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Solvability of boundary value problems for fractional order elastic beam equations

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Abstract
In this article, the existence results for solutions of a boundary value problem for nonlinear singular fractional order elastic beam equations are established. The analysis relies on the well-known Schauder's fixed point theorem.
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Keywords: solution; singular fractional differential equation; fixed point theorem; elastic beam equation

1 Introduction

Boundary value problems for fractional differential equations have been discussed by many authors; see the textbooks [1, 2], papers [3–21] and the references therein.

Fourth-order two-point boundary value problems are useful for material mechanics because the problems usually characterize the deflection of an elastic beam. The following problem

$$\begin{cases} u''''(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0 \end{cases} \tag{1}$$

describes the deflection of an elastic beam with both ends rigidly fixed. The existence of positive solutions were studied extensively; see [13–16].

In [12], the authors studied the existence of positive solutions of the following boundary value problem for the fractional order beam equation:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases} \tag{2}$$

where $3 < \alpha \le 4$, $D_{0^+}^{\alpha}$ (D^α for short) is the Riemann-Liouville fractional derivative of order α, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous or $f : [0,1] \times (0,\infty) \to [0,\infty)$ is continuous and f is singular at $x = 0$. We note that f in (1) depends on x, $t \to f(t,x)$ is continuous and the solutions obtained in [11] satisfy that both x and x' are continuous on $[0,1]$ (hence they are bounded on $[0,1]$).

Motivated by [12] and the above-mentioned example, in this paper we discuss the boundary value problem for nonlinear singular fractional order elastic beam equation of the form

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, u(t)), \quad t \in (0,1), \\ \lim_{t\to 0} t^{4-\alpha} u(t) = a, \\ \lim_{t\to 0} D_{0^+}^{\alpha-3} u(t) = b, \\ u(1) = D_{0^+}^{\alpha-3} u(1) = 0, \end{cases} \tag{3}$$

where $3 < \alpha < 4$, $D_{0^+}^{\alpha}$ (D^{α} for short) is the Riemann-Liouville fractional derivative of order α, $a, b \in R$ and $f : (0,1) \times R \to R$ is continuous. f may be singular at $t = 0$ and $t = 1$.

The purpose of this paper is to establish some existence results for solutions of BVP (3) by using Schauder's fixed point theorem. The solutions obtained in this paper may be unbounded since $\lim_{t\to 0} t^{4-\alpha} u(t) = a$. The methods used in this paper are different from the ones used in [13, contraction mapping and iterative techniques], [14, Guo-Krasnosel'skii fixed point theorem], [15, upper and lower solution methods], [16, topological degree theory in Banach space], [17, Lie symmetry group methods], [18, the contraction mapping principle and Krasnoselskii's fixed point theorem] and [19, study the asymptotic behavior of solutions], Schauder's fixed point theorem [22–24].

A function $u : (0,1] \to R$ is called a solution of BVP (3) if $u \in C^0(0,1]$ and all equations in (3) are satisfied. The remainder of the paper is divided into three sections. In Section 2, we present some preliminary results. In Section 3, we establish sufficient conditions for solvability of BVP (3). An example is given to illustrate the main result at the end of the paper.

2 Preliminary results

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and results can be found in the literature [1, 2]. Denote the gamma function and the beta function, respectively, by

$$\Gamma(\sigma_1) = \int_0^{+\infty} s^{\sigma_1 - 1} e^{-s}\, ds, \qquad \mathbf{B}(\sigma_2, \sigma_3) = \int_0^1 (1-x)^{\sigma_2 - 1} x^{\sigma_3 - 1}\, dx, \quad \sigma_i > 0, i = 1, 2, 3.$$

Definition 2.1 [1] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided that the right-hand side exists.

Definition 2.2 [1] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\, ds,$$

where $n - 1 \le \alpha < n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.1 [1] *Let* $n - 1 < \alpha \le n$, $u \in C^0(0,1) \cap L^1(0,1)$. *Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots, n$.

For our construction, we choose

$$X = \left\{ x \in C^0(0,1] : \textit{there exists the limit } \lim_{t \to 0} t^{4-\alpha} x(t) \right\}$$

with the norm

$$\|u\| = \sup_{t \in (0,1]} t^{4-\alpha} \left| u(t) \right|$$

for $u \in X$. It is easy to show that X is a real Banach space.

Lemma 2.2 *Suppose that $h \in C^0(0,1)$ and there exist $k > -1$ and $\sigma \in (3-\alpha, 0]$ such that $|h(t)| \le t^k (1-t)^\sigma$ for all $t \in (0,1)$. Then $x \in X$ is a solution of the problem*

$$\begin{cases} D^\alpha x(t) = h(t), \quad 0 < t < 1, \\ \lim_{t \to 0} t^{4-\alpha} x(t) = \lim_{t \to 0} D_{0^+}^{\alpha-3} x(t) = 0, \\ x(1) = D_{0^+}^{\alpha-3} x(1) = 0, \end{cases} \tag{4}$$

if and only if $x \in X$ satisfies

$$\begin{aligned} x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \\ & + \frac{1}{2\Gamma(\alpha-1) - \Gamma(\alpha)} \left(-\frac{2}{\alpha-1} t^{\alpha-1} + t^{\alpha-2} \right) \int_0^1 (1-s)^{\alpha-1} h(s)\, ds \\ & + \frac{1}{2\Gamma(\alpha-1) - \Gamma(\alpha)} \left(t^{\alpha-1} - t^{\alpha-2} \right) \int_0^1 (1-s)^2 h(s)\, ds \\ & + \frac{1}{2\Gamma(\alpha-1) - \Gamma(\alpha)} \left(-2(\alpha-3) t^{\alpha-1} + \alpha(\alpha-3) t^{\alpha-2} - (\alpha-2)(\alpha-3) t^{\alpha-3} \right) b \\ & + \frac{1}{2\Gamma(\alpha-1) - \Gamma(\alpha)} \left(-2\Gamma(\alpha-1) t^{\alpha-1} + \Gamma(\alpha) t^{\alpha-2} + \left(2\Gamma(\alpha-1) - \Gamma(\alpha) \right) t^{\alpha-4} \right) a. \end{aligned} \tag{5}$$

Proof Since $h \in C^0(0,1)$ and there exist $k > -1$ and $\sigma \in (3-\alpha, 0]$ such that $|h(t)| \le t^k (1-t)^\sigma$ for all $t \in (0,1)$, then

$$\begin{aligned} \left| \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\, ds \right| & \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^\sigma\, ds \le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (t-s)^\sigma\, ds \\ & \le \int_0^t \frac{(t-s)^{\alpha+\sigma-1}}{\Gamma(\alpha)} s^k\, ds = t^{\alpha+\sigma+k} \int_0^1 \frac{(1-w)^{\alpha+\sigma-1}}{\Gamma(\alpha)} w^k\, dw \\ & = t^{\alpha+\sigma+k} \frac{\mathbf{B}(\alpha+\sigma, k+1)}{\Gamma(\alpha)}. \end{aligned}$$

Similarly, we get

$$\left| \int_0^1 (1-s)^2 h(s)\, ds \right| \le \mathbf{B}(3+\sigma, k+1).$$

So, for $t \in (0,1]$, $D^\alpha u(t) = h(t)$ together with Lemma 2.1 implies that there exist constants c_i $(i = 1,2,3,4)$ such that

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + c_4 t^{\alpha-4}$$

with

$$D_{0^+}^{\alpha-3} x(t) = \int_0^t \frac{(t-s)^2}{2} h(s)\,ds + c_1 \frac{\Gamma(\alpha)}{2} t^2 + c_2 \Gamma(\alpha-1) t + c_3 \Gamma(\alpha-2).$$

Now, $\lim_{t\to 0} t^{4-\alpha} x(t) = a$ implies that $c_4 = a$.
$\lim_{t\to 0} D_{0^+}^{\alpha-3} x(t) = b$ implies $c_3 = \frac{b}{\Gamma(\alpha-2)}$.
$u(1) = D_{0^+}^{\alpha-3}(1) = 0$ implies that

$$\begin{aligned} c_1 + c_2 &= -\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds - \frac{b}{\Gamma(\alpha-2)} - a, \\ c_1 \frac{\Gamma(\alpha)}{2} + c_2 \Gamma(\alpha-1) &= -\int_0^1 \frac{(1-s)^2}{2} h(s)\,ds - b. \end{aligned}$$

It follows that

$$\begin{aligned} c_1 = {}& \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^2 h(s)\,ds - \frac{2}{\alpha-1} \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\ & - \frac{2(\alpha-3)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} b - \frac{2\Gamma(\alpha-1)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} a, \\ c_2 = {}& -\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^2 h(s)\,ds + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\ & + \frac{\alpha(\alpha-3)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} b + \frac{\Gamma(\alpha)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} a. \end{aligned}$$

Then

$$\begin{aligned} x(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\ & + t^{\alpha-1} \biggl(\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^2 h(s)\,ds \\ & - \frac{2}{\alpha-1} \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\ & - \frac{2(\alpha-3)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} b - \frac{2\Gamma(\alpha-1)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} a \biggr) \\ & + t^{\alpha-2} \biggl(-\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^2 h(s)\,ds \\ & + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} h(s)\,ds \\ & + \frac{\alpha(\alpha-3)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} b + \frac{\Gamma(\alpha)}{2\Gamma(\alpha-1)-\Gamma(\alpha)} a \biggr) + t^{\alpha-3} \frac{b}{\Gamma(\alpha-2)} + t^{\alpha-4} a \end{aligned}$$

$$\begin{aligned}
&= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-\frac{2}{\alpha-1}t^{\alpha-1}+t^{\alpha-2}\right)\int_0^1 (1-s)^{\alpha-1}h(s)\,ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(t^{\alpha-1}-t^{\alpha-2}\right)\int_0^1 (1-s)^2 h(s)\,ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2(\alpha-3)t^{\alpha-1}+\alpha(\alpha-3)t^{\alpha-2}-(\alpha-2)(\alpha-3)t^{\alpha-3}\right)b \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2\Gamma(\alpha-1)t^{\alpha-1}+\Gamma(\alpha)t^{\alpha-2}+\left(2\Gamma(\alpha-1)-\Gamma(\alpha)\right)t^{\alpha-4}\right)a.
\end{aligned}$$

It is easy to see that $x \in C^0(0,1]$. Furthermore, we have

$$t^{4-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,ds\right| \le t^{4+\sigma+k}\frac{\mathbf{B}(\alpha+\sigma,k+1)}{\Gamma(\alpha)} \to 0 \quad \text{as } t \to 0.$$

Then the following limit exists

$$\lim_{t\to 0} t^{4-\alpha}x(t).$$

Hence $x \in X$ and x satisfies (5).

On the other hand, if $x \in X$ satisfies (5), we can show that x is a solution of problem (4). The proof is completed. □

Define the operator T on X, for $x \in X$, denote $f_x(t) = f(t, x(t))$, by

$$\begin{aligned}
(Tx)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f\left(s,x(s)\right)ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-\frac{2}{\alpha-1}t^{\alpha-1}+t^{\alpha-2}\right)\int_0^1 (1-s)^{\alpha-1}f\left(s,x(s)\right)ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(t^{\alpha-1}-t^{\alpha-2}\right)\int_0^1 (1-s)^2 f\left(s,x(s)\right)ds \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2(\alpha-3)t^{\alpha-1}+\alpha(\alpha-3)t^{\alpha-2}-(\alpha-2)(\alpha-3)t^{\alpha-3}\right)b \\
&\quad + \frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2\Gamma(\alpha-1)t^{\alpha-1}+\Gamma(\alpha)t^{\alpha-2}+\left(2\Gamma(\alpha-1)-\Gamma(\alpha)\right)t^{\alpha-4}\right)a.
\end{aligned}$$

By Lemma 2.2, we have that $x \in X$ is a solution of BVP (3) if and only if $x \in X$ is a fixed point of T.

Lemma 2.3 *Suppose that*

(B0) *$f(t,x)$ is continuous on $(0,1) \times R$ and satisfies that for each $r > 0$ there exist $k > -1$, $\sigma \in (3-\alpha, 0]$ and $M_r > 0$ such that*

$$\left|f\left(t, t^{\alpha-4}x\right)\right| \le M_r t^k (1-t)^\sigma$$

holds for all $t \in (0,1)$, $|x| \le r$.

Then $T : X \to X$ is completely continuous.

Proof We divide the proof into four steps.

Step 1. We prove that $T : X \to X$ is well defined.

For $x \in X$, there exists $r > 0$ such that

$$\sup_{t\in(0,1]} t^{4-\alpha}\left|x(t)\right| < r.$$

Then there exist $k > -1$, $\sigma \in (3-\alpha, 0]$, $M_r \geq 0$ such that

$$\left|f\big(t, x(t)\big)\right| = \left|f\big(t, t^{\alpha-4}t^{4-\alpha}x(t)\big)\right| \leq M_r t^k (1-t)^\sigma \tag{6}$$

for all $t \in (0,1)$. Similarly to the proof of Lemma 2.2, we can show that $Tx \in X$. So $T : X \to X$ is well defined.

Step 2. T is continuous.

Let $\{x_n \in X\}$ be a sequence such that $x_n \to x_0$ as $n \to \infty$ in X. Then there exists $r > 0$ such that

$$\sup_{t\in(0,1]} t^{4-\alpha}\left|x_n(t)\right| \leq r$$

holds for $n = 0, 1, 2, \ldots$. Then there exist $M_r > 0$, $k > -1$ and $\sigma \in (3-\alpha, 0)$ such that

$$\left|f\big(t, x_n(t)\big)\right| = \left|f\big(t, t^{\alpha-4}t^{4-\alpha}x_n(t)\big)\right| \leq M_r t^k (1-t)^\sigma$$

holds for all $t \in (0,1)$, $n = 0, 1, 2, \ldots$. Then

$$\begin{aligned}
&t^{4-\alpha}\left|(Tx_n)(t) - (Tx_0)(t)\right| \\
&\quad\leq t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f_{x_n}(s) - f_{x_0}(s)\right| ds \\
&\qquad + \frac{1}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|}\left|-\frac{2}{\alpha-1}t^3 + t^2\right|\int_0^1 (1-s)^{\alpha-1}\left|f_{x_n}(s) - f_{x_0}(s)\right| ds \\
&\qquad + \frac{1}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|}\left|t^3 - t^2\right|\int_0^1 (1-s)^2\left|f_{x_n}(s) - f_{x_0}(s)\right| ds \\
&\quad\leq 2M_r t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (1-s)^\sigma\, ds \\
&\qquad + \frac{2M_r}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|}\left(\frac{2}{\alpha-1} + 1\right)\int_0^1 (1-s)^{\alpha-1} s^k (1-s)^\sigma\, ds \\
&\qquad + \frac{4M_r}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|}\int_0^1 (1-s)^2 s^k (1-s)^\sigma\, ds \\
&\quad\leq 2M_r t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} s^k (t-s)^\sigma\, ds \\
&\qquad + \frac{2M_r\mathbf{B}(\alpha+\sigma, k+1)}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|}\left(\frac{2}{\alpha-1} + 1\right) + \frac{4M_r\mathbf{B}(3+\sigma, k+1)}{|2\Gamma(\alpha-1) - \Gamma(\alpha)|} \\
&\quad\leq 2M_r t^{4+\sigma+k}\int_0^1 \frac{(1-w)^{\alpha+\sigma-1}}{\Gamma(\alpha)} w^k\, dw
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2M_r\mathbf{B}(\alpha+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}\left(\frac{2}{\alpha-1}+1\right)+\frac{4M_r\mathbf{B}(3+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}\\
&\le 2M_r\frac{\mathbf{B}(\alpha+\sigma,k+1)}{\Gamma(\alpha)}+\frac{2M_r\mathbf{B}(\alpha+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}\left(\frac{2}{\alpha-1}+1\right)+\frac{4M_r\mathbf{B}(3+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}.
\end{aligned}
$$

By the dominant convergence theorem, we have $\|Tx_n - Tx_0\| \to 0$ as $n \to \infty$. Then T is continuous.

Let $\Omega \subset X$ be a bounded subset. Then there exists $r > 0$ such that

$$
\sup_{t\in(0,1]} t^{4-\alpha}\left|x(t)\right| \le r, \quad x \in \Omega.
$$

Then there exist $M_r > 0$, $k > -1$ and $\sigma \in (3-\alpha, 0)$ such that

$$
\left|f\left(t,x(t)\right)\right| = \left|f\left(t,t^{\alpha-4}t^{4-\alpha}x(t)\right)\right| \le M_r t^k(1-t)^\sigma
$$

holds for all $t \in (0,1)$, $x \in \Omega$.

Step 3. Prove that $T\Omega$ is a bounded set in X.

Similarly to Step 2, we can show that

$$
\begin{aligned}
t^{4-\alpha}\left|(Tx)(t)\right| \le{}& M_r\frac{\mathbf{B}(\alpha+\sigma,k+1)}{\Gamma(\alpha)}+\frac{M_r\mathbf{B}(\alpha+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}\left(\frac{2}{\alpha-1}+1\right)\\
&+\frac{2M_r\mathbf{B}(3+\sigma,k+1)}{|2\Gamma(\alpha-1)-\Gamma(\alpha)|}.
\end{aligned}
$$

So T maps bounded sets into bounded sets in X.

Step 4. Prove that $T\Omega$ is a relatively compact set in X.

We can prove easily that $\{t^{4-\alpha}(Tu)(t) : u \in \Omega\}$ is equicontinuous on $(0,1]$. Therefore, $T\Omega$ is relatively compact.

From the above discussion, T is completely continuous. The proof is complete. □

3 Main results

In this section, we prove the main results.

Theorem 3.1 *Suppose that*

(B1) $\phi \in C^0(0,1)$ *satisfies that there exist* $k_0 > -1$, $\sigma_0 \in (3-\alpha, 0]$ *and* $M_0 > 0$ *such that* $|\phi(t)| \le M_0 t^{k_0}(1-t)^{\sigma_0}$ *for all* $t \in (0,1)$;

(B2) $f : (0,1) \times R \to R$ *is continuous and there exist numbers* $k_1 > -1$, $\sigma_1 \in (3-\alpha, 0]$, $\mu \ge 0$, $A \ge 0$ *such that*

$$
\left|f\left(t,t^{\alpha-4}x\right)-\phi(t)\right| \le At^{k_1}(1-t)^{\sigma_1}|x|^\mu
$$

holds for all $t \in (0,1)$, $x \in R$. *Let*

$$
\begin{aligned}
\Phi(t) ={}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\,ds\\
&+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-\frac{2}{\alpha-1}t^{\alpha-1}+t^{\alpha-2}\right)\int_0^1(1-s)^{\alpha-1}\phi(s)\,ds
\end{aligned}
$$

$$+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(t^{\alpha-1}-t^{\alpha-2}\right)\int_0^1(1-s)^2\phi(s)\,ds$$
$$+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2(\alpha-3)t^{\alpha-1}+\alpha(\alpha-3)t^{\alpha-2}-(\alpha-2)(\alpha-3)t^{\alpha-3}\right)b$$
$$+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-2\Gamma(\alpha-1)t^{\alpha-1}+\Gamma(\alpha)t^{\alpha-2}\right.$$
$$\left.+\left(2\Gamma(\alpha-1)-\Gamma(\alpha)\right)t^{\alpha-4}\right)a$$

and

$$P=\frac{\mathbf{B}(\alpha+\sigma_1+k_1,k_1+1)}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)-2\Gamma(\alpha-1)}\frac{\alpha+1}{\alpha-1}\mathbf{B}(\alpha+\sigma_1,k_1+1)$$
$$+\frac{2}{\Gamma(\alpha)-2\Gamma(\alpha-1)}\mathbf{B}(3+\sigma_1,k_1+1).$$

Then BVP (3) *has at least one solution if*

(i) $\mu<1$ *or*
(ii) $\mu=1$ *with* $AP<1$ *or*
(iii) $\mu>1$ *with*

$$\frac{\|\Phi\|(\mu-1)^{\mu-1}}{(\|\Phi\|\mu)^{\mu}}\geq AP.$$

Proof It is easy to show that (B1) and (B2) imply (B0). Let the Banach space X and the operator T defined on X be defined in Section 2. By Lemma 2.3, $T:X\to X$ is well defined, completely continuous, $x\in X$ is a positive solution if and only if $x\in X$ is a fixed point of T. It is easy to see that $\Phi\in X$.

For $r>0$, denote $\Omega_r=\{x\in X:\|x-\Phi\|\leq r\}$. One sees that

$$\|x\|=\sup_{t\in(0,1]}t^{4-\alpha}\left|x(t)\right|\leq\|x-\Phi\|+\|\Phi\|\leq r+\|\Phi\|,\quad x\in\Omega_r.$$

Hence for $x\in\Omega_r$, we have

$$\begin{aligned}\left|f\left(t,x(t)\right)-\phi(t)\right|&=\left|f\left(t,t^{\alpha-4}t^{4-\alpha}x(t)\right)-\phi(t)\right|\\&\leq At^{k_1}(1-t)^{\sigma_1}\left|t^{4-\alpha}x(t)\right|^{\mu}\\&\leq At^{k_1}(1-t)^{\sigma_1}\left[r+\|\Phi\|\right]^{\mu}.\end{aligned}$$

We have

$$\begin{aligned}t^{4-\alpha}\left|(Tx)(t)-\Phi(t)\right|\leq{}&t^{4-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|f\left(s,x(s)\right)-\phi(s)\right|ds\\&+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(-\frac{2}{\alpha-1}t^3+t^2\right)\\&\times\int_0^1(1-s)^{\alpha-1}\left|f\left(s,x(s)\right)-\phi(s)\right|ds\\&+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\left(t^3-t^2\right)\int_0^1(1-s)^2\left|f\left(s,x(s)\right)-\phi(s)\right|ds\end{aligned}$$

$$\begin{aligned}
&\leq t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(s,x(s)\big)-\phi(s)\big|\,ds \\
&\quad+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\frac{\alpha+1}{\alpha-1}\int_0^1 (1-s)^{\alpha-1}\big|f\big(s,x(s)\big)-\phi(s)\big|\,ds \\
&\quad+\frac{2}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\int_0^1 (1-s)^2\big|f\big(s,x(s)\big)-\phi(s)\big|\,ds \\
&\leq t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}As^{k_1}(1-s)^{\sigma_1}\big[r+\|\Phi\|\big]^{\mu}\,ds \\
&\quad+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\frac{\alpha+1}{\alpha-1}\int_0^1 (1-s)^{\alpha-1}As^{k_1}(1-s)^{\sigma_1}\big[r+\|\Phi\|\big]^{\mu}\,ds \\
&\quad+\frac{2}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\int_0^1 (1-s)^2As^{k_1}(1-s)^{\sigma_1}\big[r+\|\Phi\|\big]^{\mu}\,ds \\
&\leq A\big[r+\|\Phi\|\big]^{\mu}\bigg[t^{4-\alpha}\int_0^t \frac{(t-s)^{\alpha+\sigma_1-1}}{\Gamma(\alpha)}s^{k_1}\,ds \\
&\quad+\frac{1}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\frac{\alpha+1}{\alpha-1}\int_0^1 (1-s)^{\alpha+\sigma_1-1}s^{k_1}\,ds \\
&\quad+\frac{2}{2\Gamma(\alpha-1)-\Gamma(\alpha)}\int_0^1 (1-s)^{2+\sigma_1}s^{k_1}\,ds\bigg] \\
&= AP\big[r+\|\Phi\|\big]^{\mu}.
\end{aligned}$$

It follows that

$$\|Tx-\Phi\| \leq AP\big[r+\|\Phi\|\big]^{\mu}.$$

Case 1. $\mu < 1$.

Since there exists $r_0 > 0$ sufficiently large such that

$$AP\big[r_0+\|\Phi\|\big]^{\mu} < r_0.$$

Choose $\Omega_{r_0} = \{x \in X : \|x-\Phi\| \leq r_0\}$. From the above discussion, we have

$$\|Tx-\Phi\| \leq Ap\big[r_0+\|\Phi\|\big]^{\mu} \leq r_0.$$

Then $Tx \in \Omega_{r_0}$. By Schauder's fixed point theorem, T has at least one fixed point $x \in \Omega_{r_0}$. Then x is a solution of BVP (3).

Case 2. $\mu = 1$.

Choose

$$r_0 > \frac{AP\|\Phi\|}{1-AP}.$$

Let $\Omega_{r_0} = \{x \in X : \|x-\Phi\| \leq r_0\}$. From the above discussion, we have

$$\|Tx-\Phi\| \leq AP\big[r_0+\|\Phi\|\big] \leq r_0.$$

Then $Tx \in \Omega_{r_0}$. By Schauder's fixed point theorem, T has at least one fixed point $x \in \Omega$. Then x is a positive solution of BVP (3).

Case 3. $\mu > 1$.

Choose $r_0 = \frac{\|\Phi\|}{\mu-1}$. Let $\Omega_{r_0} = \{x \in X : \|x - \Phi\| \leq r_0\}$. From the above discussion, we have

$$\begin{aligned}
\|Tx - \Phi\| &\leq AP\bigl[r_0 + \|\Phi\|\bigr]^\mu \\
&= AP\left[\frac{\|\Phi\|}{\mu-1} + \|\Phi\|\right]^\mu \\
&\leq \frac{\|\Phi\|(\mu-1)^{\mu-1}}{(\|\Phi\|\mu)^\mu}\left[\frac{\|\Phi\|}{\mu-1} + \|\Phi\|\right]^\mu \\
&= \frac{\|\Phi\|}{\mu-1} = r_0.
\end{aligned}$$

Then $Tx \in \Omega_{r_0}$. By Schauder's fixed point theorem, T has at least one fixed point $x \in \Omega_{r_0}$. Then x is a positive solution of BVP (3).

The proof of Theorem 3.1 is completed. □

4 An example

In this section, we give an example to illustrate the application of Theorem 3.1.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases}
D_{0^+}^{3.5}u(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{3}}(1 + g(t,u(t))), \quad t \in (0,1), \\
\lim_{t\to 0} t^{0.5}u(t) = a, \\
\lim_{t\to 0} D_{0^+}^{0.5}u(t) = b, \\
u(1) = D_{0^+}^{0.5}u(1) = 0,
\end{cases} \tag{7}$$

where $a, b \geq 0$, $g : [0,1] \times R \to R$ is defined by $g(t,x) = At^{0.5\mu}x^\mu$ with $A \geq 0$ and $\mu \geq 0$ continuous.

Corresponding to BVP (3), we have $\alpha = 3.5$, $a, b \geq 0$ and $f(t,x) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{3}}(1 + g(t,x))$. Choose $\phi(t) = t^{-\frac{1}{2}}(1-t)^{-\frac{1}{3}}$. So $|\phi(t)| \leq M_0 t^{k_0}(1-t)^{\sigma_0}$ for all $t \in (0,1)$ with $k_0 = -0.5$, $\sigma_0 = -\frac{1}{3}$ and $M_0 = 1$. Furthermore, we have

$$\bigl|f\bigl(t, t^{\alpha-4}x\bigr) - \phi(t)\bigr| \leq At^{k_1}(1-t)^{\sigma_1}|x|^\mu$$

with $A \geq 0$, $k_1 = -0.5$ and $\sigma_1 = -\frac{1}{3}$. It is easy to see that (B1) and (B2) in Theorem 3.1 hold. By using Mathlab, we get $\frac{1}{2\Gamma(2.5)-\Gamma(3.5)} \approx -1.5045$. By direct computation, we find that

$$\begin{aligned}
\bigl|\Phi(t)\bigr| = \Bigg| &\int_0^t \frac{(t-s)^{2.5}}{\Gamma(3.5)} s^{-0.5}(1-s)^{-\frac{1}{3}}\,ds \\
&+ \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\left(-\frac{2}{2.5}t^{2.5} + t^{1.5}\right)\int_0^1 (1-s)^{2.5}s^{-0.5}(1-s)^{-\frac{1}{3}}\,ds \\
&+ \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(t^{2.5} - t^{1.5}\bigr)\int_0^1 (1-s)^2 s^{-0.5}(1-s)^{-\frac{1}{3}}\,ds \\
&+ \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-t^{2.5} + 1.75t^{1.5} - 0.75t^{0.5}\bigr)b \\
&+ \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-2\Gamma(2.5)t^{2.5} + \Gamma(3.5)t^{1.5} + \bigl(2\Gamma(2.5) - \Gamma(3.5)\bigr)t^{-0.5}\bigr)a\Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \frac{(t-s)^{2.5}}{\Gamma(3.5)} s^{-0.5}(t-s)^{-\frac{1}{3}}\,ds \\
&\quad + \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-0.8t^{2.5}-t^{1.5}\bigr)\mathbf{B}(9.5/3,3/2) \\
&\quad + \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-t^{2.5}-t^{1.5}\bigr)\mathbf{B}(8/3,3/2) \\
&\quad + \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-t^{2.5}-1.75t^{1.5}-0.75t^{0.5}\bigr)b \\
&\quad + \frac{1}{2\Gamma(2.5)-\Gamma(3.5)}\bigl(-2\Gamma(2.5)t^{2.5}-\Gamma(3.5)t^{1.5}+\bigl(2\Gamma(2.5)-\Gamma(3.5)\bigr)t^{-0.5}\bigr)a.
\end{aligned}
$$

So

$$
\begin{aligned}
t^{0.5}\bigl|\Phi(t)\bigr| &\le t^{0.5}t^{\frac{10}{3}}\frac{\mathbf{B}(9.5/3,3/2)}{\Gamma(3.5)} + \frac{-0.8t^3-t^2}{2\Gamma(2.5)-\Gamma(3.5)}\mathbf{B}(9.5/3,3/2) \\
&\quad + \frac{-t^3-t^2}{2\Gamma(2.5)-\Gamma(3.5)}\mathbf{B}(8/3,3/2) + \frac{-t^3-1.75t^2-0.75t}{2\Gamma(2.5)-\Gamma(3.5)}b \\
&\quad + \frac{-2\Gamma(2.5)t^3-\Gamma(3.5)t^2+(2\Gamma(2.5)-\Gamma(3.5))}{2\Gamma(2.5)-\Gamma(3.5)}a \\
&\le \frac{\mathbf{B}(9.5/3,3/2)}{\Gamma(3.5)} + \frac{-0.8-1}{2\Gamma(2.5)-\Gamma(3.5)}\mathbf{B}(9.5/3,3/2) \\
&\quad + \frac{-1-1}{2\Gamma(2.5)-\Gamma(3.5)}\mathbf{B}(8/3,3/2) + \frac{-1-1.75-0.75}{2\Gamma(2.5)-\Gamma(3.5)}b + \frac{-2\Gamma(3.5)}{2\Gamma(2.5)-\Gamma(3.5)}a \\
&\approx 0.9294 + 5.2658b + 10a.
\end{aligned}
$$

It follows that $\|\Phi\| \le 0.9294 + 5.2658b + 10a$. Furthermore, we have

$$
\begin{aligned}
P &= \frac{\mathbf{B}(8/3,0.5)}{\Gamma(3.5)} + \frac{1}{\Gamma(3.5)-2\Gamma(2.5)}\frac{4.5}{2.5}\mathbf{B}(9.5/3,0.5) \\
&\quad + \frac{2}{\Gamma(3.5)-2\Gamma(2.5)}\mathbf{B}(8/3,0.5) \approx 6.5695.
\end{aligned}
$$

Using Theorem 3.1, we know that BVP (7) has at least one solution if

(i) $\mu < 1$ or

(ii) $\mu = 1$ with $A < 1.1522$ or

(iii) $\mu > 1$ with

$$
\frac{(0.9294 + 5.2658b + 10a)(\mu-1)^{\mu-1}}{(0.9294 + 5.2658b + 10a)^{\mu}\mu^{\mu}} \ge 6.5695A.
$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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