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Periodic solutions of second order difference systems with even potential

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Abstract
In this article, we study the multiplicity and minimality of periodic solutions to difference systems, which are globally superquadratic or subquadratic. A new technique is given to detect the minimal period of periodic solutions to autonomous systems. Some weaker conditions than a globally subquadratic condition are obtained to guarantee the existence of periodic solutions with prescribed minimal period to autonomous system.

Keywords: difference systems; periodic solution; minimal period; superquadratic; subquadratic

1 Introduction

On one hand, difference equations have been widely used to describe real-life situations in computer science, economics, neural network, ecology, cybernetics, *etc.* On the other hand, they are also natural consequences of the discretization of differential equations. So it is worthwhile to explore this topic. Many tools are used to study the existence of all kinds of solutions to discrete systems. A powerful tool is critical point theory, which firstly was introduced by Guo and Yu in 2003 to study the existence of periodic solutions to a difference system (*cf.* [1–3]). Since then, the study of discrete dynamic systems has got considerable development. We refer to boundary value problems (*cf.* [4, 5]), periodic solutions (*cf.* [6, 7]), homoclinic orbits (*cf.* [8, 9]), and heteroclinic orbits (*cf.* [10, 11]).

As is well known, the minimal periodic problem is an important but difficult problem. As far as the author knows, the study of solutions with a prescribed minimal period began in 2004. In that year, by estimating the energy of a variational functional, Yu *et al.* (*cf.* [12]) studied the existence of subharmonic solutions with a prescribed minimal period to a discrete forced pendulum equation. More recently, by making use of the Clark dual method, Bin (*cf.* [13]), Long (*cf.* [14]) and Long *et al.* (*cf.* [15]) studied the existence and multiplicity of periodic solutions with a prescribed minimal period to difference systems. Because of the lack of methods, results in this field are scarce.

It is well known that the Nehari manifold has been introduced by Nehari in 1960 (*cf.* [16, 17]) and developed by Szulkin and Weth in 2010 (*cf.* [18]). It has been used widely to study the existence of ground state solutions to ordinary differential systems, partial differential systems, and difference systems (*cf.* [19–21]). A ground state solution is a solution which possesses the minimal energy of all solutions. Since such a minimality can be used to prove the minimal periods of solutions, it has been used to study the existence of period solutions with prescribed minimal period to ordinary differential equations (*cf.* [22–25]).

Motivated by the above references, in this paper, one attempts to make use of a Nehari manifold to study the multiplicity and minimality of periodic solutions to difference systems. When the systems are globally superquadratic or subquadratic, by restricting our discussion to the Nehari manifold, firstly, we study the existence of multiple periodic solutions to nonautonomous systems; secondly, we study the existence of periodic solutions with a prescribed minimal period to autonomous systems. Also, some subquadratic conditions, which are weaker than the globally subquadratic condition, are obtained to guarantee the existence of periodic solutions with prescribed minimal period to autonomous system.

For convenience, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ the sets of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$ with $a \leq b$, define $Z[a,b] = \{a, a+1, \dots, b\}$. For $m \in \mathbb{N}$, denote by $\mathbb{R}^m$ the Euclidean space with the usual inner product $(\cdot,\cdot)$ and norm $|\cdot|$.

Consider the nonautonomous difference system

$$\triangle^2 x_{n-1} + f(n, x_n) = 0, \quad n \in \mathbb{Z}, \tag{1}$$

where $x_n \in \mathbb{R}^m$, $\triangle$ is a difference operator defined by $\triangle x_n = x_{n+1} - x_n$ and $\triangle^2 x_n = \triangle(\triangle x_n)$.

Assume that:

(F1) there exists an even function $F \in C^2(\mathbb{R} \times \mathbb{R}^m, \mathbb{R})$ such that $f(t,z)$ is the gradient of $F(t,z)$ with respect to z, *i.e.*,

$$F(-t,-z) = F(t,z), \qquad f(t,z) = \nabla_z F(t,z), \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^m;$$

(F2) there exists a positive integer $T > 0$ such that

$$F(t+T,z) = F(t,z), \quad \forall (t,z) \in \mathbb{R} \times \mathbb{R}^m;$$

(F3) there exists a $\alpha > 1$ such that

$$0 < \alpha\big(f(t,z),z\big) \leq \big(f'(t,z)z,z\big), \quad \forall z \in \mathbb{R}^m \setminus \{0\},$$

where $f'(t,\cdot)$ denotes the Hermite matrix of $F(t,\cdot)$;

(F4) there exists a $\beta \in (0,1)$ such that

$$0 < \big(f'(t,z)z,z\big) \leq \beta\big(f(t,z),z\big), \quad \forall z \in \mathbb{R}^m \setminus \{0\}.$$

Remark 1 If F satisfies (F1), (F2), and (F3), without loss of generality, we can assume that $F(t,0) = 0$ for all $t \in \mathbb{R}$. Otherwise, there exists a twice continuously differentiable function $g(t)$ such that $F(t,0) = g(t)$. Let $\widehat{F}(t,z) = F(t,z) - g(t)$. Then $\widehat{F}$ also satisfies (F1), (F2), and (F3) with F replaced by $\widehat{F}$. Similarly, if F satisfies (F1), (F2), and (F4), we can assume that $F(t,0) = 0$ for all $t \in \mathbb{R}$.

For a nonautonomous system, we have the following two results.

Theorem 1 *Suppose that F satisfies* (F1), (F2), *and* (F3). *Then* (1) *has at least* $m[(T-1)/2]-1$ *distinct pairs of different T-periodic solutions.*

Theorem 2 *Suppose that F satisfies* (F1), (F2), *and* (F4). *Then* (1) *has at least* $m[(T-1)/2]-1$ *distinct pairs of different T-periodic solutions.*

Consider the autonomous difference system

$$\triangle^2 x_{n-1} + f(x_n) = 0, \quad n \in \mathbb{Z}. \tag{2}$$

Assume that F satisfies

(F5) there exists an even function $F \in C^2(\mathbb{R}^m, \mathbb{R})$ such that $f(z)$ is the gradient of $F(z)$, *i.e.*,

$$F(-z) = F(z), \qquad f(z) = \nabla_z F(z), \quad \forall z \in \mathbb{R}^m;$$

(F6) there exists a $\alpha' > 1$ such that

$$0 < \alpha'\bigl(f(z), z\bigr) \le \bigl(f'(z)z, z\bigr), \quad \forall z \in \mathbb{R}^m \setminus \{0\},$$

where $f'(\cdot)$ denotes the Hermite matrix of $F(\cdot)$;

(F7) there exists a $\beta' \in (0,1)$ such that

$$0 < \bigl(f'(z)z, z\bigr) \le \beta'\bigl(f(z), z\bigr), \quad \forall z \in \mathbb{R}^m \setminus \{0\}.$$

Remark 2 If F satisfies (F5) and (F6) (respectively, (F5) and (F7)), without loss of generality, we can assume that $F(0) = 0$.

For an autonomous system, we have the following two results.

Theorem 3 *Suppose that F satisfies* (F5) *and* (F6). *Then, for any integer $P > 1$,* (2) *possesses at least a periodic solution with minimal period P.*

Theorem 4 *Suppose that F satisfies* (F5) *and* (F7). *Then, for any integer $P > 1$,* (2) *possesses at least a periodic solution with minimal period P.*

Now, one weakens the conditions (F5) and (F7). Assume that F satisfies the following conditions:

(F8) there exists an even function $F \in C^1(\mathbb{R}^m, \mathbb{R})$ such that $f(z)$ is the gradient of $F(z)$, *i.e.*,

$$F(-z) = F(z), \qquad f(z) = \nabla_z F(z), \quad \forall z \in \mathbb{R}^m;$$

(F9) there exist constants $G_1 > 0$, $0 < \gamma < 2$ such that for all $x \in \mathbb{R}^m$ and $|x| \ge G_1$,

$$0 < \bigl(f(x), x\bigr) \le \gamma F(x);$$

(F10) there exist constants $0 < G_2 < 1$, $M_1 > 0$ and $0 < \gamma' < 2$ such that for all $x \in \mathbb{R}^m$ and $|x| < G_2$,

$$F(z) \geq M_1|z|^{\gamma'}.$$

Remark 3 Assume that F satisfies (F8) and (F9). Then there exist some positive constants M_2 and M_3 such that, for all $x \in \mathbb{R}^m$,

$$F(x) \leq M_2|x|^{\gamma} + M_3.$$

Theorem 5 *Suppose that F satisfies* (F8), (F9), *and* (F10). *Then, for any integer $P > 1$,* (2) *possesses at least a periodic solution with minimal period P.*

The rest of this paper is divided into two parts. In Section 2, we study the multiplicity of periodic solutions to a nonautonomous system. In Section 3, firstly, we study the existence of periodic solutions with a prescribed minimal period to an autonomous system, which is globally superquadratic or subquadratic; secondly, some conditions, which are weaker than the globally subquadratic condition, are given to guarantee the existence of periodic solutions with prescribed minimal period to autonomous system.

2 Nonautonomous difference system

2.1 Preparations

By a similar argument to [2], we can build the space E_T

$$E_T = \{x = (\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots) \mid x_{n+T} = x_n, x_n \in \mathbb{R}^m, \forall n \in \mathbb{Z}\}.$$

E_T is a Hilbert space equipping with the following norm and inner product

$$\|x\| := \left(\sum_{n=1}^{T}|x_n|^2\right)^{\frac{1}{2}}, \qquad \langle x, y\rangle := \sum_{n=1}^{T}(x_n, y_n), \quad \forall x, y \in E_T.$$

It is easy to check that E_T is linearly homeomorphic to $\mathbb{R}^{mT}$, which can also be identified with $\mathbb{R}^{mT}$.

For any $s > 1$, we can also define L^s norm on E_T as follows:

$$\|x\|_s := \left(\sum_{n=1}^{T}|x_n|^s\right)^{\frac{1}{s}}.$$

Obviously, $\|\cdot\| = \|\cdot\|_2$. Since the space E_T is a finite-dimensional space, all norms defined on it are equivalent. Hence, there exists a $C_{1,s} > 1$ such that

$$\frac{1}{C_{1,s}}\|x\|_s \leq \|x\| \leq C_{1,s}\|x\|_s, \quad \forall x \in E_T.$$

The variational functional corresponding to (1) defined on E_T is

$$J(x) = \sum_{n=1}^{T}\left[\frac{1}{2}|\triangle x_n|^2 - F(n, x_n)\right] = \frac{1}{2}\langle Ax, x\rangle - \sum_{n=1}^{T}F(n, x_n),$$

where $x = (x_1^\tau, x_2^\tau, \ldots, x_T^\tau)^\tau$, $A = P^{-1}CP$, $C = \operatorname{diag}(B, B, \ldots, B)_{mT\times mT}$. Here τ denotes the transposition of a vector and B, P are matrices defined in [2].

Since E_T is linearly homeomorphic to $\mathbb{R}^{mT}$, it follows from (F1) that J can be viewed as a twice continuously differentiable functional defined on a finite-dimensional Hilbert space. Thus, for any $x, y, z \in E_T$, one has

$$\langle J'(x), y\rangle = \langle Ax, y\rangle - \sum_{n=1}^{T} \bigl(f(n, x_n), y_n\bigr),$$

$$\langle J''(x)z, y\rangle = \langle Az, y\rangle - \sum_{n=1}^{T} \bigl(f'(n, x_n)z_n, y_n\bigr).$$

It is easy to check that the critical points of J are T-periodic solutions of (1).

Similarly to reference [12], the eigenvalues of B can be given by

$$\lambda_k = 4\sin^2\frac{k\pi}{T}, \quad k = 0, 1, 2, \ldots, T-1.$$

By matrix theory, A has T eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{T-1}$, each of multiplicity m. Clearly, 0 is an eigenvalue of A and set $\eta = (\eta_1, \eta_2, \ldots, \eta_m)^\tau$ the eigenvector associated to 0. If T is even, $\lambda_{T/2} = 4$ is also an eigenvalue of A and set $\eta' = (\eta'_1, \eta'_2, \ldots, \eta'_m)^\tau$, the eigenvector associated to $\lambda_{T/2}$. For any $k \in Z[1, \lfloor (T-1)/2\rfloor]$, where $\lfloor\cdot\rfloor$ denotes the upper integral function, set

$$\eta_n^{k,1} = a_k \cos\frac{2kn\pi}{T}, \qquad \eta_n^{k,2} = b_k \sin\frac{2kn\pi}{T},$$

where $a_k, b_k \in \mathbb{R}^m$. Then $\eta_n^{k,1}$, $\eta_n^{k,2}$ are eigenvectors of A corresponding to eigenvalue λ_k. Denote by W_i ($i = 1, 2, 3, 4$) the spaces defined as follows:

$$W_1 = \operatorname{span}\{\eta\}, \qquad W_2 = \operatorname{span}\left\{\eta_n^{k,1}, k = 1, 2, \ldots, \left\lfloor\frac{T-1}{2}\right\rfloor\right\},$$

$$W_3 = \operatorname{span}\left\{\eta_n^{k,2}, k = 1, 2, \ldots, \left\lfloor\frac{T-1}{2}\right\rfloor\right\}, \qquad W_4 = \operatorname{span}\{\eta'\}.$$

Then $\dim(W_1) = m$, $\dim(W_2) = \dim(W_3) = m\lfloor (T-1)/2\rfloor$ and $\dim(W_4) = m$ if T is even. Also, we have

$$E_T = W_1 \oplus W_2 \oplus W_3 \oplus W_4, \quad \text{when } T \text{ is even};$$

$$E_T = W_1 \oplus W_2 \oplus W_3, \quad \text{when } T \text{ is odd}.$$

Thus, for any $x \in E_T$, x_n has the form of

$$x_n = a + \sum_{k=1}^{\frac{T}{2}-1}\left(a_k \cos\frac{2kn\pi}{T} + b_k \sin\frac{2kn\pi}{T}\right) + (-1)^n b, \quad \text{when } T \text{ is even},$$

$$x_n = a + \sum_{k=1}^{\frac{T-1}{2}}\left(a_k \cos\frac{2kn\pi}{T} + b_k \sin\frac{2kn\pi}{T}\right), \quad \text{when } T \text{ is odd},$$

where a, b, a_k, b_k ($k = 1, 2, \ldots, \lfloor (T-1)/2\rfloor$) are all constant vectors of $\mathbb{R}^m$.

Now, we define a subspace of E_T as follows:

$$\widetilde{E}_T = \{x \in E_T \mid x_n \text{ is odd in } n\}.$$

Obviously, $\widetilde{E}_T = W_3$. Thus $\dim(\widetilde{E}_T) = m\lfloor (T-1)/2 \rfloor$. For any $x \in \widetilde{E}_T$, no matter T is even or odd, x_n has a Fourier expansion

$$x_n = \sum_{k=1}^{\lfloor \frac{T-1}{2} \rfloor} a_k \sin \frac{2kn\pi}{T}.$$

Consider the following eigenvalue problem:

$$Ax = \lambda x, \quad x \in \widetilde{E}_T. \tag{3}$$

Problem (3) has $\lfloor (T-1)/2 \rfloor$ eigenvalues, each of them of multiplicity m. The eigenvalues of (3) are $\lambda_k = 4\sin^2(k\pi/T)$, where $k = 1, 2, \ldots, \lfloor (T-1)/2 \rfloor$. Obviously, the smallest eigenvalue is λ_1 and the largest eigenvalue is $\lambda_{\lfloor (T-1)/2 \rfloor}$, which is denoted by $\lambda_{\max}$.

Now we give a useful lemma. Since the proof is standard, we omit it.

Lemma 1 *If x is a critical point of J restricted on $\widetilde{E}_T$, then x is also a critical point of J on E_T.*

At the end of this subsection, two useful lemmas are given.

Suppose that H be a real Banach space and M is a closed symmetric C^1-submanifold of H with $0 \notin M$. Suppose that $\phi \in C^1(M, \mathbb{R})$.

Lemma 2 [26] *Suppose that ϕ is even and bounded below. Define*

$$c_j := \inf_{A \in \Gamma_j} \sup_{x \in A} \phi(x), \quad j = 1, 2, \ldots,$$

where $\Gamma_j := \{A \subset M : A = -A, A \subset H \setminus \{0\}, A \text{ is compact and } \gamma(A) \geq j\}$. If $\Gamma_k \neq \emptyset$ for some $k \geq 1$ and if ϕ satisfies the $(PS)_c$ for all $c = c_j$, $j = 1, 2, \ldots, k$, then ϕ has at least k distinct pairs of critical points.

Lemma 3 [18] *If ϕ is bounded below and satisfies the (PS) condition, then $c := \inf_M \phi$ is attained and is a critical value of ϕ.*

2.2 Superquadratic case

In this subsection, we consider the existence of multiple periodic solutions of (1), where F satisfies (F1), (F2), and (F3). Arguing similarly to [25], we can prove the following lemma.

Lemma 4 *If F satisfies* (F1), (F2), *and* (F3), *then $0 < (1+\alpha)F(t,x) \leq (f(t,x), x)$ for all $x \in \mathbb{R}^m \setminus \{0\}$. Also,*

$$F(t,x) \leq \overline{M}|x|^{\alpha+1}, \quad \text{when } |x| \leq 1, \qquad F(t,x) \geq \underline{M}|x|^{\alpha+1} \quad \text{when } |x| \geq 1,$$

where $\overline{M} = \max_{\{t \mid 0 \leq t \leq T\}} \max_{|x|=1} F(t,x)$ and $\underline{M} = \min_{\{t \mid 0 \leq t \leq T\}} \min_{|x|=1} F(t,x)$.

Let $h(x) = \langle J'(x), x\rangle$. Define a Nehari manifold $\mathcal{M}$ on $\widetilde{E}_T$ as follows:

$$\mathcal{M} = \{x \in \widetilde{E}_T \setminus \{0\} \mid h(x) = 0\}.$$

By a similar argument to reference [22], one can prove the following lemma.

Lemma 5 *$\mathcal{M}$ is C^1-manifold with dimension $m\lfloor (T-1)/2\rfloor - 1$. If x_0 is a critical point of J restricted on $\mathcal{M}$, then x_0 is also a critical point of J restricted on $\widetilde{E}_T$.*

By Lemmas 1 and 5, to search for periodic solutions of (1), we need find critical points of J restricted on $\mathcal{M}$.

Lemma 6 *Fixing $x \in \widetilde{E}_T \setminus \{0\}$, there exists a unique $t_x > 0$ such that $t_x x \in \mathcal{M}$.*

Proof Fixing $x \in \widetilde{E}_T \setminus \{0\}$, define $\varphi_x(t) := J(tx)$ for $t \in (0, +\infty)$. Obviously, $\varphi_x \in C^2$. It is easy to check that $\varphi_x'(t) = 0$ if and only if $tx \in \mathcal{M}$.

If $0 < t < 1/\|x\|$ is small enough, then $|tx_n| < 1$. It follows from Lemma 4 that $|F(n, tx_n)| \leq \overline{M}|tx_n|^{\alpha+1}$. Thus

$$\varphi_x(t) = \frac{1}{2}\big\langle tx, A(tx)\big\rangle - \sum_{n=1}^{T} F(n, tx_n) \geq \frac{t^2}{2}\lambda_1\|x\|^2 - t^{1+\alpha}\overline{M}C_{1,\alpha+1}^{\alpha+1}\|x\|^{\alpha+1}.$$

Since $\alpha > 1$, there exists a $t_1 > 0$ depending only on $\|x\|$ such that

$$\varphi_x(t) > \lambda_1 t^2\|x\|^2/4, \quad \forall t \in (0, t_1).$$

Denote by $A_0(x) = \{n \in Z[1,T] \mid x_n \neq 0\}$. Obviously, $A_0(x) \neq \varnothing$. If $t > 1/\min\{|x_n| \mid n \in A_0(x)\}$, then $t|x_n| > 1$ for all $n \in A_0(x)$. It follows from Lemma 4 again that

$$\begin{aligned}
\varphi_x(t) &= \frac{1}{2}\big\langle tx, A(tx)\big\rangle - \sum_{n=1}^{T} F(n, tx_n) \\
&\leq \frac{t^2}{2}\lambda_{\max}\|x\|^2 - t^{1+\alpha}\underline{M}\sum_{n\in A_0(x)} |x_n|^{\alpha+1} \\
&\leq \frac{t^2}{2}\lambda_{\max}\|x\|^2 - t^{1+\alpha}\underline{M}C_{1,\alpha+1}^{-(\alpha+1)}\|x\|^{\alpha+1}.
\end{aligned}$$

Since $\alpha > 1$, $\varphi_x(t) < 0$ when $t > 1$ is large enough. Thus the Rolle Mean Value Theorem implies that there exists a $t_x > 0$ such that

$$\varphi_x'(t_x) = 0. \tag{4}$$

Claim: There exists a unique $t_x > 0$ satisfying (4).

Suppose, to the opposite, that there exist $0 < t_1 < t_2$ satisfying (4). By a direct computation, we have

$$\varphi_x'(t) = t\langle x, Ax\rangle - \sum_{n=1}^{T}\big(f(n, tx_n), x_n\big), \qquad \varphi_x''(t) = \langle x, Ax\rangle - \sum_{n=1}^{T}\big(f'(n, tx_n)x_n, x_n\big).$$

For $i = 1, 2$, since $\varphi_x'(t_i) = 0$, then $t_i\langle x, Ax\rangle = \sum_{n=1}^{T}(f(n, t_i x_n), x_n)$. It follows from (F3) that

$$\varphi_x''(t_i) = \frac{1}{t_i^2}\sum_{n=1}^{T}\bigl[\bigl(f(n, t_i x_n), t_i x_n\bigr) - \bigl(f'(n, t_i x_n)t_i x_n, t_i x_n\bigr)\bigr] < 0, \quad i = 1, 2. \tag{5}$$

Thus, there exists a $t_3 \in (t_1, t_2)$ satisfying $\varphi_x(t_3) = \min_{t_1 \le t_3 \le t_2}\varphi_x(t)$. Consequently, $\varphi_x'(t_3) = 0$ and $\varphi_x''(t_3) \ge 0$. However, by a similar argument to (5), $\varphi_x''(t_3) < 0$, which is a contradiction. Thus t_x is unique. □

Since $\varphi_x'(t_x) = 0$ and $\varphi_x''(t_x) < 0$, then $\varphi_x(t_x) = \max_{t\in(0,\infty)}\varphi_x(t)$. Hence $J(tx)$ restricted on $(0, \infty)$ attains its maximum at t_x.

Lemma 7 *J satisfies the (PS) condition on $\mathcal{M}$.*

Proof Assume that $\{x^k\} \subset \mathcal{M}$ is a (PS) sequence for J. Then there exists a $M_4 \ge 0$ such that $|J(x^k)| \le M_4$ for all $m \in \mathbb{N}$ and $J'(x^k) \to 0$ as $k \to \infty$. Set

$$A_1(x^k) = \bigl\{n \in Z[1, T] \mid |x_n^k| \le 1\bigr\}, \qquad A_2(x^k) = \bigl\{n \in Z[1, T] \mid |x_n^k| > 1\bigr\}.$$

Since F is continuous, there exists a $M_5 > 0$ such that

$$\bigl|\underline{M}|x|^{1+\alpha} - F(t, x)\bigr| \le M_5, \quad \forall t \in [0, T], x \in \mathbb{R}^m \text{ with } |x| \le 1.$$

Then

$$\begin{aligned} -M_4 \le J(x^k) &= \frac{1}{2}\langle x^k, Ax^k\rangle - \sum_{n=1}^{T}F(n, x_n^k) \\ &\le \frac{1}{2}\lambda_{\max}\|x^k\|^2 - \sum_{n\in A_1(x^k)}F(n, x_n^k) - \underline{M}\sum_{n\in A_2(x^k)}|x_n^k|^{1+\alpha} \\ &\le \frac{1}{2}\lambda_{\max}\|x^k\|^2 + \sum_{n\in A_1(x^k)}\bigl[\underline{M}|x_n^k|^{1+\alpha} - F(n, x_n^k)\bigr] - \underline{M}C_{1,1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha} \\ &\le \frac{1}{2}\lambda_{\max}\|x^k\|^2 + TM_5 - \underline{M}C_{1,1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha}. \end{aligned}$$

Thus $\underline{M}C_{1,1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha} - 1/2\lambda_{\max}\|x^k\|^2 \le M_4 + TM_5$. Since $\alpha > 1$, $\{\|x^k\|\}$ is bounded. Since $\widetilde{E}_T$ is a finite-dimensional space, there exists a convergent subsequence of $\{x^k\}$. □

Define a new map

$$\begin{aligned} &g : S^1 \to \mathcal{M}, \\ &x \mapsto t_x x. \end{aligned}$$

It follows from Lemma 6 that g is a bijection whose inverse g^{-1} is given by $g^{-1}(x) = x/\|x\|$.

Lemma 8 $C_1 = \inf_{x\in\mathcal{M}}J(x) > 0$.

Proof For any $x \in \mathcal{M}$, since $J(x) = \sup_{t\in(0,+\infty)} J(tx) = \sup_{t\in(0,+\infty)} J(tx/\|x\|)$, it follows that

$$\inf_{x\in\mathcal{M}} J(x) = \inf_{x\in\mathcal{M}} \sup_{t\in(0,+\infty)} J(tx) = \inf_{x\in S^1} \sup_{t\in(0,+\infty)} J(tx).$$

To prove that $C_1 > 0$, one only need to show that $\inf_{x\in S^1} \sup_{t\in(0,+\infty)} J(tx) > 0$.

Arguing similarly to Lemma 6, there exists a $t_4 > 0$, which is independent of x, such that

$$\varphi_x(t) \geq \lambda_1 t^2/4, \quad \forall 0 < t < t_4, \forall x \in S^1.$$

Setting $t = t_4/2$, one gets

$$J\left(\frac{t_4}{2}x\right) = \varphi_x\left(\frac{t_4}{2}\right) \geq \frac{\lambda_1 t_4^2}{16} > 0, \quad \forall x \in S^1.$$

Thus

$$C_1 = \inf_{x\in S^1} \sup_{t\in(0,\infty)} J(tx) \geq \frac{\lambda_1 t_4^2}{16} > 0.$$

□

Proof of Theorem 1 Because of (F2), $\mathcal{M}$ is a closed symmetric manifold and $0 \notin \mathcal{M}$. It follows from Lemma 5 that $\mathcal{M}$ is a C^1 manifold with dimension $m\lfloor (T-1)/2 \rfloor - 1$. By Lemmas 7 and 8, J is bounded from below and satisfies the (PS) condition. It is easy to check that J is even. Then Lemma 2 implies that J has at least $m\lfloor (T-1)/2 \rfloor - 1$ distinct pairs of critical points. Thus (1) possesses at least $m\lfloor (T-1)/2 \rfloor - 1$ distinct pairs of periodic solutions. □

2.3 Subquadratic case

In this subsection, we consider the multiplicity of periodic solutions of (1), where F satisfies (F1), (F2), and (F4). In order to prove Theorem 2, we consider the functional $J_1 = -J$ and the Nehari manifold $\mathcal{M}_1$ is defined as $\mathcal{M}_1 := \{x \in \widetilde{E}_T \setminus \{0\} \mid \langle J_1'(x), x\rangle = 0\}$. Since the technique of the proof of Theorem 2 is just the same as that of Theorem 1, where J and $\mathcal{M}$ are replaced by J_1 and $\mathcal{M}_1$, we omit it here.

3 Autonomous difference equations

3.1 Variational framework

In this section, we consider the existence of periodic solutions with any prescribed minimal period of (2), that is, for any given positive integer P, we search for periodic solutions of (2) with minimal period P.

The argument of this subsection is similar to that in Section 2.1. Let

$$E_P = \left\{x = \{\ldots, x_{-n}, \ldots, x_{-1}, x_0, x_1, \ldots, x_n, \ldots\} \mid x_{n+P} = x_n, x_n \in \mathbb{R}^m, n \in \mathbb{Z}\right\},$$

and equip E_P with the inner product $\langle\cdot,\cdot\rangle_P$ as follows:

$$\langle x, y\rangle_P = \sum_{n=1}^{P} (x_n, y_n), \quad \forall x, y \in E_P.$$

Then E_P is a Hilbert space, which is homeomorphic to $\mathbb{R}^{mP}$. Denote by $\|\cdot\|_P$ the norm introduced by $\langle\cdot,\cdot\rangle_P$.

Define the L^s norm on $\widetilde{E}_P$ by

$$\|x\|_{L^s} = \left(\sum_{n=1}^{P} |x|^s\right)^{\frac{1}{s}}.$$

Then L^s is equivalent to $\|\cdot\|_P$ and there exists $C_{2,s} > 0$ such that

$$\frac{1}{C_{2,s}}\|x\|_{L^s} \leq \|x\|_P \leq C_{2,s}\|x\|_{L^s}, \quad \forall x \in \widetilde{E}_P.$$

The variational functional corresponding to (2) defined on E_P is

$$I(x) = \sum_{n=1}^{P}\left[\frac{1}{2}|\triangle x_n|^2 - F(x_n)\right] = \frac{1}{2}\langle Dx, x\rangle_P - \sum_{n=1}^{P} F(x_n),$$

where $\#D$ is a matrix of order mp. Then I is twice continuously differentiable. The critical points of I are P-periodic solutions of (2).

Define a subspace of E_P as follows:

$$\widetilde{E}_P = \{x_n \in E_P \mid x_n \text{ is odd in } n\}.$$

Then the eigenvalue problem

$$Dx = \lambda x, \quad x \in \widetilde{E}_P,$$

has $\lfloor (P-1)/2 \rfloor$ solutions, each of them of multiplicity m. The smallest eigenvalue is $\overline{\lambda}_1 = 4\sin^2(\pi/P)$ and the largest eigenvalue is $\overline{\lambda}_{\lfloor (P-1)/2 \rfloor}$, which is denoted by $\overline{\lambda}_{\max}$.

Lemma 9 *If x is a critical point of I restricted on $\widetilde{E}_P$, then x is also a critical point of I on the whole space E_P.*

At the end of this subsection, we define a Nehari manifold $\overline{\mathcal{M}}$ on $\widetilde{E}_P$ as follows:

$$\overline{\mathcal{M}} = \left\{x \in \widetilde{E}_P \setminus \{0\} \mid \left\langle I'(x), x\right\rangle_P = 0\right\}.$$

3.2 Superquadratic case

In this subsection, we assume that F satisfies (F5) and (F6). The main target of this subsection is to prove Theorem 3. A similar argument to Section 2.2, one can check the following facts:

(i) $0 < (1+\alpha)F(x) \leq (f(x), x)$ for all $x \in \mathbb{R}^m \setminus \{0\}$. Also,

$$F(x) \leq \overline{M}'|x|^{\alpha+1}, \quad \text{when } |x| \leq 1, \qquad F(x) \geq \underline{M}'|x|^{\alpha+1} \quad \text{when } |x| \geq 1,$$

where $\overline{M}' = \max_{|x|=1} F(x)$ and $\underline{M}' = \min_{|x|=1} F(x)$;

(ii) $\overline{\mathcal{M}}$ is a C^1 manifold;

(iii) critical points of I restricted on $\overline{\mathcal{M}}$ are also critical points of I restricted on $\widetilde{E}_P$;
(iv) for any $x \in \widetilde{E}_P \setminus \{0\}$, there exists a unique t_x such that $t_x x \in \overline{\mathcal{M}}$ and $I(t_x x) = \max_{t\in(0,\infty)} I(tx)$;
(v) I restricted on $\overline{\mathcal{M}}$ satisfies the (PS) condition;
(vi) I restricted on $\overline{\mathcal{M}}$ is bounded from below and $C_2 = \inf_{x\in\overline{\mathcal{M}}} I(x) > 0$.

Proof of Theorem 3 It follows from Lemma 3 that C_2 is a critical value. Denote by $\widetilde{x}$ the critical point of I corresponding to C_2. Then $\widetilde{x}$ is a P-periodic solution of (2). It is easy to check that $\widetilde{x}$ is a nonconstant periodic solution.

Claim: $\widetilde{x}$ has P as its minimal period.

Suppose, to the opposite, that there exists a positive integer $k \geq 2$ such that $\widetilde{x}$ has P/k as its minimal period. Define $\widetilde{y} = \{\widetilde{y}_l \mid l \in Z[1,P]\}$ as follows:

$$\widetilde{y}_l = \widetilde{x}_{\lfloor \frac{l-1}{k} \rfloor + 1}.$$

Since $\widetilde{x} \in \overline{\mathcal{M}}$, then $\widetilde{y} \in \widetilde{E}_P \setminus \{0\}$. Thus there exists a $r_{\widetilde{y}} > 0$ such that $r_{\widetilde{y}}\widetilde{y} \in \overline{\mathcal{M}}$. Then

$$\begin{aligned}
I(r_{\widetilde{y}}\widetilde{y}) &= \sum_{n=1}^{P}\left[\frac{1}{2}|\triangle r_{\widetilde{y}}\widetilde{y}_n|^2 - F(r_{\widetilde{y}}\widetilde{y}_n)\right] \\
&= \frac{1}{2}\sum_{n=1}^{P/k}|\triangle r_{\widetilde{y}}\widetilde{x}_n|^2 - \sum_{n=1}^{P}F(r_{\widetilde{y}}\widetilde{x}_n) \\
&= \frac{1}{2k}\sum_{n=1}^{P}|\triangle r_{\widetilde{y}}\widetilde{x}_n|^2 - \sum_{n=1}^{P}F(r_{\widetilde{y}}\widetilde{x}_n) \\
&< \sum_{n=1}^{P}\left[\frac{1}{2}|\triangle r_{\widetilde{y}}\widetilde{x}_n|^2 - F(r_{\widetilde{y}}\widetilde{x}_n)\right] \\
&= I(r_{\widetilde{y}}\widetilde{x}) \leq I(\widetilde{x}) = \inf_{x\in\overline{\mathcal{M}}} I(x).
\end{aligned}$$

This contradicts $r_{\widetilde{y}}\widetilde{y} \in \overline{\mathcal{M}}$. Hence $\widetilde{x}$ has P as its minimal period. □

3.3 Subquadratic case (I)

In this subsection, we assume that F satisfies (F5) and (F7). A similar argument to Section 2.2, we can prove the following facts:

(I) $0 < (f(x), x) \leq (1+\beta')F(x)$ for all $x \in \mathbb{R}^m \setminus \{0\}$. Also,

$$F(x) \geq \underline{M}'|x|^{\beta'+1}, \quad \text{when } |x| \leq 1, \qquad F(x) \leq \overline{M}'|x|^{\beta'+1}, \quad \text{when } |x| \geq 1,$$

where $\underline{M}'$, $\overline{M}'$ are defined in Section 3.2;
(II) $\overline{\mathcal{M}}$ is a C^1 manifold;
(III) critical points of I restricted on $\overline{\mathcal{M}}$ are critical points of I restricted on $\widetilde{E}_P$;
(IV) for any $x \in \widetilde{E}_P \setminus \{0\}$, there exists a unique t_x such that $t_x x \in \overline{\mathcal{M}}$ and $I_1(t_x x) = \min_{t\in(0,\infty)} I_1(tx)$;
(V) I restricted on $\overline{\mathcal{M}}$ satisfies the (PS) condition.

Lemma 10 *Assume that F satisfies* (F5) *and* (F7). *Then I restricted on $\widetilde{E}_P$ is bounded from below.*

Proof It follows from Fact (I) that $F(x) \leq \overline{M}'|x|^{\beta'+1}$ when $|x| \geq 1$. Since F is continuous, there exists a $M_6 > 0$ such that $F(x) \leq M_6$ when $|x| \leq 1$. Consequently, $F(x) \leq \overline{M}'|x|^{\beta'+1} + M_6$ for all $x \in \mathbb{R}^m$. Thus

$$\begin{aligned} I(x) &= \frac{1}{2}\langle Dx, x\rangle_P - \sum_{n=1}^{P} F(x_n) \geq \overline{\lambda}_1 \|x\|_P^2 - \sum_{n=1}^{P}\bigl[\overline{M}'|x|^{\beta'+1} + M_6\bigr] \\ &\geq \overline{\lambda}_1 \|x\|_P^2 - \overline{M}' C_{2,\beta'+1}^{\beta'+1}\|x\|_P^{\beta'+1} - PM_6. \end{aligned} \tag{6}$$

Since $\beta' \in (0,1)$, there exists a constant C_3 such that $I_1(x) \geq C_3$ for all $x \in \widetilde{E}_P$. □

Because of Lemma 10, I restricted on $\overline{\mathcal{M}}$ is bounded from below. Denote by $C_4 = \inf_{x\in\overline{\mathcal{M}}} I(x)$. A similar argument to Lemma 8, we can prove that $C_4 < 0$.

Proof of Theorem 4 It follows from Lemma 3 that C_4 is a critical value. Denote by $\widehat{x}$ the critical point of I corresponding to C_4. Then $\widehat{x}$ is a nonconstant P-periodic solution of (2). By a similar discussion to the proof of Theorem 3, we can prove that $\widehat{x}$ has P as its minimal period. □

3.4 Subquadratic case (II)

In this subsection, we assume that F satisfies (F8), (F9), and (F10). By a similar argument to Lemma 10, we have the following lemma.

Lemma 11 *Assume that F satisfies* (F8), (F9), *and* (F10). *Then I restricted on $\widetilde{E}_P$ is bounded from below.*

Lemma 12 $C_5 = \inf_{x\in\widetilde{E}_P} I(x) < 0$.

Proof By (F10), $F(z) \geq M_1|z|^{\gamma'}$ if $|x| < G_2$. Then, for any $x \in \widetilde{E}_P$ with $\|x\|_P < G_2$, we have

$$\begin{aligned} I(x) &= \frac{1}{2}\langle Dx, x\rangle_P - \sum_{n=1}^{P} F(x_n) \leq \frac{1}{2}\overline{\lambda}_{\max}\|x\|_P^2 - \sum_{n=1}^{P}\bigl[M_1|x_n|^{\gamma'}\bigr] \\ &\leq \|x\|_P^{\gamma'}\left[\frac{1}{2}\overline{\lambda}_{\max}\|x\|_P^{2-\gamma'} - M_1 C_{3,\gamma'}^{-\gamma'}\right]. \end{aligned}$$

Since $0 < \gamma' < 2$, if $0 < \|x\|_P < \min\{G_2, [M_1 C_{3,\gamma'}^{-\gamma'}/\overline{\lambda}_{\max}]^{1/(2-\gamma')}\}$, then

$$I(x) < -\frac{(M_1 C_{3,\gamma'}^{-\gamma'})^{2/(2-\gamma')}}{2(\overline{\lambda}_{\max})^{1/(2-\gamma')}}.$$

Hence $C_5 = \inf_{x\in\widetilde{E}_P} I(x) < 0$. □

Lemma 13 *I restricted on $\widetilde{E}_P$ satisfies the (PS) condition.*

Proof Assume that $\{x^k\} \subset \widetilde{E}_P$ is a (PS) sequence for I. Then there exists a $M_7 \geq 0$ such that $|I(x^k)| \leq M_7$ for all $k \in \mathbb{N}$ and $I'(x^k) \to 0$ as $k \to \infty$. Then

$$\begin{aligned} M_7 \geq I(x^k) &= \frac{1}{2}\langle x^k, Dx^k\rangle - \sum_{n=1}^{P} F(x_n^k) \\ &\geq \frac{1}{2}\overline{\lambda}_1 \|x^k\|^2 - \sum_{n=1}^{P}(M_2|x^k|^\gamma + M_3) \\ &\geq \frac{1}{2}\overline{\lambda}_1 \|x^k\|^2 - M_2 C_{2,\gamma}^\gamma \|x^k\|^\gamma - M_3 P. \end{aligned}$$

Thus $1/2\overline{\lambda}_1\|x^k\|^2 - M_2 C_{2,\gamma}^\gamma \|x^k\|^\gamma \leq M_3 P + M_7$. Since $\gamma < 2$, it follows that $\{x_k\}$ is bounded. Since $\widetilde{E}_T$ is a finite-dimensional space, there exists a convergent subsequence of $\{x^k\}$. □

Proof of Theorem 5 Let $\{x^k\} \subset \widetilde{E}_P$ be a minimal sequence of I, that is, $I(x^k) \to C_5$ as $k \to \infty$. By the Ekeland variational principle, $I'(x^k) \to 0$ as $k \to \infty$. Since I satisfies the (PS) condition, C_5 is a critical point of I restricted on $\widetilde{E}_P$. Denote by $\overline{x}$ the critical point of I corresponding to critical value C_5. Then $\overline{x}$ is a periodic solution of (2). It is easy to check that $\overline{x}$ is a nonconstant P-periodic solution. By a similar discussion to the proof of Theorem 3, we can prove that $\overline{x}$ has P as its minimal period. □

Competing interests
The author declares that he has no competing interests.

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