$. □

Proof of Theorem 2.3 By Theorem 2.1 and $\lambda_{[p+1,q]}(f,\varphi) \leq \sigma_{[p+1,q]}(f,\varphi)$, it is easy to know that $\lambda_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A,\varphi)$ holds. It remains to show that $\sigma_{[p,q]}(A,\varphi) \leq \lambda_{[p,q]}(f,\varphi)$. Let us assume $\sigma_{[p,q]}(A,\varphi) > \lambda_{[p,q]}(f,\varphi)$. By (1.1) and a similar proof of Theorem 5.6 in [13, p.82], we have

$$T\left(r,\frac{f}{f'}\right) = O\left\{\overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{A}\right)\right\} \quad (r \notin E_3). \tag{4.13}$$

By (4.13), the assumption $\sigma_{[p,q]}(A,\varphi) > \lambda_{[p,q]}(f,\varphi)$ and $\overline{\lambda}_{[p,q]}(A,\varphi) \leq \sigma_{[p,q]}(A,\varphi)$, for some $\beta < \sigma_{[p,q]}(A,\varphi)$, we have

$$T\left(r,\frac{f}{f'}\right) = O\left\{\exp_p\left\{\beta \log_q \varphi(r)\right\}\right\}. \tag{4.14}$$

By Definition 1.6 and (4.14), we have $\sigma_{[p,q]}(\frac{f}{f'},\varphi) = \sigma_{[p,q]}(\frac{f'}{f},\varphi) \leq \beta < \sigma_{[p,q]}(A,\varphi)$. By

$$-A(z) = \left(\frac{f'}{f}\right)' + \left(\frac{f'}{f}\right)^2,$$

we have $\sigma_{[p,q]}(A,\varphi) \leq \sigma_{[p,q]}(\frac{f'}{f},\varphi) < \sigma_{[p,q]}(A,\varphi)$, this is a contradiction. Therefore, we have that $\lambda_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A,\varphi) \leq \lambda_{[p,q]}(f,\varphi)$ holds for all non-trivial solutions of (1.1). □

Proof of Theorem 2.4 As a similar proof of Theorem 3.1 in [6], we denote $F = f_1f_2$ and $F_2 = g_1g_2$. Let us assume

$$\lambda_{[p,q]}(F_2,\varphi) = \max\left\{\lambda_{[p,q]}(g_1,\varphi),\lambda_{[p,q]}(g_2,\varphi)\right\} < \sigma_1.$$

By Theorem 2.1, we have $\sigma_{[p+1,q]}(F,\varphi) \leq \max\{\sigma_{[p+1,q]}(f_1,\varphi),\sigma_{[p+1,q]}(f_2,\varphi)\} = \sigma_1$, and hence, by Lemma 3.6, for any integer $k \geq 1$ and for any $\varepsilon > 0$, we have

$$m\left(r,\frac{F^{(k)}}{F}\right) = O\left\{\exp_p\left\{(\sigma_1+\varepsilon)\log_q \varphi(r)\right\}\right\} \quad (r \notin E_3).$$

Furthermore, by Theorem 2.1, we have $\lambda_{[p,q]}(F,\varphi) = \max\{\lambda_{[p,q]}(f_1,\varphi),\lambda_{[p,q]}(f_2,\varphi)\} < \sigma_1$, and hence we have $\overline{N}(r,\frac{1}{F}) = O\{\exp_p\{\beta\log_q \varphi(r)\}\}$ for some $\beta < \sigma_1$. And the $[p,q]-\varphi$ order of the function $A(z)$ implies that

$$T(r,A) = O\left\{\exp_p\left\{(\sigma_1+\varepsilon)\log_q \varphi(r)\right\}\right\} \quad (r \to \infty).$$

By (4.9), we obtain

$$T(r,F) = O\left\{\overline{N}\left(r,\frac{1}{F}\right) + \exp_p\left\{(\sigma_1+\varepsilon)\log_q \varphi(r)\right\}\right\} = O\left\{\exp_p\left\{(\beta\log_q \varphi(r)\right\}\right\}. \tag{4.15}$$

By Definition 1.6 and (4.15), we have $\sigma_{[p,q]}(F,\varphi) \leq \sigma_1$. On the other hand, by

$$4A = \left(\frac{F'}{F}\right)^2 - 2\frac{F''}{F} - \frac{1}{F^2}, \tag{4.16}$$

we have $\sigma_{[p,q]}(A,\varphi) = \sigma_1 \leq \sigma_{[p,q]}(F,\varphi)$, hence $\sigma_{[p,q]}(F,\varphi) = \sigma_1$. The same reasoning is valid for the function F_2, we have

$$4(A + \Pi) = \left(\frac{F_2'}{F_2}\right)^2 - 2\frac{F_2''}{F_2} - \frac{1}{F_2^2}, \tag{4.17}$$

and $\sigma_{[p,q]}(F_2,\varphi) = \sigma_1$. Since $\lambda_{[p,q]}(F,\varphi) < \sigma_1$ and $\lambda_{[p,q]}(F_2,\varphi) < \sigma_1$, by Lemma 3.8, we may write

$$F = Qe^P, \qquad F_2 = Re^S, \tag{4.18}$$

where P, Q, R, S are entire functions satisfying $\sigma_{[p,q]}(Q,\varphi) = \lambda_{[p,q]}(F,\varphi) < \sigma_1$, $\sigma_{[p,q]}(R,\varphi) = \lambda_{[p,q]}(F_2,\varphi) < \sigma_1$ and $\sigma_{[p,q]}(e^P,\varphi) = \sigma_{[p,q]}(e^S,\varphi) = \sigma_1$. Substituting (4.18) into (4.16) and (4.17), we have

$$4A = -\frac{1}{Q^2 e^{2P}} + G_1(z), \tag{4.19}$$

$$4(A + \pi) = -\frac{1}{R^2 e^{2S}} + G_2(z), \tag{4.20}$$

where $G_1(z)$ and $G_2(z)$ are meromorphic functions satisfying $\sigma_{[p,q]}(G_j,\varphi) < \sigma_1$ ($j = 1,2$). Equation (4.19) subtracting (4.20), we have

$$\frac{1}{R^2 e^{2S}} - \frac{1}{Q^2 e^{2P}} = G_3(z), \tag{4.21}$$

where $G_3(z)$ is a meromorphic function satisfying $\sigma_{[p,q]}(G_3,\varphi) < \sigma_1$. From (4.21), we have

$$e^{-2S} + H_1 e^{-2P} = H_2, \tag{4.22}$$

where $H_1(z)$ and $H_2(z)$ are meromorphic functions satisfying $\sigma_{[p,q]}(H_j,\varphi) < \sigma_1$ ($j = 1,2$), and $H_1 = -\frac{R^2}{Q^2}$. Deriving (4.22), we have

$$-2S'e^{-2S} + \left(H_1' - 2P'H_1\right)e^{-2P} = H_3, \tag{4.23}$$

where $H_3(z)$ is a meromorphic function satisfying $\sigma_{[p,q]}(H_3,\varphi) < \sigma_1$. Eliminating e^{-2S} by (4.22) and (4.23), we have

$$\left(H_1' - 2\left(P' - S'\right)H_1\right)e^{-2P} = H_4, \tag{4.24}$$

where $H_4(z)$ is a meromorphic function satisfying $\sigma_{[p,q]}(H_4,\varphi) < \sigma_1$. Since $\sigma_{[p,q]}(e^P,\varphi) = \sigma_1$, therefore by (4.24), we have $H_1' - 2(P' - S')H_1 \equiv 0$, thus we have $H_1 = ce^{2(P-S)}$, $c \neq 0$. Hence

$$\frac{F^2}{F_2^2} = \frac{Q^2}{R^2}e^{2(P-S)} = -\frac{1}{c}. \tag{4.25}$$

From (4.16), (4.17) and (4.25), we have

$$4\left(A+\Pi+\frac{1}{c}A\right)=\left(\frac{F_2'}{F_2}\right)^2-2\frac{F_2''}{F_2}+\frac{1}{c}\left(\frac{F'}{F}\right)^2-\frac{2}{c}\frac{F''}{F}.$$

By Lemma 3.6, we obtain

$$\begin{aligned}T\left(r,\left(1+\frac{1}{c}\right)A+\Pi\right)&=m\left(r,\left(1+\frac{1}{c}\right)A+\Pi\right)\\&=O\bigl\{\exp_{p-1}\bigl\{(\sigma_1+\varepsilon)\log_q\varphi(r)\bigr\}\bigr\}\quad(r\to\infty).\end{aligned}$$

This implies

$$\sigma_{[p,q]}\left(\left(1+\frac{1}{c}\right)A+\Pi,\varphi\right)=0.$$

Hence, by Proposition 1.1, we have $c=-1$. Since $F^2=F_2^2$, we have

$$\frac{F'}{F}=\frac{F_2'}{F_2},\qquad\frac{F''}{F}=\frac{F_2''}{F_2}.$$

From (4.13) and (4.17), we have $\Pi\equiv 0$, this is a contradiction. Therefore, we obtain the conclusion of Theorem 2.4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XS, JT and HYX completed the main part of this article, JT and HYX corrected the main theorems. All authors read and approved the final manuscript.

Author details
[1]College of Science, Jiujiang University, JiuJiang, 332005, China. [2]College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, 330022, China. [3]Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, 333403, China.

Acknowledgements
The authors thank the referee for his/her valuable suggestions to improve the present article. This work was supported by the NSFC (11301233), the Natural Science Foundation of Jiang-Xi Province in China (20132BAB211001, 20132BAB211002), and the Foundation of Education Department of Jiangxi (GJJ14272, GJJ14644) of China.

Received: 22 February 2014 Accepted: 22 May 2014 Published: 23 Jul 2014

References

1. Hayman, W: Meromorphic Functions. Clarendon, Oxford (1964)
2. Yang, L: Vaule Distribution Theory and Its New Research. Science Press, Beijing (1988) (in Chinese)
3. Yang, CC, Yi, HX: The Uniqueness Theory of Meromorphic Function. Mathematics and Its Applications, vol. 557. Kluwer Academic, Dordrecht (2003)
4. Bank, S, Laine, I: On the oscillation theory of $f''+Af=0$ where A is entire. Trans. Am. Math. Soc. **273**, 352-363 (1982)
5. Bank, S, Laine, I: On the zeros of meromorphic solutions of second-order linear differential equations. Comment. Math. Helv. **58**, 656-677 (1983)
6. Bank, S, Laine, I, Langley, J: Oscillation results for solutions of linear differential equations in the complex domain. Results Math. **16**, 3-15 (1989)
7. Gundersen, GG: Finite order solutions of second order linear differential equations. Trans. Am. Math. Soc. **305**, 415-429 (1988)
8. Juneja, OP, Kapoor, GP, Bajpai, SK: On the (p,q)-order and lower (p,q)-order of an entire function. J. Reine Angew. Math. **282**, 53-67 (1976)
9. Juneja, OP, Kapoor, GP, Bajpai, SK: On the (p,q)-type and lower (p,q)-type of an entire function. J. Reine Angew. Math. **290**, 180-190 (1977)

10. Li, LM, Cao, TB: Solutions for linear differential equations with meromorphic coefficients of (p, q)-order in the plane. Electron. J. Differ. Equ. **2012**, 1-15 (2012)
11. Liu, J, Tu, J, Shi, LZ: Linear differential equations with entire coefficients of $[p, q]$-order in the complex plane. J. Math. Anal. Appl. **372**, 55-67 (2010)
12. Schönhage, A: Über das wachstum zusammengesetzter funktionen. Math. Z. **73**, 22-44 (1960)
13. Kinnunen, L: Linear differential equations with solutions of finite iterated order. Southeast Asian Bull. Math. **22**(4), 385-405 (1998)
14. Cao, TB, Li, LM: Oscillation results on meromorphic solutions of second order differential equations in the complex plane. Electron. J. Qual. Theory Differ. Equ. **68**, 1-13 (2010)
15. Xu, HY, Tu, J: Oscillation of meromorphic solutions to linear differential equations with coefficients of $[p, q]$-order. Electron. J. Differ. Equ. **2014**, 73 (2014)
16. Chyzhykov, I, Heittokangas, J, Rättyä, J: Finiteness of φ-order of solutions of linear differential equations in the unit disc. J. Anal. Math. **109**, 163-198 (2009)
17. Hayman, W: The local growth of power series: a survey of the Wiman-Valiron method. Can. Math. Bull. **17**, 317-358 (1974)
18. He, YZ, Xiao, XZ: Algebroid Functions and Ordinary Differential Equations. Science Press, Beijing (1988) (in Chinese)
19. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
20. Gao, SA, Chen, ZX, Chen, TW: The Complex Oscillation Theory of Linear Differential Equations. Huazhong Univ. Sci. Tech. Press, Wuhan (1998) (in Chinese)
21. Jank, G, Volkmann, L: Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen. Birkhäuser, Basel (1985)

10.1186/1687-1847-2014-200
Cite this article as: Shen et al.: **Complex oscillation of a second-order linear differential equation with entire coefficients of** $[p, q] - \varphi$ **order.** *Advances in Difference Equations* 2014, 2014:200