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# Stochastic modified Boussinesq approximate equation driven by fractional Brownian motion

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**Abstract**

The current paper is devoted to the dynamics of a stochastic modified Boussinesq approximate equation driven by fractional Brownian motion with $H \in (\frac{1}{2}, 1)$. Based on the different diffusion operators $\triangle^2$ and $-\triangle$ in the stochastic system, we combine two types of operators $\Phi_1 = I$ and a Hilbert-Schmidt operator $\Phi_2$ to guarantee the convergence of the corresponding Wiener-type stochastic integrals. Then the existence and regularity of the stochastic convolution for the corresponding additive linear stochastic equation can be shown. By the Banach modified fixed point theorem in the selected intersection space, the existence and uniqueness of the global mild solution are obtained. Finally, the existence of a random attractor for the random dynamical system generated by the mild solution for the modified Boussinesq approximation equation is also established.

**MSC:** 35B40; 35Q35; 76D05

**Keywords:** fractional Brownian motion; stochastic modified Boussinesq approximate equation; stochastic convolution; mild solution; random attractor

## 1 Introduction

The modified Boussinesq approximation equation is a reasonable model to describe the essential phenomena of the highly viscous incompressible fluid in the Earth's mantle. We refer to Hills and Roberts [1] and Padula [2] for a derivation of the following Boussinesq approximation equation:

$$\begin{cases} u_t + u \cdot \nabla u - \nabla \cdot \tau(e(u)) = -\nabla \pi + f(x) + e_2\theta, \\ \theta_t + (u \cdot \nabla)\theta - \triangle\theta = g(x), \end{cases} \tag{1.1}$$

where the vector function $u$ represents the velocity of the fluid, $\theta$ is the scalar temperature, function $f(x)$ and $g(x)$ are periodic external forces with respect to space variable $x$, the vector $e_2 = (0,1)$ is a unit vector in $R^2$, the scalar function $\pi$ is the pressure and $\tau_{ij}(e(u))$ is a symmetric stress tensor with the following form:

$$\tau_{ij}\big(e(u)\big) = 2\mu_0\big(\epsilon + |e|^2\big)^{\frac{p-2}{2}} e_{ij} - 2\mu_1\triangle e_{ij}, \quad \epsilon > 0, i,j = 1,2,$$

$$e_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad \big|e(u)\big|^2 = \sum_{i,j=1}^{2}\big|e_{ij}(u)\big|^2,$$

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where $\mu_0$ and $\mu_1$ are positive constants. There are many papers concerning the existence and uniqueness of the solution, attractors, and manifold for the modified Boussinesq approximation equation. We refer to [3–6] for the deterministic non-Newtonian flow (in the absence of $\theta$). The well-posedness and long-time behavior of the modified Boussinesq approximation equation can be referred to [7].

The fractional Brownian motion (FBM) is a family of Gaussian process which is indexed by the Hurst parameter $H \in (0,1)$. For $H \neq \frac{1}{2}$, the FBM is not a semi-martingale and the increments of the process are not independent. The application of the classical Itô stochastic integral to FBM fails. The stochastic integral of FBM has been studied in many papers (see [8–10] and the references therein). Due to the properties of FBM such as the self similar and long range dependence, the data in the fields like financial markets, traffic networks, and climate systems can be described suitably by FBM. In [11], the equilibrium fluctuation of the distance between an electron transfer donor and acceptor pair within a protein that spans a broad range of time scales can be explained by the generalized Langevin equation driven by fractional Gaussian noise. This result is in excellent agreement with a single-molecule experiment. So it is worth to study the well-posedness and long-time behavior of the stochastic partial differential equation (SPDE) driven by FBM. We also refer to [9, 10, 12–14] for the well-posedness and dynamics of the stochastic PDE driven by FBM.

Recently, Guo [15] showed the existence of a random attractor for the stochastic Boussinesq approximation equation driven by Gaussian white noise in domain $D = [0,L] \times [0,L]$:

$$\begin{cases} du + (u \cdot \nabla u - \nabla \cdot \tau(e(u)) + \nabla\pi)\,dt \\ \quad = (f(x) + e_2\theta)\,dt + \Phi_1(t)\,dW(t), \quad x \in D, t > 0, \\ d\theta_t + ((u \cdot \nabla)\theta - \triangle\theta)\,dt = g(x)\,dt + \Phi_2(t)\,dW(t), \quad x \in D, t > 0, \\ \nabla \cdot u(x,t) = 0, \quad x \in D, t > 0, \\ u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in D, \\ u_i(x,t) = u_i(x + L\chi_j, t), \qquad \theta(x,t) = \theta(x + L\chi_j, t), \quad i = 1,2, \end{cases} \tag{1.2}$$

where $\{\chi_j\}_{j=1}^2$ is the natural basis of $R^2$, $W(t) = \sum_i \beta_i(t)h_i$ is the cylindrical Wiener process for white noise, $\beta_i(t)$ is a family of mutually independent real-valued standard Wiener process, $h_i(x)$ is an orthonormal complete basis in Hilbert space $L^2(D)$, $\Phi_1(t)$ is vector value predictable process, while $\Phi_2(t)$ is scalar predictable process, which are linear mappings and are assumed to be Hilbert-Schmidt operators.

Motivated by the ideas in [4] and [15], we consider the following stochastic modified Boussinesq equation driven by fractional Brownian motion with $H \in (\frac{1}{2}, 1)$:

$$\begin{cases} du(t) + (u \cdot \nabla u - \nabla \cdot \tau(e(u)) + \nabla\pi)\,dt \\ \quad = (f(x) + e_2\theta)\,dt + \Phi_1\,dB^H(t), \quad x \in \mathcal{O}, t > 0, \\ d\theta(t) + ((u \cdot \nabla)\theta - \triangle\theta)\,dt = g(x)\,dt + \Phi_2(t)\,dB^H(t), \quad x \in \mathcal{O}, t > 0, \\ \nabla \cdot u(x,t) = 0, \quad x \in \mathcal{O}, t > 0, \\ u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x), \quad x \in \mathcal{O}, \\ u_i(x,t) = u_i(x + L\chi_j, t), \qquad \theta(x,t) = \theta(x + L\chi_j, t), \quad i = 1,2, \end{cases} \tag{1.3}$$

where $\mathcal{O} \subset R^2$ is a bounded domain with smooth boundary $\partial\mathcal{O}$.

Due to the regularity of the stochastic convolution for FBM depends on the value of Hurst parameter $H \in (0,1)$, the stochastic Wiener-type integral is quite different for $H \in (1/2,1)$ and $H \in (0,1/2)$. For $H \in (0,1/2)$, on account of the lower regularity, it needs the fractional Riemann-Liouville integral to transfer the fractional Brownian motion to be represented in terms of the standard cylindrical Brownian motion. The well-posedness and dynamics of equation (1.3) with $H \in (\frac{1}{4},\frac{1}{2})$ have been studied in [16].

However, for $H \in (1/2,1)$, we can use Wiener stochastic integrals to deal with fractional Brownian motion directly. In this paper, we focus on the case $H \in (\frac{1}{2},1)$. For the different diffusion operators $\triangle^2$ and $-\triangle$, let $\Phi_1 = I$ and $\Phi_2$ be a Hilbert-Schmidt operator. Then the existence and regularity of the stochastic convolution for the corresponding additive linear stochastic equation can be guaranteed. By the modified Banach fixed point theorem in the selected intersection space, the existence and uniqueness of the mild solution for equation (1.3) are obtained. Finally, the existence of a random attractor for the random dynamical system generated by the mild solution for equation (1.3) is also presented.

The contribution of the current paper is to establish the well-posedness and dynamics of the stochastic modified Boussinesq approximation equation with $H \in (\frac{1}{2},1)$, and reveals the difference between dynamics of modified Boussinesq approximation equation with differential Hurst parameter. Since the computation for the regularity is different for $H \in (\frac{1}{4},\frac{1}{2})$ and $H \in (\frac{1}{2},1)$, the conditions (Hyper-4) and (Hyper-5) are different from that in [16]. For the technical reasons, there is no result on the computation for the regularity for $H \in (0,\frac{1}{4})$. This shows that the Hurst parameter $H$ determines the conditions which ensure the regularity of the stochastic convolution and the existence of a random attractor generated by the mild solution for equation (1.3). If the temperature variable $\theta = 0$, then the result in the present paper and [16] will reduce to that in [17] and [4], respectively.

The rest of the paper is organized as follows: In Section 2, we present the function space and operators. Then the definitions and criteria for the random dynamical system are presented. In Section 3, we introduce the definition of the infinite-dimensional fractional Brownian motion and its stochastic integral. Then some proper conditions which can ensure the existence and regularity of the stochastic convolution for the corresponding additive linear stochastic equation are shown. In Section 4, the proper selected intersection space is constructed and the existence and uniqueness of the global mild solution are obtained in the space by the modified Banach fixed point theorem. Finally, the existence of a random attractor for the random dynamical system generated by the mild solution of equation (1.3) is shown in Section 5.

## 2 Preliminaries

In this section, we will present some notations for the working function space and operators, and then rewrite equation (1.3) as a stochastic evolution equation by the standard mathematical setting.

Firstly, we introduce some notations as follows:

$$H_1 = \{u \in (L^2(\mathcal{O}))^2 : \nabla \cdot u = 0, u \cdot \mathbf{n}|_{\partial\mathcal{O}} = 0\}, \qquad H_2 = L^2(\mathcal{O}).$$

Denote $H = H_1 \times H_2$ endowed with the norm

$$|\phi|_H^2 := |u|_{H_1}^2 + |\theta|_{H_2}^2,$$

for any $\phi = (u, \theta) \in H$, where $u \in H_1$ and $\theta \in H_2$. For ease of notation, we use the notation $|\cdot|$ to represent the norm for space $H_1$, $H_2$, and $H$, respectively. It is easy to verify that $H_1$, $H_2$, and $H$ are Hilbert space with the inner product $(\cdot,\cdot)$.

Denote

$$V_1 = \{u \in (H_0^2(\mathcal{O}))^2 : \nabla \cdot u = 0\}, \qquad V_2 = H_0^1(\mathcal{O}), \qquad V = V_1 \times V_2,$$

where $V$ is endowed with the norm

$$|\phi|_V^2 := |u|_{V_1}^2 + |\theta|_{V_2}^2.$$

Define bilinear operator $a_1(\cdot,\cdot) : V_1 \times V_1 \to \mathbb{R}$ and $a_2(\cdot,\cdot) : V_2 \times V_2 \to \mathbb{R}$ by

$$a_1(u, v) = (u, v)_{V_1}, \qquad a_2(\theta, \xi) = (\theta, \xi)_{V_2}.$$

By the Lax-Milgram lemma, we can use the bilinear operators $a_1(\cdot,\cdot)$ and $a_2(\cdot,\cdot)$ to define the following linear operators $A_1 \in \mathscr{L}(V_1, V_1')$ and $A_2 \in \mathscr{L}(V_2, V_2')$:

$$\langle A_1 u, v\rangle = a_1(u, v), \qquad \langle A_2\theta, \xi\rangle = a_2(\theta, \xi),$$

where $V_i'$ is the dual space of $V_i$.

Similar to the arguments in [18] and [4], the operator $A_i$ is an isometry from $V_i$ to $V_i'$ for $i = 1, 2$.

Denote

$$D(A_1) = V_1 \cap \{H^4(\mathcal{O})\}^2, \qquad D(A_2) = V_2 \cap H^2(\mathcal{O}),$$

then $A_i \in \mathscr{L}(D(A_i), H_i)$ is an isometry from $D(A_i)$ to $H_i$, and $A_i$ is a self-adjoint positive operator with compact inverse $A_i^{-1}$, where $i = 1, 2$.

It follows from the Hilbert-Schmidt theorem that there exist eigenvalues $\{\lambda_j\}_{j=1}^\infty$, $\{\hat{\lambda}_j\}_{j=1}^\infty$ and the corresponding eigenvectors $\{e_j\}_{j=1}^\infty \subset D(A_1)$, $\{\hat{e}_j\}_{j=1}^\infty \subset D(A_2)$ such that

$$A_1 e_j = \lambda_j e_j, \quad j = 1, 2, \ldots, 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \cdots, \lambda_j \to \infty \ (j \to \infty),$$
$$A_2 \hat{e}_j = \hat{\lambda}_j \hat{e}_j, \quad j = 1, 2, \ldots, 0 < \hat{\lambda}_1 \le \hat{\lambda}_2 \le \cdots \le \hat{\lambda}_j \le \cdots, \hat{\lambda}_j \to \infty \ (j \to \infty).$$

Moreover, $\{e_j\}_{j=1}^\infty$ and $\{\hat{e}_j\}_{j=1}^\infty$ are the orthonormal basis for $H_1$ and $H_2$, respectively.

Since $A_i$ $(i = 1, 2)$ is the densely defined, self-adjoint, and bounded below operator in Hilbert space $H_i$ $(i = 1, 2)$, then $A_i$ $(i = 1, 2)$ is a sectional operator, and $S_i(t) \in \mathscr{L}(H_i)$ is an analytic semigroup generated by $A_i$ $(i = 1, 2)$ in the following form:

$$S_i(t) := e^{-tA_i} = \int_0^\infty e^{-t\lambda}\, dE_{i,\lambda}, \quad i = 1, 2,$$

where $\{E_{i,\lambda}\}$ is the spectrum of the operator $A_i$, $i = 1, 2$.

For any $\phi = (u, \theta) \in V$, denote

$$A\phi = \begin{pmatrix} 2\mu_1 A_1 u \\ A_2\theta \end{pmatrix}, \qquad S(t)\phi = \begin{pmatrix} S_1(t)u \\ S_2(t)\theta \end{pmatrix},$$

and define the trilinear operator $b(\cdot,\cdot,\cdot)$ by

$$b(\phi_1,\phi_2,\phi_3) = b_1(u_1,u_2,u_3) + b_2(u_1,\theta_2,\theta_3), \quad \forall \phi_i = (u_i,\theta_i) \in V,$$

where

$$b_1(y,v,w) = \sum_{i,j=1}^{2} \int_{\mathcal{O}} y_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad \forall y,v,w \in \left(H^1(\mathcal{O})\right)^2,$$

$$b_2(y,\theta,\xi) = \sum_{i=1}^{2} \int_{\mathcal{O}} y_i \frac{\partial \theta}{\partial x_i} \xi \, dx, \quad \forall y \in \left(H^1(\mathcal{O})\right)^2, \theta,\xi \in H^1(\mathcal{O}).$$

For any $\phi_i = (u_i,\theta_i)$, we define the continuous bilinear functionals $B(\phi_1,\phi_2) \in V'$, $B_1(u_1,u_2) \in V_1'$, and $B_2(u_1,\theta_2) \in V_2'$ by

$$\langle B(\phi_1,\phi_2),\phi_3\rangle = b(\phi_1,\phi_2,\phi_3),$$
$$\langle B_1(u_1,u_2),u_3\rangle = b_1(u_1,u_2,u_3),$$
$$\langle B_2(u_1,\theta_2),\theta_3\rangle = b_2(u_1,\theta_2,\theta_3).$$

In what follows, we abbreviate $B(\phi,\phi)$ as $B(\phi)$ for any $\phi \in V$.

Define the functional $N(u) \in V_1'$ by

$$\langle N(u),v\rangle = \int_{\mathcal{O}} \mu(u) e_{ij}(u) e_{ij}(v) \, dx, \quad \forall v \in V_1$$

and

$$\widetilde{N}(\phi) = \begin{pmatrix} N(u) \\ 0 \end{pmatrix}, \quad \forall \phi = (u,\theta) \in V.$$

We also denote $\widetilde{N}$ as $N$ without any confusion, and

$$R\phi = \begin{pmatrix} -\theta\chi_2 \\ 0 \end{pmatrix}, \qquad \Phi \, dB^H(t) = \begin{pmatrix} \Phi_1 \, dB^H(t) \\ \Phi_2 \, dB^H(t) \end{pmatrix}.$$

Under the above notations, the stochastic modified Boussinesq approximation equation (1.3) can be rewritten as the following abstract stochastic evolution equation:

$$\begin{cases} d\phi(t) + (A\phi(t) + B(\phi(t)) + N(\phi)(t) + R(\phi(t))) \, dt = \Phi \, dB^H(t), \\ \phi(0) = (u_0,\theta_0). \end{cases} \tag{2.1}$$

Finally, we recall some definitions and criteria of the random dynamical system and random attractor which are taken from [19].

Due to the properties of the stationary increments, we can switch FBM and additive white noise to the equivalent canonical realization. Consider the Borel set

$$\Omega = C_0(\mathbb{R}) := \left\{\omega \in C(\mathbb{R},\mathbb{R}) : \omega(0) = 0\right\} \tag{2.2}$$

with the compact open topology and let $\mathscr{F}$ be the associated incomplete Borel-$\sigma$-algebra. The operator $\theta_t$ can form the flow which is given by the Wiener shift:

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}. \tag{2.3}$$

**Definition 2.1** Let $E$ be a complete and separable metric space. A random dynamical system (RDS) with space $E$ carried by a metric dynamical system $(\Omega, \mathscr{F}, P, \theta)$ is given by the mapping

$$\varphi : \mathbb{R}_+ \times \Omega \times E \to E, \tag{2.4}$$

which is $(\mathscr{B}(\mathbb{R}_+) \times \mathscr{F} \times \mathscr{B}(E); \mathscr{B}(E))$-measurable and possesses the cocycle property:

$$\varphi(t + \tau, \omega, x) = \varphi\big(\tau, \theta_t\omega, \varphi(t, \omega, x)\big), \quad \forall t, \tau \in \mathbb{R}_+, x \in E, \omega \in \Omega, \tag{2.5}$$

$$\varphi(0, \omega, \cdot) = id_\Omega. \tag{2.6}$$

**Definition 2.2**

(i) A set-valued mapping $K : \Omega \to 2^E$ taking value in the closed subsets of $E$ is said to be measurable if for each $x \in E$ the mapping $\omega \mapsto d(x, K(\omega))$ is measurable, where

$$d(A, B) = \sup_{x\in A} \inf_{y\in B} d(x, y). \tag{2.7}$$

A measurable set-valued mapping $K$ is called a random set.

(ii) Let $A$, $B$ be random sets. $A$ is said to attract $B$ if

$$d\big(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), A(\omega)\big) \to 0, \quad \text{as } t \to \infty \text{ P-a.s.} \tag{2.8}$$

$A$ is said to absorb $B$ if P-a.s. there exists an absorption time $t_B(\omega)$ such that, for all $t \geq t_B(\omega)$,

$$\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega) \subset A(\omega). \tag{2.9}$$

(iii) The $\Omega$-limit set of a random set $K$ is defined by

$$\Omega_K(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \theta_{-t}\omega)K(\theta_{-t}\omega)}. \tag{2.10}$$

**Definition 2.3** A random attractor for an RDS $\varphi$ is a compact random set $\mathcal{A}$ P-a.s. satisfying:

(i) $\mathcal{A}$ is invariant, *i.e.*, $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t\omega)$ for all $t > 0$;

(ii) $\mathcal{A}$ attracts all the deterministic bounded sets in $E$.

The following proposition (*cf.* [19], Theorem 3.11) yields a sufficient criterion for the existence of a random attractor.

**Theorem 2.1** *Let $\varphi$ be an RDS and assume that there exists a compact random set $K$ absorbing every deterministic bounded set $B \subset E$. Then the set*

$$\mathcal{A}(\omega) = \overline{\bigcup_{B \subset E} \Omega_B(\omega)} \tag{2.11}$$

*is a random attractor for $\varphi$.*

## 3 Fractional Brownian motion and its stochastic convolution

In this section, we introduce the definition of the infinite-dimensional fractional Brownian motion only with $H \in (\frac{1}{2}, 1)$ and its Wiener-type stochastic integral with respect to infinite-dimensional FBM. Then some conditions which ensure the existence and regularity of the infinite-dimensional stochastic convolution for the corresponding additive linear stochastic equation are presented. Finally, the existence and regularity of the stochastic convolution are obtained under these conditions.

Since the derivative of FBM exists almost nowhere, equation (2.1) should be understood in the integral form. There are several approaches to define an integral for one-dimensional FBM and each has its advantage (*cf.* [8] for a useful summary). Wiener integrals are introduced since they deal with the simplest case of deterministic integrands by using FBM's Gaussianity in [17]. We refer to [17] for the general framework of the Wiener-type stochastic integral with respect to infinite-dimensional FBM.

As a conclusion, we have the following relationship between the Wiener integral with respect to FBM and the Wiener integral with respect to the Wiener process:

$$\int_0^t \varphi(s)\, d\beta^H(s) = \int_0^t \big(K_H^* \varphi\big)(s)\, dW(s), \tag{3.1}$$

for every $t \leq T$ and $\varphi \in \mathscr{H}$ if and only if $K_H^* \varphi \in L^2(0, T; V)$.

The infinite-dimensional FBM and corresponding stochastic integration are defined now. Let $Q$ be a self-adjoint and positive linear operator on $H_i$. Assume that there exists a sequence of nonnegative numbers $\{\widetilde{\lambda}_i\}_{i \in \mathbb{N}}$ such that

$$Qe_i = \widetilde{\lambda}_i e_i, \quad i = 1, 2, \ldots. \tag{3.2}$$

The infinite-dimensional FBM on $H$ with covariance operator $Q$ is formally defined by

$$B^H(t) = \sum_{i=1}^{\infty} \sqrt{\widetilde{\lambda}_i} e_i \beta_i^H(t), \tag{3.3}$$

where $\{\beta_i^H(t)\}_{i \in \mathbb{N}}$ is a sequence of real stochastically independent one-dimensional FBM. This process, if convergence, is an $H$-valued Gaussian process. It starts from 0, has zero mean and covariance

$$\mathbb{E}\big(B^H(t) B^H(s)\big) = R(t, s) Q. \tag{3.4}$$

Let $(\Phi_s)_{0 \leq s \leq T}$ be a deterministic function with values in $\mathscr{L}(H_i)$ $(i = 1, 2)$, the space of all bounded linear operators from $H_i$ to $H_i$. The stochastic integral of $\Phi_i$ with respect to $B^H$

is formally defined by

$$\int_0^t \Phi_s \, dB^H(s) := \sum_{i=1}^{\infty} \sqrt{\widetilde{\lambda}_i} \int_0^t \Phi_s e_i \, d\beta_i^H(s) = \sum_{i=1}^{\infty} \sqrt{\widetilde{\lambda}_i} \int_0^t \big(K_H^*(\Phi e_i)\big)_s \, d\beta_i(s), \tag{3.5}$$

where $\beta_i$ is the standard Brownian motion. The above sum may not converge. However, as we are about to see, the linear additive stochastic equation can have a mild solution even if $\int_0^t \Phi_s \, dB^H(s)$ is not properly defined as an $H$-valued Gaussian random variable.

Consider the following stochastic convolution:

$$z_2(t) := \int_0^t S_2(t-s)\Phi_2 \, dB^H(s). \tag{3.6}$$

Then $z_2$, if it is well defined, is the unique mild solution of the following linear stochastic evolution equation:

$$dz_2(t) = A_2 z_2(t) \, dt + \Phi_2 \, dB^H(t), \qquad z_2(0) = 0 \in V_2. \tag{3.7}$$

Here are three kinds of condition on the stochastic convolution:

(Hyper-1) $Q \in \mathscr{L}_1(H_2)$, $\Phi_2 \equiv id_{H_2}$;

(Hyper-2) $Q \equiv id_{H_2}$, $\Phi_2 \in \mathscr{L}_2(H_2)$;

(Hyper-3) $Q \equiv id_{H_2}$, $\Phi_2 \in \mathscr{L}(H_2)$ such that $\Phi_2\Phi_2^* \in \mathscr{L}_1(H_2)$,

where $\mathscr{L}_1(H_2)$ is the space of all nuclear operators on $H_2$ and $\mathscr{L}_2(H_2)$ is the space of all Hilbert-Schmidt operators on $H_2$ (*cf.* [20], Appendix C). These conditions have been proposed by Maslowski and Schmalfuss in [10], Duncan *et al.* in [12] and Tindel *et al.* in [9], respectively. We mention that the Hilbert-Schmidt operators (elements of $\mathscr{L}_2(H_2)$) is compact. Indeed, the key feature of these conditions is the compactness which guarantees that we can handle the infinite-dimensional problem in a finite-dimensional manner.

Then for the linear stochastic evolution equation (3.7), let $\Phi_2$ satisfy the condition of (Hyper-2) and we have the following lemma.

**Lemma 3.1** ([10]) *If $H \in (\frac{1}{2},1)$ and $\Phi_2$ satisfies the condition of* (Hyper-2), *then there is a version of the stochastic convolution $z_2(t) = \int_0^t S_2(t-s)\Phi_2 \, dB^H(s)$, $t \in [0,T]$ with $C([0,T];V_2)$ sample paths.*

Next we consider another stochastic linear differential equation:

$$\begin{cases} dz_1 = A_1 z_1 \, dt + \Phi_1 \, dB^H, \\ z_1(0) = 0 \in V_1, \end{cases} \tag{3.8}$$

where $\Phi_1 = I$.

As noted in [12] and [9], the stochastic integral $\int_0^t I_d \, dB^H(s)$ is not well defined as a $V_1$-valued random variable since the identity operator $I_d \notin \mathscr{L}_2(V_1)$. Then we consider the following condition on the stochastic convolution which is taken from [17]:

(Hyper-4) $\mathcal{O} = [-\pi,\pi] \times [-\pi,\pi]$, $Q \equiv \Phi \equiv id_H$.

If the condition (Hyper-4) is satisfied, we have the following lemma from [17].

**Lemma 3.2** ([17]) *If $H \in (\frac{1}{2}, 1)$ and condition* (Hyper-4) *is satisfied, there is a version of the stochastic convolution $z_1(t) := \int_0^t S_1(t-s)\,dB^H(s)$ with $C([0,T];H_1)$ sample paths. If additionally $H \in (\frac{3}{4}, 1)$, then the process $z_1$ has an $L^\infty(0,T;V_1)$ version.*

Noticing that the sample orbit of fractional Brownian motion is not differentiable almost everywhere in the classical sense, we consider the stochastic convolution in the product space $H$:

$$z(t) = \int_0^t S(t-s)\Phi\,dB^H(t) := \begin{pmatrix} \int_0^t S_1(t-s)\,dB^H(t) \\ \int_0^t S_2(t-s)\Phi_2\,dB^H(t) \end{pmatrix} \triangleq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}. \tag{3.9}$$

## 4 Existence and uniqueness of the mild solution

In this section, we will apply the modified Banach fixed point theorem to show the existence and uniqueness of the mild solution for equation (2.1) in the space $E = C([0,T];H) \cap L^2(0,T;V)$ with the norm $|\cdot|_E = |\cdot|_{C([0,T];H)} + |\cdot|_{L^2(0,T;V)}$.

The notion of the mild solution for equation (2.1) is given as follows.

**Definition 4.1** An $H$-valued random process $(\phi(t), t \geq 0)$ on a fixed probability space $(\Omega, \mathscr{F}, P)$ with a given infinite-dimensional fractional Brownian motion is called a mild solution of stochastic equation (2.1) if $(\phi(t), t \geq 0)$ satisfies the following equation:

$$\begin{aligned} \phi(t) = S(t)\phi_0 &- \int_0^t S(t-s)B\big(\phi(s)\big)\,ds - \int_0^t S(t-s)N\big(\phi(s)\big)\,ds \\ &- \int_0^t S(t-s)R\big(\phi(s)\big)\,ds + \int_0^t S(t-s)\Phi\,dB^H(s), \end{aligned} \tag{4.1}$$

where the first three terms are operator-valued Bochner integrals, and the last one is the Wiener-type stochastic integral defined by equation (3.9).

Denote

$$E_1 = C\big([0,T];H_1\big) \cap L^2(0,T;V_1), \qquad E_2 = C\big([0,T];H_2\big) \cap L^2(0,T;V_2),$$

with the norm

$$\begin{aligned} &|\cdot|_{E_1} = |\cdot|_{C([0,T];H_1)} + |\cdot|_{L^2(0,T;V_1)}, \qquad |\cdot|_{E_2} = |\cdot|_{C([0,T];H_2)} + |\cdot|_{L^2(0,T;V_2)}, \\ &|\cdot|_E^2 = |\cdot|_{E_1}^2 + |\cdot|_{E_2}^2. \end{aligned}$$

It is easy to verify that $E_1$, $E_2$, and $E$ are Banach spaces. In order to apply the modified Banach fixed point theorem, it is necessary to estimate each term of the integral equation (4.1) in the space $E$.

For any $\phi \in E$, denote

$$J_1(\phi) := -\int_0^\cdot S(\cdot - s)B\big(\phi(s)\big)\,ds, \tag{4.2}$$

$$J_2(\phi) := -\int_0^\cdot S(\cdot - s)N\big(\phi(s)\big)\,ds, \tag{4.3}$$

$$J_3(\phi) := -\int_0^{\cdot} S(\cdot - s)R(\phi(s))\,ds, \tag{4.4}$$

then the operators $J_1$, $J_2$, and $J_3$ satisfy the following properties.

**Lemma 4.1** *$J_1 : E \to E$, and for any $\phi, \psi \in E$, it follows that*

$$\begin{aligned}
&|J_1(\phi)|_E^2 \le c_1|\phi|_E^4,\\
&|J_1(\phi) - J_1(\psi)|_E^2\\
&\quad\le c_2\left(4c_3|\phi|^2_{C([0,T];H)}|\phi|^2_{L^2(0,T;V)} + 4c_3|\psi|^2_{C([0,T];H)}|\psi|^2_{L^2(0,T;V)} + \frac{1}{4c_3}\right)|\phi - \psi|_E^2.
\end{aligned}$$

*Proof* The proof is the same as Lemma 4.1 in [16], and it is omitted here. □

**Lemma 4.2** *$J_2 : E \to E$, and for any $\phi, \psi \in E$, it follows that*

$$|J_2(\phi)|_E^2 \le c_4|\phi|_E^2, \tag{4.5}$$

$$|J_2(\phi) - J_2(\phi)|_E^2 \le c_5 T^{\frac{1}{2}}|\phi - \psi|_E^2. \tag{4.6}$$

*Proof* The proof is the same as Lemma 4.2 in [16], and it is omitted here. □

**Lemma 4.3** *$J_3 : E \to E$, and for any $\phi, \psi \in E$, it follows that*

$$|J_3(\phi)|_E^2 \le c_6|\phi|_E^2, \tag{4.7}$$

$$|J_3(\phi) - J_3(\psi)|_E^2 \le c_7 T|\phi - \psi|_E^2. \tag{4.8}$$

*Proof* The proof is the same as Lemma 4.3 in [16], and it is omitted here. □

Since the process $z(t)$ ($t \in [0,T]$) has a $V$-valued continuous modification, then we can obtain the existence and uniqueness of the mild solution for equation (1.3).

**Theorem 4.1** *If $H \in (\frac{1}{2},1)$,* (Hyper-2) *and* (Hyper-4) *hold, then for any initial value $\phi_0 \in H$ and $T > 0$, equation* (1.3) *has a unique mild solution in the space $C([0,T];H) \cap L^2(0,T;V)$.*

*Proof* The proof is the same as Theorem 4.1 in [16], and it is omitted here. □

Next, we will show the existence of the global mild solution for equation (1.3).

Let $\phi$ be the local mild solution of equation (1.3) on $[0,T_0]$, and denote $\psi(t) = \phi(t) - z(t)$, then $\psi(t)$ is the mild solution for the following equation:

$$\begin{aligned}
\psi(t) = {}& S(t)\phi_0 - \int_0^t S(t-s)B(\psi(s) + z(s))\,ds - \int_0^t S(t-s)N(\psi(s) + z(s))\,ds\\
&- \int_0^t S(t-s)R(\psi(s) + z(s))\,ds.
\end{aligned} \tag{4.9}$$

It is easy to see that $\psi(t)$ is also the weak solution of the following evolution equation with random coefficients:

$$\begin{cases}
\frac{d}{dt}\psi(t) + A\psi(t) + B(\psi(t) + z(t)) + N(\psi(t) + z(t)) + R(\psi(s) + z(s)) = 0,\\
\psi(0) = \phi_0.
\end{cases} \tag{4.10}$$

Following the arguments in [20], Section 15.3, we can get a upper boundedness for $\psi$ in the space $E$.

**Lemma 4.4** *Let $\psi$ be the local solution of the stochastic evolution equation* (4.9) *on* $[0,T]$, *then*

$$\sup_{t\in[0,T]}\left|\psi(t)\right|^2 \le e^{c_8\int_0^T |z(s)|_1^2\,ds}|\phi_0|^2 + \int_0^T e^{c_8\int_s^T |z(r)|_1^2\,dr}h_1(s)\,ds, \tag{4.11}$$

$$\int_0^T \left|\psi(t)\right|_V^2\,dt \le c_9|\phi_0|^2 + c_8c_9\sup_{t\in[0,T]}\left|\psi(t)\right|^2\int_0^T\left(\left|z(s)\right|_1^2\right)ds + c_9\int_0^T h_1(s)\,ds, \tag{4.12}$$

*where $c_8$ and $c_9$ are positive constants which depend on the domain $\mathcal{O}$, and the integral function $h_1$ depends on $z$.*

*Proof* Integrating both sides of equation (4.10) with $\psi(t)$ over $\mathcal{O}$, and applying the facts that $\langle N(\psi),\psi\rangle \ge 0$, $\langle R(\psi),\psi\rangle \ge 0$, and that we have the orthogonality of the trilinear term $b$ leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d|\psi(t)|^2}{dt} + \left|\psi(t)\right|_V^2 \\
&\quad = -b\big(\psi(t)+z(t),\psi(t)+z(t),\psi(t)\big) - \big\langle N\big(\psi(t)+z(t)\big),\psi(t)\big\rangle \\
&\qquad - \big\langle R\big(\psi(t)+z(t)\big),\psi(t)\big\rangle \\
&\quad \le \left|b\big(\psi+z(t),z(t),\psi+z(t)\big)\right| - \big\langle N\big(z(t)\big),\psi(t)\big\rangle - \big\langle R\big(z(t)\big),\psi(t)\big\rangle.
\end{aligned} \tag{4.13}$$

It follows from $\lambda_1 > 1$ (due to Lemma 2.3 in [17]) that

$$\begin{aligned}
&b_1(v+z_1,z_1,v+z_1) \\
&\quad \le C_1|v+z_1|\cdot|z_1|_1\cdot|v+z_1|_1 \\
&\quad \le \frac{C_1}{2C_2}|z_1|_1^2\cdot|v+z_1|^2 + \frac{C_1C_2}{2}|v+z_1|_1^2 \\
&\quad \le \frac{C_1}{C_2}|z_1|_1^2\cdot|v|^2 + C_1C_2|v|_1^2 + \frac{C_1}{C_2}|z_1|^2\cdot|z_1|_1^2 + C_1C_2|z_1|_1^2 \\
&\quad \le \frac{C_1}{C_2}|z_1|_1^2\cdot|v|^2 + C_1C_2\frac{|v|_{V_1}^2}{\lambda_1^{\frac{1}{2}}} + \frac{C_1}{C_2}|z_1|^2\cdot|z_1|_1^2 + C_1C_2|z_1|_1^2 \\
&\quad \le \frac{C_1}{C_2}|z_1|_1^2\cdot|v|^2 + C_1C_2|v|_{V_1}^2 + \frac{C_1}{C_2}|z_1|^2\cdot|z_1|_1^2 + C_1C_2|z_1|_1^2
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&b_2(v+z_1,z_2,\eta+z_2) \\
&\quad \le C_1|v+z_1|\cdot|z_2|_1\cdot|\eta+z_2|_1 \\
&\quad \le \frac{C_1}{2C_2}|z_2|_1^2\cdot|v+z_1|^2 + \frac{C_1C_2}{2}|\eta+z_2|_1^2 \\
&\quad \le \frac{C_1}{C_2}|z_2|_1^2\cdot|v|^2 + C_1C_2|\eta|_{V_2}^2 + \frac{C_1}{C_2}|z_1|^2\cdot|z_2|_1^2 + C_1C_2|z_2|_1^2.
\end{aligned} \tag{4.15}$$

Hence, combining equations (4.14) and (4.15), we get

$$\begin{aligned} &b\bigl(\psi(t)+z(t),\psi(t)+z(t),\psi(t)\bigr) \\ &\quad\le b_1(\nu+z_1,z_1,\nu+z_1)+b_2(\nu+z_1,z_2,\eta+z_2) \\ &\quad\le \frac{C_1}{C_2}|z|_1^2\cdot|\nu|^2+C_1C_2|\psi|_V^2+\frac{C_1}{C_2}|z_1|^2\cdot|z|_1^2+C_1C_2|z|_1^2 \\ &\quad\le \frac{C_1}{C_2}|z|_1^2\cdot|\psi|^2+C_1C_2|\psi|_1^2+\frac{C_1}{C_2}|z|^2\cdot|z|_1^2+C_1C_2|z|_1^2. \end{aligned} \tag{4.16}$$

Similarly, direct calculations show that

$$\begin{aligned} -\bigl\langle N(z),\psi\bigr\rangle &= -\bigl\langle N(z_1),\nu\bigr\rangle \le \mu_0\epsilon^{-\alpha/2}|z_1|_1\cdot|\nu|_1 \le r_1\lambda_1^{\frac{1}{2}}|\nu|_1^2+\frac{\mu_0^2}{4r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|z_1|_1^2 \\ &\le r_1|\nu|_{V_1}^2+\frac{\mu_0^2}{4r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|z_1|_1^2 \le r_1|\psi|_V^2+\frac{\mu_0^2}{4r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|z|_1^2 \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} -\bigl\langle R(z),\psi\bigr\rangle &\le \bigl|(e_2z_2,\nu)\bigr| \le \frac{\lambda_1^{\frac{1}{2}}}{4}|\nu|_1^2+\frac{1}{\lambda_1^{\frac{1}{2}}}|z_2|^2 \\ &\le \frac{1}{4}|\nu|_{V_1}^2+\frac{1}{\lambda_1^{\frac{1}{2}}}|z|^2 \le \frac{1}{4}|\nu|_V^2+\frac{1}{\lambda_1^{\frac{1}{2}}}|z|^2. \end{aligned} \tag{4.18}$$

Then let $\lambda=\min\{\lambda_1,\hat{\lambda}_1\}$ and we can obtain

$$|\psi|_V^2=|\nu|_{V_1}^2+|\theta|_{V_2}^2 \ge \lambda_1|\nu|^2+\hat{\lambda}_1|\theta|^2 \ge \lambda|\psi|_V^2. \tag{4.19}$$

Combining equations (4.16), (4.17), (4.18), and (4.19) gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\psi|^2+\frac{\lambda}{2}|\psi|^2+\frac{1}{2}|\psi|_V^2 \\ &\quad\le \frac{C_1}{C_2}|z|_1^2|\psi|^2+\left(C_1C_2+r_1+\frac{1}{4}\right)|\psi|_V^2+\left(\frac{C_1}{C_2}|z|_1^2+\frac{1}{\lambda_1^{\frac{1}{2}}}\right)|z|^2 \\ &\qquad+C_1C_2|z|_1^2+\frac{\mu_0^2}{4r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|z|_1^2, \end{aligned} \tag{4.20}$$

where $C_2$ is a positive constant determined later.

Let

$$h_1=\left(2\frac{C_1}{C_2}|z|_1^2+\frac{2}{\lambda_1^{\frac{1}{2}}}\right)|z|^2+2C_1C_2|z|_1^2+\frac{\mu_0^2}{2r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|z|_1^2,$$

then

$$\frac{d}{dt}|\psi|^2+\left(\frac{1}{2}-2(C_1C_2+r_1)\right)|\psi|_V^2+\left(\lambda-2\frac{C_1|z|_1^2}{C_2}\right)|\psi|^2 \le h_1. \tag{4.21}$$

Choosing $C_2 < \frac{1}{4C_1}$, let $r_1$ be small enough such that $C_1C_2 + r_1 < \frac{1}{4}$, then we deduce

$$\frac{d}{dt}|\psi|^2 + \left(\lambda - 2\frac{C_1|z|_1^2}{C_2}\right)|\psi|^2 \le h_1. \tag{4.22}$$

Applying the Gronwall lemma leads to

$$\left|\psi(t)\right|^2 \le \left|\phi(0)\right|^2 e^{-\int_0^t (\lambda - 2\frac{C_1|z(s)|_1^2}{C_2})\,ds} + \int_0^t h_1(s_1) e^{-\int_{s_1}^t (\lambda - 2\frac{C_1|z(s_2)|_1^2}{C_2})\,ds_2}\,ds_1,$$

which implies that

$$\sup_{t\in[0,T]} \left|\psi(t)\right|^2 \le e^{2\frac{C_1}{C_2}\int_0^T |z(s)|_1^2\,ds}|\phi_0|^2 + \int_0^T e^{2\frac{C_1}{C_2}\int_s^T |z(r)|_1^2\,dr} h_1(s)\,ds.$$

Let $c_8 = 2C_1/C_2$, then the inequality (4.11) holds.

Integrating both sides of equation (4.21) over $[0,T]$, we have

$$\begin{aligned} &\left|\psi(T)\right|^2 - \left|\psi(0)\right|^2 + \left(\frac{1}{2} - 2(C_1C_2 + r_1)\right)\int_0^T \left|\psi(s)\right|_V^2\,ds \\ &\quad \le \int_0^T 2\frac{C_1}{C_2}\left|z(s)\right|_1^2\left|\Phi(s)\right|^2\,ds + \int_0^T h_1(s)\,ds. \end{aligned} \tag{4.23}$$

Let $c_9 = (\frac{1}{2} - 2(C_1C_2 + r_1))^{-1}$, then the inequality (4.12) holds. Thus, we complete the proof of Lemma 4.4. □

Based on Theorem 4.1 for the existence of the local mild solution and Lemma 4.4 for the extension of the local mild solution, we state the existence of the global mild solution for equation (1.3).

**Theorem 4.2** *Assume the conditions* (Hyper-2) *and* (Hyper-4) *hold, then for* $H \in (\frac{1}{2}, 1)$, $\phi_0 \in H$, *and* $T > 0$, *equation* (1.3) *has a unique global mild solution in the space* $C([0,T];H) \cap L^2(0,T;V)$.

**Remark 4.1** In fact, we define a stopping time

$$\tau_n = T \wedge \inf\left\{t \in [t_0,T] : \left|\psi(t)\right| > n\right\}. \tag{4.24}$$

Then for some given $\omega \in \Omega$, $\psi(t)$ is bounded on $[t_0,T]$, $|\psi(t)| < n$ for large enough $n$, and $\tau_n = T$, which implies that $\tau_n \to T$, $t \wedge \tau_n \to t$ as $n \to \infty$ and $t \in [t_0,T]$. We replace $t$ in the argument of Lemma 4.4 by $t \wedge \tau_n$, we can obtain the existence of the global mild solution.

## 5 Existence of a random attractor

In this section, we will show the existence of a random attractor for the random dynamical system generalized by equation (4.1).

To the end, it suffices to prove the absorbing set in the space $\dot{H}^1 = \dot{H}_1^1 \times \dot{H}_2^1$ with the norm $|\phi|_1^2 = |v|_1^2 + |\theta|_1^2$ where

$$\dot{H}_1^1 = \text{the closure of } \mathscr{V} \text{ in space } \left(H^1(\mathcal{O})\right)^2,$$

$$\dot{H_2}^1 = V_2 = \text{the closure of } \mathscr{V} \text{ in space } H^1(\mathcal{O}).$$

If we follow the method employed for Lemma 4.4 (which is based on the change of variable $\psi = \phi - z$), we end up with the problem of finding a uniform bound for $\{\int_{t_0}^t |z(s)|_1^2\,ds, t_0 \in \mathbb{R}\}$ (*cf.* equations (4.11)-(4.12)). To overcome this difficulty, the fractional Ornstein-Uhlenbeck process is introduced. By the stationary property of the fractional Ornstein-Uhlenbeck process, we can use the Birkhoff-Chintchin ergodic theorem to convert the integration over time variable to the integration over sample space. This yields a uniform bound for those integrals, so that we can deduce a bound in $H$ for $|\psi(t)|^2$.

Denote

$$Z(t,\omega) = \int_{-\infty}^t S(t-r)\,dB^H(r,\omega) = \begin{pmatrix} \int_{-\infty}^t S_1(t-r)\,dB^H(r,\omega) \\ \int_{-\infty}^t S_2(t-r)\Phi_2\,dB^H(r,\omega) \end{pmatrix} \triangleq \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix}. \tag{5.1}$$

Then $Z$ is a stationary solution of the following fractional Ornstein-Uhlenbeck equation:

$$dZ(t) = AZ(t) + \Phi\,dB^H(t), \quad t \in \mathbb{R}, \tag{5.2}$$

where

$$\Phi = \begin{pmatrix} I \\ \Phi_2 \end{pmatrix}. \tag{5.3}$$

Now the existence of the stationary solution $Z(t)$ ($t \in \mathbb{R}$) will be proved. Assume Lemma 3.1 and Lemma 3.2 hold, it suffices to prove that $Z(0)$ converges in the space $L^2(\Omega; H^1)$.

For $H \in (\frac{1}{2}, 1)$, it follows that

$$\begin{aligned}
\mathbb{E}\big|Z(0)\big|_1^2 &= \mathbb{E}\left|\lim_{t\to\infty}\sum_{i=1}^\infty \int_{-t}^0 S(-s)\,dB^H(s)\right|_1^2 \\
&\le \limsup_{t\to\infty}\mathbb{E}\left(\left|\sum_{i=1}^\infty \int_{-t}^0 S_1(-s)e_i\,d\beta_i(s)\right|_1^2 + \left|\sum_{i=1}^\infty \int_{-t}^0 S_2(-s)\Phi_2\hat{e}_i\,d\beta_i(s)\right|_1^2\right) \\
&= \limsup_{t\to\infty}\sum_{i=1}^\infty \left(\big|S_1(\cdot)e_i\big|^2_{\mathscr{H}(0,t;H^1)} + \big|S_2(\cdot)\Phi_2\hat{e}_i\big|^2_{\mathscr{H}(0,t;H^1)}\right) \\
&= \limsup_{t\to\infty}\sum_{i=1}^\infty \int_0^t\int_0^t \big\langle S_1(u)e_i, S_1(v)e_i\big\rangle_{H^1}|u-v|^{2H-2}\,du\,dv \\
&\quad + \int_0^t\int_0^t \big\langle S_2(\theta)\Phi_2\hat{e}_i, S_2(\xi)\Phi_2\hat{e}_i\big\rangle_{H^1}|\theta-\xi|^{2H-2}\,d\theta\,d\xi \\
&= 2\limsup_{t\to\infty}\sum_{i=1}^\infty \lambda_i^{\frac{1}{2}}\int_0^t\int_0^u e^{-\lambda_i(u+v)}(u-v)^{2H-2}\,du\,dv \\
&\quad + \int_0^t\int_0^\theta e^{-\hat{\lambda}_1(\theta+\xi)}|\Phi_2\hat{e}_i|^2_{V_2}(\theta-\xi)^{2H-2}\,d\theta\,d\xi \\
&= 2\limsup_{t\to\infty}\sum_{i=1}^\infty \lambda_i^{\frac{1}{2}}\int_0^t\int_0^u e^{-\lambda_i(2u-x)}x^{2H-2}\,dx\,du \\
&\quad + |\Phi_2\hat{e}_i|^2_{V_2}\int_0^t\int_0^\theta e^{-\lambda_1(2\theta-x)}x^{2H-2}\,dx\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&= 2\limsup_{t\to\infty}\sum_{i=1}^{\infty}\lambda_i^{-\frac{1}{2}}\int_0^t e^{-\lambda_i(2u-x)}x^{2H-2}\bigl(1-e^{-2\lambda_i(t-x)}\bigr)\,dx \\
&\quad + |\Phi_2\hat{e}_i|_{V_2}^2\int_0^t e^{-\lambda_1(2\theta-x)}x^{2H-2}\bigl(1-e^{-2\lambda_1(t-x)}\bigr)\,dx \\
&\le \limsup_{t\to\infty}\sum_{i=1}^{\infty}\lambda_i^{\frac{1}{2}-2H}\int_0^{\lambda_i t}e^{-y}y^{2H-2}\,dy + |\Phi_2|_{\mathscr{L}(V_2)}^2\hat{\lambda}_1^{-2H}\int_0^{\lambda_1 t}e^{-y}y^{2H-2}\,dy \\
&\le \Gamma(2H-1)\cdot\bigl(\beta_D(4H-1)\zeta(4H-1)-\zeta(8H-2)+|\Phi_2|_{\mathscr{L}(V_2)}^2\hat{\lambda}_1^{-2H}\bigr)<\infty.
\end{aligned}
$$

Now, let us consider the real-valued continuous function $|Z(\theta.\omega)|_1^2$. Notice that $(\Omega,\mathscr{F},\{\theta(t)\}_{t\in\mathbb{R}})$ is the metric dynamical system and it follows from the Birkhoff-Chintchin ergodic theorem that

$$
\lim_{n\to\pm\infty}\frac{1}{n}\int_0^n\bigl|Z(\theta_t\omega)\bigr|_1^2\,dt=\mathbb{E}\bigl|Z(\omega)\bigr|_1^2<\infty. \tag{5.4}
$$

Next, we will verify that the random dynamical system can be generated by the mild solution of equation (1.3).

It follows from Theorem 4.2 that, for $t_0\in\mathbb{R}$, $\phi(t,\omega;t_0,\phi_0)$ is the unique mild solution of the equation:

$$
\begin{aligned}
\phi(t;t_0) &= S(t)\phi_0-\int_0^t S(t-s)B\bigl(\phi(s)\bigr)\,ds-\int_0^t S(t-s)N\bigl(\phi(s)\bigr)\,ds \\
&\quad -\int_0^t S(t-s)R\bigl(\phi(s)\bigr)\,ds+\int_0^t S(t-s)\Phi\,dB^H(s).
\end{aligned} \tag{5.5}
$$

By the change of variable $\phi(t,\omega;t_0)=\psi(t,\omega;t_0)+Z(t,\omega)$, it follows that $\psi(t,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega))$ is the solution of the following integral equation:

$$
\begin{aligned}
\psi(t) &= S(t)\bigl(\phi_0-Z(\theta_{t_0}\omega)\bigr)-\int_0^t S(t-s)B\bigl(\psi(s)+Z(s)\bigr)\,ds \\
&\quad -\int_0^t S(t-s)N\bigl(\psi(s)+Z(s)\bigr)\,ds-\int_0^t S(t-s)R\bigl(\psi(s)+Z(s)\bigr)\,ds.
\end{aligned}
$$

Thus, $\psi$ is the weak solution of the following differential equation with random coefficients:

$$
\begin{cases}
\frac{d\psi}{dt}+A(\psi)+B(\psi+Z)+N(\psi+Z)+R(\psi+Z)=0, \\
\psi(t_0)=\phi_0-Z(\theta_{t_0}\omega).
\end{cases} \tag{5.6}
$$

Define a continuous map:

$$
\varphi(t,\omega,\phi_0)=\psi\bigl(t,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega)\bigr)+Z(\theta_t\omega),\quad \forall(t,\omega,\phi_0)\in\mathbb{R}\times\Omega\times H.
$$

Then it can be verified that the measurability of $\varphi$ follows from the continuous dependence of the initial values, and the cocycle property follows from the uniqueness of the solution. Hence, the random dynamical system can be generated by the mild solution $\varphi$ of equation (4.1).

Next, we will show the following two important lemmas, which give the existence of the absorbing set in the spaces $H$ and $H^1$, respectively. For simplicity of presentation, we introduce the following condition:

(Hyper-5) $\Gamma(2H-1)\cdot(\beta_D(4H-1)\zeta(4H-1)-\zeta(8H-2)+|\Phi_2|^2_{\mathscr{L}(V_2)}\hat{\lambda}_1^{-2H}) < \frac{\lambda_1}{4C_1^2}$.

**Lemma 5.1** *Assume that the conditions* (Hyper-2), (Hyper-4), *and* (Hyper-5) *are satisfied. Then for* $H\in(\frac{1}{2},1)$, *there exist random radii* $\rho_H(\omega)$ *and* $\rho_1(\omega)$ *such that, for any constant* $M>0$, *there exists* $t_2(\omega)<-1$, *such that, for any* $t_0<t_2$ *and* $|\phi_0|<M$,

$$\left|\psi\left(t,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega)\right)\right|^2 \le \rho_H(\omega), \quad \forall t\in[-1,0], \tag{5.7}$$

$$\left|\phi(t,\omega;t_0,\phi_0)\right|^2 \le \rho_H(\omega), \quad \forall t\in[-1,0], \tag{5.8}$$

$$\int_{-1}^0 \left|\psi(t)\right|_V^2\,dt \le \rho_1(\omega), \qquad \int_{-1}^0 \left|\psi(t)+Z(t)\right|_V^2\,dt \le \rho_1(\omega). \tag{5.9}$$

*Proof* Firstly, we show that both $|\phi(t)|^2$ and $|\psi(t)|^2$ are bounded in the space $H$. Similar to the argument in Lemma 4.4, we have

$$\frac{d}{dt}|\psi|^2+\left(\frac{1}{2}-2(C_1C_2+r_1)\right)|\psi|_V^2+\left(\lambda-2\frac{C_1|Z|_1^2}{C_2}\right)|\psi|^2 \le h_2, \tag{5.10}$$

where

$$h_2=\left(2\frac{C_1}{C_2}|Z|_1^2+\frac{2}{\lambda_1^{\frac{1}{2}}}\right)|Z|^2+2C_1C_2|Z|_1^2+\frac{\mu_0^2}{2r_1\epsilon^\alpha\lambda_1^{\frac{1}{2}}}|Z|_1^2.$$

Choosing

$$C_2\in\left(2C_1\lambda^{-1}\Gamma(2H-1)\cdot\left(\beta_D(4H-1)\zeta(4H-1)-\zeta(8H-2)+|\Phi_2|^2_{\mathscr{L}(V_2)}\hat{\lambda}_1^{-2H}\right),\frac{1}{2C_1}\right),$$

let $r_1$ be small enough such that the following inequality holds:

$$\frac{d}{dt}|\psi|^2+\left(\lambda-2\frac{C_1|Z|_1^2}{C_2}\right)|\psi|^2 \le h_2. \tag{5.11}$$

By the Gronwall lemma, it follows that, for any $t\in[-1,0]$ and $t_0<-1$,

$$\begin{aligned}\left|\psi(t)\right|^2 &\le \left|\psi(t_0)\right|^2 e^{-\int_{t_0}^t(\lambda-2\frac{C_1|Z(s)|_1^2}{C_2})\,ds}+\int_{t_0}^t h_2(s_1)e^{-\int_{s_1}^t(\lambda-2\frac{C_1|Z(s_2)|_1^2}{C_2})\,ds_2}\,ds_1\\ &\le \left|\psi(t_0)\right|^2 e^{-\int_{t_0}^0(\lambda-2\frac{C_1|Z(s)|_1^2}{C_2})\,ds}+\int_{t_0}^0 h_2(s_1)e^{-\int_{s_1}^0(\lambda-2\frac{C_1|Z(s_2)|_1^2}{C_2})\,ds_2}\,ds_1.\end{aligned} \tag{5.12}$$

Applying the ergodic theorem gives

$$\lim_{t_0\to-\infty}\frac{1}{-t_0}\int_{t_0}^0\left|Z(s)\right|_1^2\,ds=\mathbb{E}\left|Z(\omega)\right|_1^2. \tag{5.13}$$

Let $r_2$ be small enough such that

$$\frac{C_1}{C_2}\Gamma(2H-1)\cdot\big(\beta_D(4H-1)\zeta(4H-1)-\zeta(8H-2)+|\Phi_2|^2_{\mathscr{L}(V_2)}\hat{\lambda}_1^{-2H}\big)<\frac{\lambda}{2}-\frac{r_2}{2}. \quad (5.14)$$

Then there exists $t_1(\omega)<-1$ such that, for any $t_0<t_1$ and $t\in[-1,0]$,

$$\big|\psi(t)\big|^2\le e^{(1+t_0)r_2}|\phi_0|^2+\int_{t_0}^0 e^{(1+t_0)r_2}h_2(s)\,ds. \quad (5.15)$$

Noticing that the term $h_2$ has at most polynomial growth as $t_0\to-\infty$ for P-a.s. $\omega\in\Omega$, we derive

$$\int_{t_0}^0 h_2(s)e^{(1+s)r_2}\,ds\le\int_{-\infty}^0 h_2(s)e^{(1+s)r_2}\,ds\le\infty,\quad \text{P-a.s.}$$

Let

$$\rho_H=4\int_{-\infty}^0 h_2(s)e^{(1+s)r_2}\,ds+2\sup_{t\in[-1,0]}\big|Z(t)\big|^2.$$

Then there exists $t_2(\omega)<t_1(\omega)<-1$ such that, for any $|\phi_0|\le M$, $t_0<t_2$, and $t\in[-1,0]$,

$$\big|\psi\big(-1,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega)\big)\big|^2\le 2\int_{-\infty}^0 h_2(s)e^{(1+s)r_2}\,ds$$

and

$$\big|\phi(-1,\omega;t_0,\phi_0)\big|^2\le 2\big|\psi\big(-1,\omega;t_0,\phi_0-Z(\theta_{t_0}\omega)\big)\big|^2+2\sup_{t\in[-1,0]}\big|Z(t)\big|^2\le\rho_H(\omega).$$

Next, we prove that both $\int_{-1}^0|\psi(t)|_V^2\,dt$ and $\int_{-1}^0|\phi(t)|_V^2\,dt$ are bounded. Integrating both sides of equation (5.10) over $[-1,0]$ leads to

$$\begin{aligned}&\big|\psi(0)\big|^2-\big|\psi(-1)\big|^2+c_{11}^{-1}\int_{-1}^0\big|\psi(t)\big|_V^2\,dt\\&\quad\le\int_{-1}^0 h_2(t)\,dt+\int_{-1}^0 2\frac{C_1}{C_2}\big|Z(t)\big|_1^2\cdot\big|\psi(t)\big|^2\,dt.\end{aligned} \quad (5.16)$$

It follows that, for $t_0<t_2$,

$$\int_{-1}^0\big|\psi(t)\big|_V^2\,dt\le c_9\bigg(\int_{-1}^0 h_2(t)\,dt+\frac{2C_1\rho_H}{C_2}\int_{-1}^0\big|Z(t)\big|_1^2\,dt+\big|\psi(-1)\big|^2\bigg)\triangleq C(\omega).$$

Similarly,

$$\begin{aligned}&\int_{-1}^0\big|\psi(t)+Z(t)\big|_V^2\,dt\\&\quad\le 2c_{11}\bigg(\int_{-1}^0 h_2(t)\,dt+\frac{2C_1\rho_H}{C_2}\int_{-1}^0\big|Z(t)\big|_1^2\,dt+\big|\psi(-1)\big|^2\bigg)+2\int_{-1}^0\big|Z(t)\big|_V^2\,dt\triangleq\widetilde{C}(\omega).\end{aligned}$$

Denote $\rho_1(\omega)=\max\{C(\omega),\widetilde{C}(\omega)\}$. Thus, the proof is completed. □

**Lemma 5.2** *Assume that the conditions* (Hyper-2), (Hyper-4), *and* (Hyper-5) *are satisfied. Then for* $H \in (\frac{1}{2}, 1)$, *there exists a random radius* $\rho_2(\omega)$ *such that, for any* $M > 0$, *there exists* $t_2(\omega) < -1$ *such that, for* $|\phi_0| < M$, $t_0 < t_2$ *and* $t \in [-\frac{1}{2}, 0]$, *the following inequalities hold* $P$*-a.s.*:

$$\left|\psi\left(t, \omega; t_0, \phi_0 - Z(\theta_{t_0}\omega)\right)\right|_1^2 \le \rho_2(\omega), \tag{5.17}$$

$$\left|\phi(t, \omega; t_0, \phi_0)\right|_1^2 \le \rho_2(\omega). \tag{5.18}$$

*Proof* By integrating equation (5.6) with $-\triangle v$ over $\mathcal{O}$, it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|\psi|_1^2 + |v|_3^2 + |\theta|_2^2 \le{}& \left|b(\psi + Z, \psi + Z, \triangle\psi)\right| - \left\langle N(\psi + Z), -\triangle\psi\right\rangle \\ &- \left\langle R\left(\psi(t) + z(t)\right), -\triangle\psi\right\rangle + \left\langle G(x), -\triangle\psi\right\rangle.\end{aligned} \tag{5.19}$$

By Gagliardo-Nirenberg's inequality and Young's inequality, we have

$$\begin{aligned}&\left|b_1(v + Z_1, v + Z_1, \triangle v)\right| \\ &\quad \le C|v + Z_1|^{1/2} \cdot |v + Z_1|_2^{1/2} \cdot |v + Z_1|_1 \cdot |v|_2 \\ &\quad \le C|v + Z_1|^{1/2} \cdot |v + Z_1|_1 \cdot |v|_2^{3/2} + C|v + Z_1|^{1/2} \cdot |v + Z_1|_1 \cdot |Z_1|_2^{1/2} \cdot |v|_2 \\ &\quad \le \frac{1}{8}|v|_2^2 + 64C^4|v + Z_1|^2 \cdot |v + Z_1|_1^4 + \frac{1}{8}|v|_2^2 + 2C^2|v + Z_1| \cdot |v + Z_1|_1^2 \cdot |Z_1|_2\end{aligned}$$

and

$$\begin{aligned}&\left|b_2(v + Z_1, \eta + Z_2, \triangle\eta)\right| \\ &\quad \le C|v + Z_1|^{1/2} \cdot |\eta + Z_2|_2^{1/2} \cdot |\eta + Z_2|_1 \cdot |\eta|_2 \\ &\quad \le C|v + Z_1|^{1/2} \cdot |\eta + Z_2|_1 \cdot |\eta|_2^{3/2} + C|v + Z_1|^{1/2} \cdot |\eta + Z_2|_1 \cdot |Z_2|_2^{1/2} \cdot |\eta|_2 \\ &\quad \le \frac{1}{8}|\eta|_2^2 + 64C^4|v + Z_1|^2 \cdot |\eta + Z_2|_1^4 + \frac{1}{8}|\eta|_2^2 + 2C^2|v + Z_1| \cdot |\eta + Z_2|_1^2 \cdot |Z_2|_2.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|b(\psi + Z, \psi + Z, \triangle\psi)\right| \\ &\quad \le \left|b_1(v + Z_1, v + Z_1, \triangle v)\right| + \left|b_2(v + Z_1, \eta + Z_2, \triangle\eta)\right| \\ &\quad \le \frac{1}{4}|\psi|_2^2 + 64C^4|\psi + Z|^2 \cdot |\psi + Z|_1^4 + 2C^2|\psi + Z| \cdot |\psi + Z|_1^2 \cdot |Z|_2.\end{aligned}$$

Finally, we estimate the following two terms in equation (5.19):

$$\begin{aligned}-\left\langle N(\psi + Z), \triangle\psi\right\rangle &= -\left\langle N(v + Z_1), \triangle v\right\rangle \le \mu_0\epsilon^{-\frac{\alpha}{2}}\int_D \left|e_{ij}(v + Z_1)e_{ij}(\triangle v)\right| dx \\ &\le \mu_0\epsilon^{-\frac{\alpha}{2}}|v + Z_1|_1 \cdot |v|_3 \le \frac{1}{4}|v|_3^2 + \frac{\mu_0^2}{\epsilon^\alpha}|v + Z_1|_1^2 \le \frac{1}{4}|v|_3^2 + \frac{\mu_0^2}{\epsilon^\alpha}|\psi + Z|_1^2\end{aligned}$$

and

$$-\left\langle R(\psi + Z), \triangle\psi\right\rangle = -\left\langle e_2(\eta + Z_2), \triangle v\right\rangle \le \frac{1}{2}|v|_2^2 + \frac{1}{2}|\eta + Z_2|^2 \le \frac{1}{2}|\psi|_2^2 + \frac{1}{2}|\psi + Z|^2.$$

Denote

$$h_3(t) = 2\left(64C^4|\psi+Z|^2\cdot|Z|_1^4 + 2C^2|\psi+Z|\cdot|\psi+Z|_1^2\cdot|Z|_2 + \frac{\mu_0^2}{\epsilon^\alpha}|\psi+Z|_1^2 + \frac{1}{2}|\psi+Z|^2\right)$$

and

$$h_4(t) = 128C^4|\psi+Z|^2\cdot|\psi|_1^2.$$

It follows from $\lambda_1 > 4$ (Lemma 2.3 in [17]) that

$$\frac{3}{4}|\nu|_3^2 + |\theta|_2^2 \geq \frac{3}{4}\lambda_1|\nu|_2^2 + |\theta|_2^2 \geq |\nu|_2^2 + |\theta|_2^2 = |\psi|_2^2. \tag{5.20}$$

Then the inequality (5.19) can be rewritten as the following inequality:

$$\frac{d}{dt}\left|\psi(t)\right|_1^2 + \frac{1}{4}\left|\psi(t)\right|_2^2 \leq h_3(t) + h_4(t)\left|\psi(t)\right|_1^2. \tag{5.21}$$

Thus,

$$\frac{d}{dt}\left|\psi(t)\right|_1^2 \leq h_3(t) + h_4(t)\left|\psi(t)\right|_1^2. \tag{5.22}$$

By the variation of constant formula, it follows from equation (5.22) that, for any $-1 \leq s \leq t \leq 0$,

$$\begin{aligned}\left|\psi(t)\right|_1^2 &\leq \left|\psi(s)\right|_1^2\cdot e^{\int_s^t h_4(s_1)\,ds_1} + e^{\int_s^t h_4(s_1)\,ds_1}\cdot\int_s^t h_3(s_2)e^{-\int_s^{s_2} h_4(s_1)\,ds_1}\,ds_2 \\ &\leq \left(\left|\psi(s)\right|_1^2 + \int_{-1}^0 h_3(s_2)\,ds_2\right)\cdot e^{\int_{-1}^0 h_4(s_1)\,ds_1}.\end{aligned} \tag{5.23}$$

Integrating inequality (5.23) with respect to $s$ over $[-1,t]$, we obtain

$$(1+t)\left|\psi(t)\right|_1^2 \leq \left(\int_{-1}^0\left|\psi(s)\right|_1^2\,ds + \int_{-1}^0 h_3(s)\,ds\right)\cdot e^{\int_{-1}^0 h_4(s)\,ds}. \tag{5.24}$$

Since all the terms $\int_{-1}^0 h_3(s)\,ds$, $\int_{-1}^0 h_4(s)\,ds$ and $\int_{-1}^0|\psi(s)|_1^2\,ds$ are bounded as $t_0 \to -\infty$. Therefore, for any $t_0 < t_2$ and $t \in [-\frac{1}{2},0]$, we have

$$\left|\psi(t)\right|_1^2 \leq C(\omega). \tag{5.25}$$

Finally, we will prove that the second inequality (5.17) holds. There exists a random radius $\rho_2(\omega)$ such that

$$\begin{aligned}\left|\phi(t,\omega;t_0,\phi_0)\right|_1^2 &\leq 2\left|\psi\left(t,\omega;t_0,\phi_0 - Z(\theta_{t_0}\omega)\right)\right|_1^2 + \sup_{t\in[-\frac{1}{2},0]}\left|Z(t)\right|_1^2 \\ &\leq \rho_2(\omega), \quad \forall t_0 < t_2, t \in \left[-\frac{1}{2},0\right].\end{aligned} \tag{5.26}$$

Especially for $t = 0$, it follows that

$$\left|\phi(0,\omega;t_0,\phi_0)\right|_1^2 \leq \rho_2(\omega), \quad \forall t_0 < t_2.$$

Thus, the proof has been completed. □

Since $\dot{H}^1$ is compactly embedded in $H$, then it follows from Lemmas 5.1 and 5.2 that there exists a compact random absorbing set in the space $H$. So we get the following existence of a random attractor for equation (1.3) from Theorem 2.1.

**Theorem 5.1** *Assume that the conditions* (Hyper-2), (Hyper-4), *and* (Hyper-5) *are satisfied. Then for* $H \in (\frac{1}{2}, 1)$, *the stochastic modified Boussinesq approximate equation* (1.3) *possesses a random attractor.*

**Remark 5.1** Since the computation for the regularity is different for $H \in (\frac{1}{4}, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$, the conditions (Hyper-4) and (Hyper-5) are different from that in [16]. This shows that the Hurst parameter $H$ determines the conditions which ensure the regularity of the stochastic convolution and the existence of a random attractor generated by the mild solution for equation (1.3). If the temperature variable $\theta = 0$, then the result in the present paper and [16] will reduce to that in [17] and [4], respectively.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors read and approved the final manuscript.

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