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Multiple periodic solutions for a class of nonlinear difference systems with classical or bounded (ϕ_1, ϕ_2)-Laplacian

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Abstract

In this paper, we consider the multiplicity of periodic solutions for a class of difference systems involving the (ϕ_1, ϕ_2)-Laplacian in the cases when the gradient of the nonlinearity has a sublinear growth. By using the variational method, some existence results are obtained. Our results generalize some recent results in (Mawhin in Discrete Contin. Dyn. Syst. 6:1065-1076, 2013).

Keywords: difference systems; periodic solutions; critical point theorem; variational method

1 Introduction and main results

Let $\mathbb{R}$ denote the real numbers and $\mathbb{Z}$ the integers. Given $a < b$ in $\mathbb{Z}$. Let $\mathbb{Z}[a, b] = \{a, a + 1, \dots, b\}$. Let $T > 1$ and N be fixed positive integers.

In this paper, we investigate the multiplicity of periodic solutions for the following nonlinear difference systems:

$$\begin{cases} \Delta\phi_1(\Delta u_1(t-1)) = \nabla_{u_1} F(t, u_1(t), u_2(t)) + h_1(t), \\ \Delta\phi_2(\Delta u_2(t-1)) = \nabla_{u_2} F(t, u_1(t), u_2(t)) + h_2(t), \end{cases} \tag{1.1}$$

where $F : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and ϕ_m, $m = 1, 2$, satisfy the following condition:

($\mathcal{A}0$) ϕ_m is a homeomorphism from $\mathbb{R}^N$ onto $B_a \subset \mathbb{R}^N$ ($a \in (0, +\infty]$), such that $\phi_m(0) = 0$, $\phi_m = \nabla\Phi_m$, with $\Phi_m \in C^1(\mathbb{R}^N, [0, +\infty])$ strictly convex and $\Phi_m(0) = 0$, $m = 1, 2$.

Remark 1.1 Assumption ($\mathcal{A}0$) is given in [1], which is used to characterize the classical homeomorphism and the bounded homeomorphism. ϕ_m is called classical when $a = +\infty$ and bounded when $a < +\infty$. If furthermore $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ is coercive (*i.e.* $\Phi_m(x) \to +\infty$ as $|x| \to \infty$), there exists $\delta_m > 0$ such that

$$\Phi_m(x) \geq \delta_m\bigl(|x| - 1\bigr), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\delta_m = \min_{|x|=1} \Phi_m(x)$, $m = 1, 2$ (see [1]).

It is well known that the variational method has been an important tool to study the existence and multiplicity of solutions for various difference systems. Lots of contributions have been obtained (for example, see [1–13]). However, to the best of our knowledge, few people investigated system (1.1). Recently, in [1] and [14], by using the variational approach, Mawhin investigated the following second order nonlinear difference systems with ϕ-Laplacian:

$$\Delta\phi\bigl[\Delta u(n-1)\bigr] = \nabla_u F\bigl[n, u(n)\bigr] + h(n) \quad (n \in \mathbb{Z}), \tag{1.3}$$

where $\phi = \nabla\Phi$, Φ strictly convex, is a homeomorphism of $\mathbb{R}^N$ onto the ball $B_a \subset \mathbb{R}^N$ or of B_a onto $\mathbb{R}^N$. By using the variational approach, under different conditions, the author found that system (1.3) has at least one or $N+1$ geometrically distinct T-periodic solutions. It is interesting that Mawhin considered three kinds of ϕ: (1) $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a classical homeomorphism, for example, $\phi(x) = |x|^{p-1}x$ for some $p > 1$ and all $x \in \mathbb{R}^N$; (2) $\phi : \mathbb{R}^N \to B_a$ ($a < +\infty$) is a bounded homeomorphism, for example, $\phi(x) = \frac{x}{\sqrt{1+|x|^2}} \in B_1$ for all $x \in \mathbb{R}^N$; (3) $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$ is a singular homeomorphism, for example, $\phi(x) = \frac{x}{\sqrt{1-|x|^2}}$ for all $x \in B_1$.

For a classical and bounded homeomorphism, in [14], Mawhin obtained the following multiplicity results.

Theorem A (see [14], Theorem 4.1) *Assume that the following assumptions hold*:

(HB) *ϕ is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$, such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi \in C^1(\mathbb{R}^N, [0, +\infty])$ strictly convex and $\Phi(0) = 0$.*

(HF) *$F \in C(\mathbb{Z} \times \mathbb{R}^N, \mathbb{R})$, $F(n, \cdot) \in C^1(\mathbb{R}^N, \mathbb{R})$, and there exist an integer $T > 0$ and real numbers $\omega_1 > 0, \omega_2 > 0, \ldots, \omega_N > 0$ such that*

$$F(t + T, u_1 + \omega_1, u_2 + \omega_2, \ldots, u_N + \omega_N) = F(t, u_1, u_2, \ldots, u_N)$$

for all $t \in \mathbb{R}$ and $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$.

If there exist $\gamma > 0$ and $p > 1$ such that

$$\bigl|\Phi(u)\bigr| \geq \gamma |u|^p \quad \bigl(u \in \mathbb{R}^N\bigr).$$

Then, for any $h \in H_T$ such that $\frac{1}{T}\sum_{t=1}^{T} h(t) = 0$ (the definition of H_T can be seen in [14]*), system* (1.3) *has at least $N+1$ geometrically distinct T-periodic solutions.*

Theorem B (see [14], Theorem 4.2) *Assume that assumption* (HF) *and the following condition hold*:

(HB)′ *ϕ is a homeomorphism from $\mathbb{R}^N$ onto $B_a \subset \mathbb{R}^N$ ($a \in (0, +\infty)$), such that $\phi(0) = 0$, $\phi = \nabla\Phi$, with $\Phi \in C^1(\mathbb{R}^N, [0, +\infty])$ strictly convex and $\Phi(0) = 0$.*

If $\Phi : \mathbb{R}^N \to \mathbb{R}$ is coercive, $h \in H_T$ such that $\frac{1}{T}\sum_{t=1}^{T} h(t) = 0$ and $|H|_\infty < \delta$, system (1.3) *has at least $N+1$ geometrically distinct T-periodic solutions, where $\delta > 0$ is given by* (5) *in* [14] *and $H = (H(n))_{n\in\mathbb{Z}} \in H_T$ is such that $\Delta H(n) = h(n)$, $n \in \mathbb{Z}$.*

Obviously, (HF) implies that F is periodic on all variables $u_1, \ldots, u_N$. Hence, a natural question is that what will occur if F is periodic on some of variables $u_1, \ldots, u_N$. For differential systems, in [15] and [16], the arguments on this question have been given. In [15], Tang and Wu considered the second order Hamiltonian system

$$\begin{cases} \ddot{u}(t) + \nabla F(t, u(t)) = e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.4}$$

and in [16], Zhang and Tang generalized and improved the results in [15]. They considered the following ordinary p-Laplacian system:

$$\begin{cases} (|u'(t)|^{p-2}u'(t))' + \nabla F(t, u(t)) = e(t), \quad \text{a.e. } t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0. \end{cases} \tag{1.5}$$

Inspired by [1, 14, 15] and [16], in this paper, we investigate system (1.1), which is different from (1.3), and consider the case that $F(t, x_1, x_2)$ is periodic on some of the variables $x_1^{(1)}, \ldots, x_N^{(1)}$ and some of the variables $x_1^{(1)}, \ldots, x_N^{(2)}$, where $x_1 = (x_1^{(1)}, \ldots, x_N^{(1)})^\tau$ and $x_2 = (x_1^{(1)}, \ldots, x_N^{(2)})^\tau$. We generalize Theorem A and Theorem B.

Next, in order to present our main results, we consider two decompositions $\mathbb{R}^N = \mathcal{R}_1 \oplus \mathcal{S}_1$ and $\mathbb{R}^N = \mathcal{R}_2 \oplus \mathcal{S}_2$ with

$$\begin{aligned} &\mathcal{R}_1 = \operatorname{span}\langle e_{i_1}, \ldots, e_{i_{r_1}} \rangle, \qquad \mathcal{S}_1 = \operatorname{span}\langle e_{i_{r_1+1}}, \ldots, e_{i_N} \rangle, \\ &\mathcal{R}_2 = \operatorname{span}\langle e_{j_1}, \ldots, e_{j_{r_2}} \rangle, \qquad \mathcal{S}_2 = \operatorname{span}\langle e_{j_{r_2+1}}, \ldots, e_{j_N} \rangle, \end{aligned}$$

where e_{i_k} and e_{j_s} are the canonical basis of $\mathbb{R}^N$ for $1 \le k \le N$, $1 \le s \le N$, $1 \le r_1 \le N$, and $1 \le r_2 \le N$.

In this paper, we make the following assumptions:

($\mathcal{A}1$) *Let* $p > 1$, $q > 1$, $\beta_1 \in [0, p)$, *and* $\beta_2 \in [0, q)$. *Assume that there exist positive constants* γ_1, γ_2, γ_3, γ_4 *such that*

$$\Phi_1(x) \ge \gamma_1 |x|^p - \gamma_2 |x|^{\beta_1}, \qquad \Phi_2(y) \ge \gamma_3 |y|^q - \gamma_4 |y|^{\beta_2}, \quad \forall x, y \in \mathbb{R}^N.$$

($\mathcal{A}2$) *There exist positive constants* d_1, d_2, d_3, d_4 *with* $d_1 > \frac{1}{p}$ *and* $d_3 > \frac{1}{q}$, $\beta_3 \in [0, p)$, *and* $\beta_4 \in [0, q)$ *such that*

$$\big(\phi_1(x), x\big) \ge d_1 |x|^p - d_2 |x|^{\beta_3}, \qquad \big(\phi_2(x), x\big) \ge d_3 |x|^q - d_4 |x|^{\beta_4}, \quad \forall x \in \mathbb{R}^N.$$

($\mathcal{A}3$) *There exist constants* $c_{m0} > 0$, $k_{m1} > 0$, $k_{m2} > 0$, $\alpha_1 \in [0, p-1)$, $\alpha_2 \in [0, q-1)$, *and two nonnegative functions* $w_m \in C([0, +\infty), [0, +\infty))$, *where* $m = 1, 2$, *with the properties*:
  (i) $w_m(s) \le w_m(t)$ $\forall s \le t$, $s, t \in [0, +\infty)$,
  (ii) $w_m(s + t) \le c_{m0}(w_m(s) + w_m(t))$ $\forall s, t \in [0, +\infty)$,
  (iii) $0 \le w_1(t) \le k_{11} t^{\alpha_1} + k_{12}$, $0 \le w_2(t) \le k_{21} t^{\alpha_2} + k_{22}$, $\forall t \in [0, +\infty)$,
  (iv) $w_m(t) \to +\infty$, *as* $t \to +\infty$.

($\mathcal{F}1$) *$F : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $(t, x_1, x_2) \longrightarrow F(t, x_1, x_2)$ is T-periodic in t for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and continuously differentiable in (x_1, x_2) for every $t \in \mathbb{Z}[1, T]$, where $x_1 = (x_1^{(1)}, \dots, x_N^{(1)})^\tau$, $x_2 = (x_1^{(2)}, \dots, x_N^{(2)})^\tau$.*

($\mathcal{F}2$) *$F(t, x_1, x_2)$ is $T_{i_k}^{(1)}$-periodic in $x_{i_k}^{(1)}$, where $x_{i_k}^{(1)}$ is a component of vector x_1 and $T_{i_k}^{(1)} > 0$, $1 \le k \le r_1$, and $T_{j_s}^{(2)}$-periodic in $x_{j_s}^{(2)}$, where $x_{j_s}^{(2)}$ is a component of vector x_2 and $T_{j_s}^{(2)} > 0$, $1 \le s \le r_2$.*

($\mathcal{F}3$) *There exist $f_m, g_m : \mathbb{Z}[1, T] \to \mathbb{R}$, $m = 1, 2$, such that*

$$\begin{aligned} \left|\nabla_{x_1} F(t, x_1, x_2)\right| &\le f_1(t) w_1\left(|x_1|\right) + g_1(t), \\ \left|\nabla_{x_2} F(t, x_1, x_2)\right| &\le f_2(t) w_2\left(|x_2|\right) + g_2(t) \end{aligned}$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and $t \in \mathbb{Z}[1, T]$.

($\mathcal{E}$)

$$\sum_{t=1}^{T} h_1(t) = \sum_{t=1}^{T} h_2(t) = 0.$$

Remark 1.2 A condition similar to ($\mathcal{A}3$) and ($\mathcal{F}3$) was given first in [17] for the second order Hamiltonian systems

$$\begin{cases} \ddot{u}(t) = \nabla F(t, u(t)), \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T). \end{cases} \tag{1.6}$$

The condition presented some advantages over the following subquadratic condition: there exist $\alpha \in [0, 1)$ and $f, g \in L^1([0, T]; \mathbb{R}^N)$ such that

$$\left|\nabla F(t, x)\right| \le f(t)|x|^\alpha + g(t).$$

We refer readers to [17] for more details.

Moreover, assume that $p' > 1$ and $q' > 1$ satisfying $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$. Let

$$C(p') = \min\left\{\frac{(T-1)^{(p'+1)/p'}}{T}, \left(\frac{(T+1)^{p'+1} - 2}{T^{p'}(p'+1)}\right)^{1/p'}\right\}, \tag{1.7}$$

$$C(q') = \min\left\{\frac{(T-1)^{(q'+1)/q'}}{T}, \left(\frac{(T+1)^{q'+1} - 2}{T^{q'}(q'+1)}\right)^{1/q'}\right\}, \tag{1.8}$$

$$C(p, p') = \min\left\{\frac{(T-1)^{2p-1}}{T^{p-1}}, \frac{T^{p-1}\Theta(p', p)}{(p'+1)^{p/p'}}\right\}, \tag{1.9}$$

$$C(q, q') = \min\left\{\frac{(T-1)^{2q-1}}{T^{q-1}}, \frac{T^{q-1}\Theta(q', q)}{(q'+1)^{q/q'}}\right\}, \tag{1.10}$$

$$\Theta(p', p) = \sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{p'+1} + \left(1 - \frac{t}{T} + \frac{1}{T}\right)^{p'+1} - \frac{2}{T^{p'+1}}\right]^{p/p'},$$

$$\Theta(q', q) = \sum_{t=1}^{T}\left[\left(\frac{t}{T}\right)^{q'+1} + \left(1 - \frac{t}{T} + \frac{1}{T}\right)^{q'+1} - \frac{2}{T^{q'+1}}\right]^{q/q'}.$$

Next, we present our main results.

(I) For classical homeomorphism

Theorem 1.1 *Assume that* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, $(\mathcal{A}3)$, $(\mathcal{F}1)$-$(\mathcal{F}3)$, *and* $(\mathcal{E})$ *hold. Assume that F satisfies the following condition*:

$(\mathcal{F}4)$ *For* $(x_1, x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$,

$$\lim_{|x_1|+|x_2|\to+\infty} \frac{\sum_{t=1}^T F(t,x_1,x_2)}{w_1^{p'}(|x_1|) + w_2^{q'}(|x_2|)} > \max\left\{ \frac{[c_{10}C(p')]^{p'}}{[p\gamma_1]^{p'-1}p'} \left(\sum_{t=1}^T f_1(t)\right)^{p'}, \frac{[c_{20}C(q')]^{q'}}{[q\gamma_3]^{q'-1}q'} \left(\sum_{t=1}^T f_2(t)\right)^{q'} \right\}.$$

Then system (1.1) *has at least* $r_1 + r_2 + 1$ *geometrically distinct solutions in* $\mathcal{H}$, *where the definition of* $\mathcal{H}$ *is given in Section* 2 *below.*

Theorem 1.2 *Assume that* $(\mathcal{A}0)$ *with* $a = +\infty$, $(\mathcal{A}1)$, $(\mathcal{A}2)$, $(\mathcal{A}3)$, $(\mathcal{F}1)$-$(\mathcal{F}3)$, *and* $(\mathcal{E})$ *hold. Assume that F satisfies the following condition*:

$(\mathcal{A}1)'$ *Let* $\theta_1 \in [0, p)$ *and* $\theta_2 \in [0, q)$. *Assume that there exist positive constants* ζ_1, ζ_2, ζ_3, ζ_4 *such that*

$$\Phi_1(x) \le \zeta_1|x|^p + \zeta_2|x|^{\theta_1}, \qquad \Phi_2(y) \le \zeta_3|y|^q + \zeta_4|y|^{\theta_2}, \quad \forall x, y \in \mathbb{R}^N;$$

$(\mathcal{F}4)'$ *For* $(x_1, x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$,

$$\begin{aligned} &\lim_{|x_1|+|x_2|\to+\infty} \frac{\sum_{t=1}^T F(t,x_1,x_2)}{w_1^{p'}(|x_1|) + w_2^{q'}(|x_2|)} \\ &\quad < -\max\left\{ \frac{[C(p')c_{10}]^{p'}}{p'} \left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right] \left(\sum_{t=1}^T f_1(t)\right)^{p'}, \right. \\ &\qquad \left. \frac{[C(q')c_{20}]^{q'}}{q'} \left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right] \left(\sum_{t=1}^T f_2(t)\right)^{q'} \right\}. \end{aligned}$$

Then system (1.1) *has at least* $r_1 + r_2 + 1$ *geometrically distinct solutions in* $\mathcal{H}$.

(II) For bounded homeomorphism

Theorem 1.3 *Assume that* $(\mathcal{A}0)$ *with* $a < +\infty$, $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive*, $m = 1, 2$, $(\mathcal{F}1)$, $(\mathcal{F}2)$, *and* $(\mathcal{E})$ *hold. Assume that F satisfies the following conditions*:

$(\mathcal{F}5)$ *There exists a nonnegative* $b_m : \mathbb{Z}[1, T] \to \mathbb{R}^+$, $m = 1, 2$, *such that*

$$\left|\nabla_{x_1} F(t, x_1, x_2)\right| \le b_1(t),$$
$$\left|\nabla_{x_2} F(t, x_1, x_2)\right| \le b_2(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ *and* $t \in \mathbb{Z}[1, T]$;

($\mathcal{F}6$) *For* $(x_1, x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$,

$$\lim_{|x_1|+|x_2| \to +\infty} \sum_{t=1}^{T} F(t, x_1, x_2) = +\infty;$$

($\mathcal{F}7$)

$$\sum_{t=1}^{T} b_1(t) + \sum_{t=1}^{T} \left|h_1(t)\right| < \frac{\delta_1}{C(p')},$$
$$\sum_{t=1}^{T} b_2(t) + \sum_{t=1}^{T} \left|h_2(t)\right| < \frac{\delta_2}{C(q')},$$

where δ_m, $m = 1, 2$ *are given in* (1.2). *Then system* (1.1) *has at least* $r_1 + r_2 + 1$ *geometrically distinct solutions in* $\mathcal{H}$.

Theorem 1.4 *Assume that* ($\mathcal{A}0$) *with* $a < +\infty$, $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ *are coercive*, $m = 1, 2$, ($\mathcal{F}1$), ($\mathcal{F}2$), ($\mathcal{F}5$), *and* ($\mathcal{E}$) *hold. If F satisfies the following conditions*:

($\mathcal{F}6$)$'$ *For* $(x_1, x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$,

$$\lim_{|x_1|+|x_2| \to +\infty} \sum_{t=1}^{T} F(t, x_1, x_2) = -\infty;$$

($\mathcal{F}7$)$'$

$$C(p') \sum_{t=1}^{T} b_1(t) + C(p') \sum_{t=1}^{T} \left|h_1(t)\right| + \left(C(p, p') + 1\right)^{1/p} < \delta_1,$$
$$C(q') \sum_{t=1}^{T} b_2(t) + C(q') \sum_{t=1}^{T} \left|h_2(t)\right| + \left(C(q, q') + 1\right)^{1/q} < \delta_2,$$

where δ_m, $m = 1, 2$ *are given in* (1.2), *then system* (1.1) *has at least* $r_1 + r_2 + 1$ *geometrically distinct solutions in* $\mathcal{H}$.

2 Preliminaries

First, we present some basic notations. We use $|\cdot|$ to denote the usual Euclidean norm in $\mathbb{R}^N$. Define

$$\begin{aligned}\mathcal{V} = \{u = (u_1, u_2)^\tau = \{u(t)\} \mid u(t) = \left(u_1(t), u_2(t)\right)^\tau \in \mathbb{R}^{2N},\\ u_m = \{u_m(t)\}, u_m(t) \in \mathbb{R}^N, m = 1, 2, t \in \mathbb{Z}\}.\end{aligned}$$

$\mathcal{H}$ is defined as a subspace of $\mathcal{V}$ by

$$\mathcal{H} = \{u = \{u(t)\} \in \mathcal{V} \mid u(t + T) = u(t), t \in \mathbb{Z}\}.$$

Define

$$\mathcal{H}_m = \{u_m = \{u_m(t)\} \mid u_m(t + T) = u_m(t), u_m(t) \in \mathbb{R}^N, t \in \mathbb{Z}\}, \quad m = 1, 2.$$

Then $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. For $u_m \in \mathcal{H}_m$, set

$$\|u_m\|_r = \left(\sum_{t=1}^{T} |u_m(t)|^r\right)^{1/r} \quad \text{and} \quad \|u_m\|_\infty = \max_{t\in\mathbb{Z}[1,T]} |u_m(t)|, \quad m = 1, 2, r > 1.$$

Obviously, we have

$$\|u_m\|_\infty \leq \|u_m\|_2, \quad m = 1, 2. \tag{2.1}$$

For $1 < p, q < +\infty$, on $\mathcal{H}_1$, we define

$$\|u_1\|_p = \left(\sum_{t=1}^{T} |\Delta u_1(t)|^p + \sum_{t=1}^{T} |u_1(t)|^p\right)^{1/p}$$

and, on $\mathcal{H}_2$, we define

$$\|u_2\|_q = \left(\sum_{t=1}^{T} |\Delta u_2(t)|^q + \sum_{t=1}^{T} |u_2(t)|^q\right)^{1/q}.$$

For $u = (u_1, u_2)^\tau \in \mathcal{H}$, we define

$$\|u\| = \|u_1\|_p + \|u_2\|_q.$$

Let

$$\mathcal{W} = \left\{ u = (u_1, u_2)^\tau \in \mathcal{H} \,\middle|\, u_m(1) = \cdots = u_m(T) = \frac{1}{T}\sum_{t=1}^{T} u_m(t), m = 1, 2 \right\}$$

and

$$\tilde{\mathcal{H}} = \left\{ u = (u_1, u_2)^\tau \in \mathcal{H} \,\middle|\, \sum_{t=1}^{T} u_m(t) = 0, m = 1, 2 \right\}.$$

Then $\mathcal{H}$ can be decomposed into the direct sum $\mathcal{H} = \mathcal{W} \oplus \tilde{\mathcal{H}}$. So, for any $u \in \mathcal{H}$, u can be expressed in the form $u = \tilde{u} + \bar{u}$, where $\tilde{u} = (\tilde{u}_1, \tilde{u}_2)^\tau \in \mathcal{V}$ and $\bar{u} = (\bar{u}_1, \bar{u}_2)^\tau \in \mathcal{W}$. Obviously, $u_m = \tilde{u}_m + \bar{u}_m$, $m = 1, 2$.

For $u = (u_1, u_2)^\tau \in \tilde{\mathcal{H}}$, let

$$\|\Delta u_m\|_r = \left(\sum_{t=1}^{T} |\Delta u_m(t)|^r\right)^{1/r},$$

where $m = 1, 2$, $r > 1$. It is easy to verify that

$$\|\Delta u\| = \|\Delta u_1\|_p + \|\Delta u_2\|_q$$

is also a norm on $\tilde{\mathcal{H}}$. Since $\tilde{\mathcal{H}}$ is finite-dimensional, the norm $\|\Delta u\|$ is equivalent to the norm $\|u\|$ in $\mathcal{H}$ if $u \in \tilde{\mathcal{H}}$.

Lemma 2.1 (see [12]) *Let* $u = (u_1, u_2) \in \tilde{\mathcal{H}}$. *Then*

$$\max_{t\in\mathbb{Z}[1,T]} |u_m(t)| \le C(p') \left(\sum_{s=1}^{T} |\Delta u_m(s)|^p \right)^{1/p}, \quad m = 1, 2, \tag{2.2}$$

$$\max_{t\in\mathbb{Z}[1,T]} |u_m(t)| \le C(q') \left(\sum_{s=1}^{T} |\Delta u_m(s)|^q \right)^{1/q}, \quad m = 1, 2, \tag{2.3}$$

and

$$\sum_{t=1}^{T} |u_m(t)|^p \le C(p, p') \sum_{s=1}^{T} |\Delta u_m(s)|^p, \quad m = 1, 2, \tag{2.4}$$

$$\sum_{t=1}^{T} |u_m(t)|^q \le C(q, q') \sum_{s=1}^{T} |\Delta u_m(s)|^q, \quad m = 1, 2, \tag{2.5}$$

where $C(p')$, $C(q')$, $C(p, p')$, *and* $C(q, q')$ *are defined by* (1.7)-(1.10).

Lemma 2.2 (see [16]) *Let* $a > 0$, $b, c \ge 0$, $\varepsilon > 0$.
(i) *If* $\alpha \in (0, 1]$, *then* $(a + b + c)^\alpha \le a^\alpha + b^\alpha + c^\alpha$;
(ii) *if* $\alpha \in (1, +\infty)$, *then there exists* $B(\varepsilon) > 1$ *such that*

$$(a + b + c)^\alpha \le (1 + \varepsilon)a^\alpha + B(\varepsilon)b^\alpha + B(\varepsilon)c^\alpha.$$

Lemma 2.3 *For any* $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{H}$, *the following two equalities hold*:

$$-\sum_{t=1}^{T} \bigl(\Delta\phi_1\bigl(\Delta u_1(t-1)\bigr), v_1(t)\bigr) = \sum_{t=1}^{T} \bigl(\Delta\phi_1\bigl(\Delta u_1(t)\bigr), \Delta v_1(t)\bigr), \tag{2.6}$$

$$-\sum_{t=1}^{T} \bigl(\Delta\phi_2\bigl(\Delta u_2(t-1)\bigr), v_2(t)\bigr) = \sum_{t=1}^{T} \bigl(\Delta\phi_2\bigl(\Delta u_2(t)\bigr), \Delta v_2(t)\bigr). \tag{2.7}$$

Proof In fact, since $u_1(t) = u_1(t + T)$ and $v_1(t) = v_1(t + T)$ for all $t \in \mathbb{Z}$, we have

$$\begin{aligned}
&-\sum_{t=1}^{T} \bigl(\Delta\phi_1\bigl(\Delta u_1(t-1)\bigr), v_1(t)\bigr) \\
&\quad = -\sum_{t=1}^{T} \bigl(\phi_1\bigl(\Delta u_1(t)\bigr), v_1(t)\bigr) + \sum_{t=1}^{T} \bigl(\phi_1\bigl(\Delta u_1(t-1)\bigr), v_1(t)\bigr) \\
&\quad = -\sum_{t=1}^{T} \bigl(\phi_1\bigl(\Delta u_1(t)\bigr), v_1(t)\bigr) + \sum_{t=1}^{T-1} \bigl(\phi_1\bigl(\Delta u_1(t)\bigr), v_1(t+1)\bigr) + \bigl(\phi_1\bigl(\Delta u_1(0)\bigr), v_1(1)\bigr) \\
&\quad = \sum_{t=1}^{T} \bigl(\phi_1\bigl(\Delta u_1(t)\bigr), \Delta v_1(t)\bigr) + \bigl(\phi_1\bigl(\Delta u_1(0)\bigr), v_1(1)\bigr) - \bigl(\phi_1\bigl(\Delta u_1(T)\bigr), v_1(T+1)\bigr) \\
&\quad = \sum_{t=1}^{T} \bigl(\phi_1\bigl(\Delta u_1(t)\bigr), \Delta v_1(t)\bigr).
\end{aligned}$$

Hence, (2.6) holds. Similarly, it is easy to obtain (2.7). The proof is complete. □

Lemma 2.4 *Let $L : \mathbb{Z}[1,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $(t,x_1,x_2,y_1,y_2) \longrightarrow L(t,x_1,x_2,y_1,y_2)$ and assume that L is continuously differential in (x_1,x_2,y_1,y_2) for all $t \in \mathbb{Z}[1,T]$. Then the function $\varphi : \mathcal{H} \to \mathbb{R}$ defined by*

$$\varphi(u) = \varphi(u_1,u_2) = \sum_{t=1}^{T} L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big)$$

is continuously differentiable on $\mathcal{H}$ and

$$\begin{aligned}
\langle \varphi'(u),v\rangle &= \langle \varphi'(u_1,u_2),(v_1,v_2)\rangle \\
&= \sum_{t=1}^{T}\big[\big(D_{x_1}L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big),v_1(t)\big) \\
&\quad + \big(D_{y_1}L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big),\Delta v_1(t)\big) \\
&\quad + \big(D_{x_2}L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big),v_2(t)\big) \\
&\quad + \big(D_{y_2}L\big(t,u_1(t),u_2(t),\Delta u_1(t),\Delta u_2(t)\big),\Delta v_2(t)\big)\big],
\end{aligned}$$

where $u,v \in \mathcal{H}$.

Proof Define $G : [-1,1] \times \mathbb{Z}[1,T] \to \mathbb{R}$, $(\lambda,t) \to G(\lambda,t)$ by

$$G(\lambda,t) = L\big(t,u_1(t)+\lambda v_1(t),u_2(t)+\lambda v_2(t),\Delta u_1(t)+\lambda\Delta v_1(t),\Delta u_2(t)+\lambda\Delta v_2(t)\big).$$

Since L is continuously differential in (x_1,x_2,y_1,y_2) for all $t \in \mathbb{Z}[1,T]$, $G(\lambda,t)$ is differential in λ and

$$\begin{aligned}
&G'(\lambda,t) \\
&\quad = \big(D_{x_1}L\big(t,u_1(t)+\lambda v_1(t),u_2(t)+\lambda v_2(t),\Delta u_1(t)+\lambda\Delta v_1(t),\Delta u_2(t)+\lambda\Delta v_2(t)\big),v_1(t)\big) \\
&\qquad + \big(D_{x_2}L\big(t,u_1(t)+\lambda v_1(t),u_2(t)+\lambda v_2(t), \\
&\qquad \Delta u_1(t)+\lambda\Delta v_1(t),\Delta u_2(t)+\lambda\Delta v_2(t)\big),v_2(t)\big) \\
&\qquad + \big(D_{y_1}L\big(t,u_1(t)+\lambda v_1(t),u_2(t)+\lambda v_2(t), \\
&\qquad \Delta u_1(t)+\lambda\Delta v_1(t),\Delta u_2(t)+\lambda\Delta v_2(t)\big),\Delta v_1(t)\big) \\
&\qquad + \big(D_{y_2}L\big(t,u_1(t)+\lambda v_1(t),u_2(t)+\lambda v_2(t), \\
&\qquad \Delta u_1(t)+\lambda\Delta v_1(t),\Delta u_2(t)+\lambda\Delta v_2(t)\big),\Delta v_2(t)\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\langle \varphi'(u),v\rangle &= \lim_{\lambda\to 0}\frac{\varphi(u+\lambda v)-\varphi(u)}{\lambda} \\
&= \lim_{\lambda\to 0}\frac{\sum_{t=1}^{T}G(\lambda,t)-\sum_{t=1}^{T}G(0,t)}{\lambda} \\
&= \sum_{t=1}^{T}\lim_{\lambda\to 0}\frac{G(\lambda,t)-G(0,t)}{\lambda}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{T} G'(0,t) \\
&= \sum_{t=1}^{T} \bigl[\bigl(D_{x_1} L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), v_1(t)\bigr) \\
&\quad + \bigl(D_{y_1} L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), \Delta v_1(t)\bigr) \\
&\quad + \bigl(D_{x_2} L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), v_2(t)\bigr) \\
&\quad + \bigl(D_{y_2} L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), \Delta v_2(t)\bigr) \bigr].
\end{aligned}
$$

The proof is complete. □

Let

$$
L(t, x_1, x_2, y_1, y_2) = \Phi_1(y_1) + \Phi_2(y_2) + F(t, x_1, x_2) + \bigl(h_1(t), x_1\bigr) + \bigl(h_2(t), x_2\bigr).
$$

Then

$$
\begin{aligned}
\varphi(u) &= \varphi(u_1, u_2) \\
&= \sum_{t=1}^{T} \bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + F\bigl(t, u_1(t), u_2(t)\bigr) \\
&\quad + \bigl(h_1(t), u_1(t)\bigr) + \bigl(h_2(t), u_2(t)\bigr) \bigr].
\end{aligned}
\tag{2.8}
$$

It follows from $(\mathcal{A}0)$, $(\mathcal{F}1)$, and Lemma 2.4 that

$$
\begin{aligned}
\bigl\langle \varphi'(u), v \bigr\rangle &= \bigl\langle \varphi'(u_1, u_2), (v_1, v_2) \bigr\rangle \\
&= \sum_{t=1}^{T} \bigl[\bigl(\phi_1\bigl(\Delta u_1(t)\bigr), \Delta v_1(t)\bigr) + \bigl(\phi_2\bigl(\Delta u_2(t)\bigr), \Delta v_2(t)\bigr) \\
&\quad + \bigl(\nabla_{u_1} F\bigl(t, u_1(t), u_2(t)\bigr), v_1(t)\bigr) + \bigl(\nabla_{u_2} F\bigl(t, u_1(t), u_2(t)\bigr), v_2(t)\bigr) \\
&\quad + \bigl(h_1(t), v_1(t)\bigr) + \bigl(h_2(t), v_2(t)\bigr) \bigr], \quad \forall u, v \in \mathcal{H}.
\end{aligned}
\tag{2.9}
$$

By Lemma 2.3, it is easy to see that the critical points of φ in $\mathcal{H}$ are periodic solutions of system (1.1).

Next, we recall a definition. Let G be a discrete subgroup of a Banach space X and let $\pi : X \to X/G$ be the canonical surjection. A subset A of X is G-invariant if $\pi^{-1}(\pi(A)) = A$. A function f defined on X is G-invariant if $f(u+g) = f(u)$ for every $u \in X$ and every $g \in G$ (see [18]).

Definition 2.1 (see [18], Definition 4.2) A G-invariant differentiable functional $\varphi : X \to \mathbb{R}$ satisfies the $(\mathrm{PS})_G$ condition, if for every sequence $\{u_k\}$ in X such that $\varphi(u_k)$ is bounded and $\varphi'(u_k) \to 0$, the sequence $\{\pi(u_k)\}$ contains a convergent subsequence.

We will use the following two lemmas to obtain the critical points of φ.

Lemma 2.5 (see [18], Theorem 4.12) *Let $\varphi \in C^1(x, \mathbb{R})$ be a G-invariant functional satisfying the* $(\mathrm{PS})_G$ *condition. If φ is bounded from below and if the dimension N of the space generated by G is finite, then φ has at least $N+1$ critical orbits.*

Lemma 2.6 (see [19]) *Let X be a Banach space and have a decomposition: $X = Y + Z$ where Y and Z are two subspaces of X with* $\dim Y < +\infty$. *Let V be a finite-dimensional, compact C^2-manifold without boundary. Let $f : X \times V \to \mathbb{R}$ be a C^1-function and satisfy the* (PS) *condition. Suppose that f satisfies*

$$\inf_{u \in Z \times V} f(u) \geq a, \qquad \sup_{u \in S \times V} f(u) \leq b < a,$$

where $S = \partial D$, $D = \{u \in Y \mid \|u\| \leq R\}$, R, a, and b are constants. Then the function f has at least cuplength$(V) + 1$ critical points.

Let

$$\hat{u}_m(t) = P_m \bar{u}_m + Q_m \bar{u}_m + \tilde{u}_m, \quad m = 1, 2,$$

where

$$P_1 \bar{u}_1 = \sum_{k=r_1+1}^{N} (\bar{u}_1, e_{i_k}) e_{i_k}, \qquad Q_1 \bar{u}_1 = \sum_{k=1}^{r_1} \big[(\bar{u}_1, e_{i_k}) - m_{i_k} T_{i_k}\big] e_{i_k},$$
$$P_2 \bar{u}_2 = \sum_{s=r_2+1}^{N} (\bar{u}_2, e_{j_s}) e_{j_s}, \qquad Q_2 \bar{u}_2 = \sum_{s=1}^{r_2} \big[(\bar{u}_2, e_{j_s}) - m_{j_s} T_{j_s}\big] e_{j_s},$$

and m_{i_k}, m_{j_s} are the unique integers such that

$$0 \leq (\bar{u}_1, e_{i_k}) - m_{i_k} T_{i_k} < T_{i_k}, \quad 1 \leq k \leq r_1,$$
$$0 \leq (\bar{u}_2, e_{j_s}) - m_{j_s} T_{j_s} < T_{j_s}, \quad 1 \leq s \leq r_2.$$

Let

$$G = \left\{ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \mathbf{R}^N \times \mathbf{R}^N \,\middle|\, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{r_1} m_{i_k} T_{i_k} e_{i_k} \\ \sum_{s=1}^{r_2} m_{j_s} T_{j_s} e_{j_s} \end{pmatrix}, \right.$$
$$\left. m_{i_k} \text{ and } m_{j_s} \text{ are integers}, 1 \leq k \leq r_1, 1 \leq s \leq r_2 \right\}. \tag{2.10}$$

Let $Z = \tilde{\mathcal{H}}$, $Y = \mathcal{S}_1 \times \mathcal{S}_2$, $X = Y + Z$, and V is the quotient space $(\mathcal{R}_1 \times \mathcal{R}_2)/G$ which is isomorphic to the torus $T^{r_1+r_2}$. Now define $\Psi : X \times T^{r_1+r_2} \to \mathbb{R}$ by

$$\Psi\big((y + z(t), v)\big) = \varphi\big(y + v + z(t)\big), \quad \forall (y, z, v) \in Y \times Z \times T^{r_1+r_2}, \tag{2.11}$$

that is,

$$\Psi\big((y + z(t), v)\big) = \varphi\big(y_1 + v_1 + z_1(t), y_2 + v_2 + z_2(t)\big), \tag{2.12}$$

where

$$y = (y_1, y_2)^\tau \in Y, \qquad v = (v_1, v_2)^\tau \in V, \qquad z = z(t) = \big(z_1(t), z_2(t)\big)^\tau \in Z,$$
$$y + v + z(t) = \big(y_1 + v_1 + z_1(t), y_2 + v_2 + z_2(t)\big)^\tau.$$

It is easy to verify that Ψ is continuously differentiable and that

$$\begin{aligned}
&\big\langle \Psi'\big(\big(y^{[1]} + z^{[1]}(t), v^{[1]}\big)\big), \big(y^{[2]} + z^{[2]}(t), v^{[2]}\big)\big\rangle \\
&\quad = \big\langle \varphi'\big(y^{[1]} + v^{[1]} + z^{[1]}(t)\big), y^{[2]} + v^{[2]} + z^{[2]}(t)\big\rangle \\
&\quad = \big\langle \varphi'\big(y_1^{[1]} + v_1^{[1]} + z_1^{[1]}(t), y_2^{[1]} + v_2^{[1]} + z_2^{[1]}(t)\big), \\
&\qquad \big(y_1^{[2]} + v_1^{[2]} + z_1^{[2]}(t), y_2^{[2]} + v_2^{[2]} + z_2^{[2]}(t)\big)\big\rangle, \\
&\quad \forall \big(y^{[m]}, z^{[m]}, v^{[m]}\big) \in Y \times Z \times T^{r_1+r_2}, m = 1, 2.
\end{aligned} \tag{2.13}$$

Then $(\mathcal{F}2)$ implies that

$$F(t + T, x_1 + g_1, x_2 + g_2) = F(t, x_1, x_2), \quad \forall t \in \mathbb{Z} \text{ and } \forall g \in G.$$

Hence, we have

$$\begin{aligned}
F\big(t, u_1(t), u_2(t)\big) &= F\left(t, \hat{u}_1(t) + \sum_{k=1}^{r_1} m_{i_k} T_{i_k} e_{i_k}, \hat{u}_2(t) + \sum_{s=1}^{r_2} m_{j_s} T_{j_s} e_{j_s}\right) \\
&= F\big(t, \hat{u}_1(t), \hat{u}_2(t)\big),
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\nabla F\big(t, u_1(t), u_2(t)\big) &= \nabla F\left(t, \hat{u}_1(t) + \sum_{k=1}^{r_1} m_{i_k} T_{i_k} e_{i_k}, \hat{u}_2(t) + \sum_{s=1}^{r_2} m_{j_s} T_{j_s} e_{j_s}\right) \\
&= \nabla F\big(t, \hat{u}_1(t), \hat{u}_2(t)\big)
\end{aligned} \tag{2.15}$$

and, by $(\mathcal{E})$, we have

$$\begin{aligned}
\sum_{t=1}^{T}\big(h_1(t), u_1(t)\big) &= \sum_{t=1}^{T}\left(h_1(t), \hat{u}_1(t) + \sum_{k=1}^{r_1} m_{i_k} T_{i_k} e_{i_k}\right) \\
&= \sum_{t=1}^{T}\big(h_1(t), \hat{u}_1(t)\big),
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\sum_{t=1}^{T}\big(h_2(t), u_2(t)\big) &= \sum_{t=1}^{T}\left(h_2(t), \hat{u}_2(t) + \sum_{k=1}^{r_1} m_{i_k} T_{i_k} e_{i_k}\right) \\
&= \sum_{t=1}^{T}\big(h_2(t), \hat{u}_2(t)\big).
\end{aligned} \tag{2.17}$$

Hence $\varphi(u) = \varphi(\hat{u})$ and $\varphi'(u) = \varphi'(\hat{u})$.

3 Proofs

For the sake of convenience, we denote by C_{ij} and D_{ij}, $i = 1,2$, $j = 0,1,\ldots,9$ below the various positive constants, by $C_{ij}(\varepsilon)$ and $D_{ij}(\varepsilon)$, $i = 1,2$, $j = 0,1,\ldots,9$ below the various positive constants depending on ε and

$$\begin{aligned}
&M_{11} = \sum_{t=1}^{T} f_1(t), \qquad M_{12} = \sum_{t=1}^{T} g_1(t), \qquad M_{13} = \left(\sum_{k=1}^{r_1} T_{i_k}^2\right)^{1/2}, \\
&M_{14} = \sum_{t=1}^{T} \bigl|h_1(t)\bigr|, \qquad M_{15} = \sum_{t=1}^{T} b_1(t), \\
&M_{21} = \sum_{t=1}^{T} f_2(t), \qquad M_{22} = \sum_{t=1}^{T} g_2(t), \qquad M_{23} = \left(\sum_{s=1}^{r_2} T_{j_s}^2\right)^{1/2}, \\
&M_{24} = \sum_{t=1}^{T} \bigl|h_2(t)\bigr|, \qquad M_{25} = \sum_{t=1}^{T} b_2(t).
\end{aligned}$$

Proof of Theorem 1.1 It follows from $(\mathcal{F}4)$ that there exist $a_1 > \frac{C(p')}{p\gamma_1}$ and $a_2 > \frac{C(q')}{q\gamma_3}$ such that

$$\lim_{|x_1|+|x_2|\to\infty} \frac{F(t,x_1,x_2)}{w_1^{p'}(|x_1|) + w_2^{q'}(|x_2|)} > \max\left\{\frac{c_{10}^{p'} M_{11}^{p'} a_1^{p'/p} C(p')}{p'}, \frac{c_{20}^{q'} M_{21}^{q'} a_2^{q'/q} C(q')}{q'}\right\}, \tag{3.1}$$

for $(x_1,x_2) \in \mathcal{S}_1 \times \mathcal{S}_2$. It follows from $(\mathcal{A}3)$, $(\mathcal{F}3)$, Lemma 2.1, and Lemma 2.2 that

$$\begin{aligned}
&\sum_{t=1}^{T} \bigl|F\bigl(t,\hat{u}_1(t),\hat{u}_2(t)\bigr) - F(t,P_1\bar{u}_1,P_2\bar{u}_2)\bigr| \\
&\quad\le \sum_{t=1}^{T} \bigl|F\bigl(t,\hat{u}_1(t),\hat{u}_2(t)\bigr) - F\bigl(t,P_1\bar{u}_1,\hat{u}_2(t)\bigr)\bigr| \\
&\qquad + \sum_{t=1}^{T} \bigl|F\bigl(t,P_1\bar{u}_1,\hat{u}_2(t)\bigr) - F(t,P_1\bar{u}_1,P_2\bar{u}_2)\bigr| \\
&\quad\le \sum_{t=1}^{T} \left|\int_0^1 \bigl(\nabla_{x_1} F\bigl(t,P_1\bar{u}_1 + s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr),\hat{u}_2(t)\bigr), Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr)\,ds\right| \\
&\qquad + \sum_{t=1}^{T} \left|\int_0^1 \bigl(\nabla_{x_2} F\bigl(t,P_1\bar{u}_1,P_2\bar{u}_2 + s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr), Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\,ds\right| \\
&\quad\le \bigl(|Q_1\bar{u}_1| + \|\tilde{u}_1\|_\infty\bigr) \sum_{t=1}^{T} \int_0^1 \bigl|\nabla_{x_1} F\bigl(t,P_1\bar{u}_1 + s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr),\hat{u}_2(t)\bigr)\bigr|\,ds \\
&\qquad + \bigl(|Q_2\bar{u}_2| + \|\tilde{u}_2\|_\infty\bigr) \sum_{t=1}^{T} \int_0^1 \bigl|\nabla_{x_2} F\bigl(t,P_1\bar{u}_1,P_2\bar{u}_2 + s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr)\bigr|\,ds \\
&\quad\le \bigl(M_{13} + \|\tilde{u}_1\|_\infty\bigr) \sum_{t=1}^{T} \int_0^1 \bigl[f_1(t) w_1\bigl(\bigl|P_1\bar{u}_1 + s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr)\bigr|\bigr) + g_1(t)\bigr]\,ds \\
&\qquad + \bigl(M_{23} + \|\tilde{u}_2\|_\infty\bigr) \sum_{t=1}^{T} \int_0^1 \bigl[f_2(t) w_2\bigl(\bigl|P_2\bar{u}_2 + s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr|\bigr) + g_2(t)\bigr]\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} \int_0^1 \left[c_{10} f_1(t) w_1\left(|P_1 \bar{u}_1|\right) + c_{10} f_1(t) w_1\left(s\left|Q_1 \bar{u}_1 + \tilde{u}_1(t)\right|\right)\right] ds \\
&\quad + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} \int_0^1 \left[c_{20} f_2(t) w_2\left(|P_2 \bar{u}_2|\right) + c_{20} f_2(t) w_2\left(s\left|Q_2 \bar{u}_2 + \tilde{u}_2(t)\right|\right)\right] ds \\
&\quad + \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} \int_0^1 g_1(t)\, ds + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} \int_0^1 g_2(t)\, ds \\
&\leq \left(M_{13} + \|\tilde{u}_1\|_\infty\right) w_1\left(|P_1 \bar{u}_1|\right) \sum_{t=1}^{T} c_{10} f_1(t) + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) w_2\left(|P_2 \bar{u}_2|\right) \sum_{t=1}^{T} c_{20} f_2(t) \\
&\quad + \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} \int_0^1 \left[c_{10} f_1(t) k_{11} \left|s\left(Q_1 \bar{u}_1 + \tilde{u}_1(t)\right)\right|^{\alpha_1} + f_1(t) c_{10} k_{12} + g_1(t)\right] ds \\
&\quad + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} \int_0^1 \left[c_{20} f_2(t) k_{21} \left|s\left(Q_2 \bar{u}_2 + \tilde{u}_2(t)\right)\right|^{\alpha_2} + f_2(t) c_{20} k_{22} + g_2(t)\right] ds \\
&\leq \left(M_{13} + \|\tilde{u}_1\|_\infty\right) w_1\left(|P_1 \bar{u}_1|\right) \sum_{t=1}^{T} c_{10} f_1(t) \\
&\quad + \frac{1 + \varepsilon_1}{\alpha_1 + 1} \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} f_1(t) c_{10} k_{11} |Q_1 \bar{u}_1|^{\alpha_1} \\
&\quad + \frac{B(\varepsilon_1)}{\alpha_1 + 1} \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} f_1(t) c_{10} k_{11} \left|\tilde{u}_1(t)\right|^{\alpha_1} \\
&\quad + \left(M_{13} + \|\tilde{u}_1\|_\infty\right) \sum_{t=1}^{T} \left[f_1(t) c_{10} k_{12} + g_1(t)\right] \\
&\quad + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) w_2\left(|P_2 \bar{u}_2|\right) \sum_{t=1}^{T} c_{20} f_2(t) \\
&\quad + \frac{1 + \varepsilon_2}{\alpha_2 + 1} \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} f_2(t) c_{20} k_{21} |Q_2 \bar{u}_2|^{\alpha_2} \\
&\quad + \frac{B(\varepsilon_2)}{\alpha_2 + 1} \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} f_2(t) c_{20} k_{21} \left|\tilde{u}_2(t)\right|^{\alpha_2} \\
&\quad + \left(M_{23} + \|\tilde{u}_2\|_\infty\right) \sum_{t=1}^{T} \left[f_2(t) c_{20} k_{22} + g_2(t)\right] \\
&\leq M_{11} c_{10} w_1\left(|P_1 \bar{u}_1|\right) \|\tilde{u}_1\|_\infty + M_{11} M_{13} c_{10} w_1\left(|P_1 \bar{u}_1|\right) \\
&\quad + \frac{(1 + \varepsilon_1) M_{13}^{\alpha_1 + 1} c_{10} k_{11} M_{11}}{\alpha_1 + 1} + \frac{(1 + \varepsilon_1) M_{13}^{\alpha_1} c_{10} k_{11} M_{11}}{\alpha_1 + 1} \|\tilde{u}_1\|_\infty \\
&\quad + \frac{B(\varepsilon_1) M_{13} c_{10} k_{11} M_{11}}{\alpha_1 + 1} \|\tilde{u}_1\|_\infty^{\alpha_1} + \frac{B(\varepsilon_1) c_{10} k_{11} M_{11}}{\alpha_1 + 1} \|\tilde{u}_1\|_\infty^{\alpha_1 + 1} \\
&\quad + M_{13} \sum_{t=1}^{T} \left[f_1(t) c_{10} k_{12} + g_1(t)\right] + \|\tilde{u}_1\|_\infty \sum_{t=1}^{T} \left[f_1(t) c_{10} k_{12} + g_1(t)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ M_{21}c_{20}w_2\bigl(|P_2\bar{u}_2|\bigr)\|\tilde{u}_2\|_\infty + M_{21}M_{23}c_{20}w_2\bigl(|P_2\bar{u}_2|\bigr) \\
&+ \frac{(1+\varepsilon_2)M_{23}^{\alpha_2+1}c_{20}k_{21}M_{21}}{\alpha_2+1} + \frac{(1+\varepsilon_2)M_{23}^{\alpha_2}c_{20}k_{21}M_{21}}{\alpha_2+1}\|\tilde{u}_2\|_\infty \\
&+ \frac{B(\varepsilon_2)M_{23}c_{20}k_{21}M_{21}}{\alpha_2+1}\|\tilde{u}_2\|_\infty^{\alpha_2} + \frac{B(\varepsilon_2)c_{20}k_{21}M_{21}}{\alpha_2+1}\|\tilde{u}_2\|_\infty^{\alpha_2+1} \\
&+ M_{23}\sum_{t=1}^{T}\bigl[c_{20}f_2(t)k_{22}+g_2(t)\bigr] + \|\tilde{u}_2\|_\infty\sum_{t=1}^{T}\bigl[c_{20}f_2(t)k_{22}+g_2(t)\bigr] \\
\le{}& \frac{1}{pa_1}\left(\frac{1}{C(p')}\right)^{p/p'}\|\tilde{u}_1\|_\infty^{p} + \frac{M_{11}^{p'}c_{10}^{p'}a_1^{p'/p}C(p')}{p'}w_1^{p'}\bigl(|P_1\bar{u}_1|\bigr) \\
&+ C_{11}(\varepsilon_1)\|\tilde{u}_1\|_\infty^{\alpha_1+1} + C_{12}(\varepsilon_1)\|\tilde{u}_1\|_\infty^{\alpha_1} + C_{13}(\varepsilon_1)\|\tilde{u}_1\|_\infty + C_{14} \\
&+ \frac{1}{qa_2}\left(\frac{1}{C(q')}\right)^{q/q'}\|\tilde{u}_2\|_\infty^{q} + \frac{M_{21}^{q'}c_{20}^{q'}a_2^{q'/q}C(q')}{q'}w_2^{q'}\bigl(|P_2\bar{u}_2|\bigr) \\
&+ C_{21}(\varepsilon_2)\|\tilde{u}_2\|_\infty^{\alpha_2+1} + C_{22}(\varepsilon_2)\|\tilde{u}_2\|_\infty^{\alpha_2} + C_{23}(\varepsilon_2)\|\tilde{u}_2\|_\infty + C_{24} \\
&+ M_{11}M_{13}c_{10}w_1\bigl(|P_1\bar{u}_1|\bigr) + M_{21}M_{23}c_{20}w_2\bigl(|P_2\bar{u}_2|\bigr) \\
\le{}& \frac{C(p')}{pa_1}\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p + \frac{M_{11}^{p'}c_{10}^{p'}a_1^{p'/p}C(p')}{p'}w_1^{p'}\bigl(|P_1\bar{u}_1|\bigr) \\
&+ C_{11}(\varepsilon_1)\bigl[C(p')\bigr]^{\alpha_1+1}\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{\frac{\alpha_1+1}{p}} + C_{14} \\
&+ C_{13}(\varepsilon_1)C(p')\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{\frac{1}{p}} + C_{12}(\varepsilon_1)\bigl[C(p')\bigr]^{\alpha_1}\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{\frac{\alpha_1}{p}} \\
&+ \frac{C(q')}{qa_2}\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q + \frac{M_{21}^{q'}c_{20}^{q'}a_2^{q'/q}C(q')}{q'}w_2^{q'}\bigl(|P_2\bar{u}_2|\bigr) \\
&+ C_{21}(\varepsilon_2)\bigl[C(q')\bigr]^{\alpha_2+1}\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{\frac{\alpha_2+1}{q}} + C_{24} \\
&+ C_{23}(\varepsilon_2)C(q')\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{\frac{1}{q}} + C_{22}(\varepsilon_2)\bigl[C(q')\bigr]^{\alpha_2}\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{\frac{\alpha_2}{q}} \\
&+ M_{11}M_{13}c_{10}w_1\bigl(|P_1\bar{u}_1|\bigr) + M_{21}M_{23}c_{20}w_2\bigl(|P_2\bar{u}_2|\bigr).
\end{aligned}
\tag{3.2}
$$

By $(\mathcal{A}1)$, (3.2), and Lemma 2.1, we have

$$
\begin{aligned}
\varphi(u) &= \varphi(\hat{u}_1,\hat{u}_2) \\
&= \sum_{t=1}^{T}\bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + F\bigl(t,\hat{u}_1(t),\hat{u}_2(t)\bigr) \\
&\quad + \bigl(h_1(t),\hat{u}_1(t)\bigr) + \bigl(h_2(t),\hat{u}_2(t)\bigr)\bigr] \\
&= \sum_{t=1}^{T}\bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + F\bigl(t,\hat{u}_1(t),\hat{u}_2(t)\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - F(t, P_1\bar{u}_1, P_2\bar{u}_2) + F(t, P_1\bar{u}_1, P_2\bar{u}_2) + \big(h_1(t), \hat{u}_1(t)\big) + \big(h_2(t), \hat{u}_2(t)\big)\big] \\
&\geq \sum_{t=1}^{T} \big(\gamma_1 \big|\Delta u_1(t)\big|^p + \gamma_3 \big|\Delta u_2(t)\big|^q - \gamma_2 \big|\Delta u_1(t)\big|^{\beta_1} - \gamma_4 \big|\Delta u_2(t)\big|^{\beta_2}\big) \\
&\quad - \frac{C(p')}{pa_1} \sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p - \frac{M_{11}^{p'} c_{10}^{p'} a_1^{p'/p} C(p')}{p'} w_1^{p'}\big(|P_1\bar{u}_1|\big) \\
&\quad - C_{11}(\varepsilon_1)\big[C\big(p'\big)\big]^{\alpha_1+1} \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{\alpha_1+1}{p}} - C_{14} - \|\tilde{u}_1\|_\infty \sum_{t=1}^{T} \big|h_1(t)\big| \\
&\quad - C_{13}(\varepsilon_1) C\big(p'\big) \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{1}{p}} - C_{12}(\varepsilon_1)\big[C\big(p'\big)\big]^{\alpha_1} \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{\alpha_1}{p}} \\
&\quad - \frac{C(q')}{qa_2} \sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q - \frac{M_{21}^{q'} c_{20}^{q'} a_2^{q'/q} C(q')}{q'} w_2^{q'}\big(|P_2\bar{u}_2|\big) \\
&\quad - C_{21}(\varepsilon_2)\big[C\big(q'\big)\big]^{\alpha_2+1} \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{\alpha_2+1}{q}} - C_{24} - \|\tilde{u}_2\|_\infty \sum_{t=1}^{T} \big|h_2(t)\big| \\
&\quad - C_{23}(\varepsilon_2) C\big(q'\big) \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{1}{q}} - C_{22}(\varepsilon_2)\big[C\big(q'\big)\big]^{\alpha_2} \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{\alpha_2}{q}} \\
&\quad - M_{11} M_{13} c_{10} w_1\big(|P_1\bar{u}_1|\big) - M_{21} M_{23} c_{20} w_2\big(|P_2\bar{u}_2|\big) \\
&\geq \left(\gamma_1 - \frac{C(p')}{pa_1}\right) \sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p - M_{11} M_{13} c_{10} w_1\big(|P_1\bar{u}_1|\big) \\
&\quad - C_{11}(\varepsilon_1)\big[C\big(p'\big)\big]^{\alpha_1+1} \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{\alpha_1+1}{p}} \\
&\quad - C_{12}(\varepsilon_1)\big[C\big(p'\big)\big]^{\alpha_1} \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{\alpha_1}{p}} \\
&\quad - C_{13}(\varepsilon_1) C\big(p'\big) \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{1}{p}} - \gamma_2 T^{1-\frac{\beta_1}{p}} \left(\sum_{t=1}^{T} \big|\Delta u_1(t)\big|^p\right)^{\frac{\beta_1}{p}} \\
&\quad - C_{14} + \left(\gamma_3 - \frac{C(q')}{qa_2}\right) \sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q - M_{21} M_{23} c_{20} w_2\big(|P_2\bar{u}_2|\big) \\
&\quad - C_{21}(\varepsilon_2)\big[C\big(q'\big)\big]^{\alpha_2+1} \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{\alpha_2+1}{q}} \\
&\quad - C_{22}(\varepsilon_2)\big[C\big(q'\big)\big]^{\alpha_2} \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{\alpha_2}{q}} \\
&\quad - C_{23}(\varepsilon_2) C\big(q'\big) \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{1}{q}} - \gamma_4 T^{1-\frac{\beta_2}{q}} \left(\sum_{t=1}^{T} \big|\Delta u_2(t)\big|^q\right)^{\frac{\beta_2}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&- C_{24} + \left[w_1^{p'}\left(|P_1\bar{u}_1|\right) + w_2^{q'}\left(|P_2\bar{u}_2|\right)\right]\left[\frac{F(t,P_1\bar{u}_1,P_2\bar{u}_2)}{w_1^{p'}(|P_1\bar{u}_1|) + w_2^{q'}(|P_2\bar{u}_2|)}\right. \\
&\left.- \max\left\{\frac{M_{11}^{p'}c_{10}^{p'}a_1^{p'/p}C(p')}{p'}, \frac{M_{21}^{q'}c_{20}^{q'}a_2^{q'/q}C(q')}{q'}\right\}\right] \\
&- C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{\frac{1}{p}}\sum_{t=1}^{T}\left|h_1(t)\right| \\
&- C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{\frac{1}{q}}\sum_{t=1}^{T}\left|h_2(t)\right|. \qquad (3.3)
\end{aligned}
$$

It follows from (3.1), (3.3), $a_1 > \frac{C(p')}{p\gamma_1}$, and $a_2 > \frac{C(q')}{q\gamma_3}$ that φ is bounded from below. Let G be a discrete subgroup of $\mathcal{H}$ defined by (2.10) and let $\pi : \mathcal{H} \to \mathcal{H}/G$ be the canonical surjection. By (2.14)-(2.17), it is easy to verify that φ is G-invariant. In what follows, we show that the functional φ satisfies the $(\mathrm{PS})_G$ condition, that is, for every sequence $\{u_m\}$ in $\mathcal{H}$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$, the sequence $\{\pi(u_n)\}$ has a convergent subsequence. In fact, the boundedness of $\varphi(u_n)$, (3.1), (3.3), and the facts that $a_1 > \frac{C(p')}{p\gamma_1}$ and $a_2 > \frac{C(q')}{q\gamma_3}$ imply that $(P\bar{u}_n)$ and $\sum_{t=1}^{T}|\Delta u_n(t)|^2$ are bounded. Furthermore, by Lemma 2.1, we know that $(\tilde{u}_n)$ is also bounded. Hence $\{\hat{u}_n\}$ is bounded in $\mathcal{H}$. Since $\dim\mathcal{H} < \infty$, we know that $\{\hat{u}_n\}$ has a convergent subsequence. Since $\pi(u_n) = \pi(\hat{u}_n)$, $\{\pi(u_n)\}$ also has a convergent subsequence. Thus, by Lemma 2.5, we know that φ has $r_1 + r_2 + 1$ critical orbits. Hence, system (1.1) has at least $r_1 + r_2 + 1$ geometrically distinct solutions in $\mathcal{H}$. The proof is complete. □

Proof of Theorem 1.2 First, we prove that Ψ defined by (2.11) satisfies the (PS) condition. Assume that $\{(y^{[n]} + z^{[n]}, v^{[n]})\}_{n=1}^{\infty} \subset X \times T^{r_1+r_2}$ is (PS) sequence for Ψ, that is, $\{\Psi((y^{[n]} + z^{[n]}, v^{[n]}))\}$ is bounded and $\Psi'((y^{[n]} + z^{[n]}, v^{[n]})) \to 0$, where $y^{[n]} = (y_1^{[n]}, y_2^{[n]})^\tau \in Y$, $z^{[n]} = z^{[n]}(t) = (z_1^{[n]}(t), z_2^{[n]}(t))^\tau \in Z$, $v^{[n]} = (v_1^{[n]}, v_2^{[n]})^\tau \in T^{r_1+r_2}$ for $n = 1, 2, \ldots$. Let

$$
u^{[n]} = y^{[n]} + v^{[n]} + z^{[n]} = \left(y_1^{[n]} + v_1^{[n]} + z_1^{[n]}, y_2^{[n]} + v_2^{[n]} + z_2^{[n]}\right)^\tau, \quad n = 1, 2 \ldots.
$$

Then it is easy to see that

$$
y_m^{[n]} = P_m\bar{u}_m^{[n]}, \qquad v_m^{[n]} = Q_m\bar{u}_m^{[n]}, \qquad z_m^{[n]}(t) = \tilde{u}_m^{[n]}(t), \quad m = 1, 2, n = 1, 2 \ldots.
$$

By (2.12) and (2.13), we find that $\{\varphi(u_1^{[n]}, u_2^{[n]})\}$ is bounded and $\varphi'(u_1^{[n]}, u_2^{[n]}) \to 0$. Then there exists a positive constant D_0 such that

$$
\left|\varphi\left(u_1^{[n]}, u_2^{[n]}\right)\right| \le D_0, \qquad \left\|\varphi'\left(u_1^{[n]}, u_2^{[n]}\right)\right\| \le D_0, \quad \forall n \in \mathbb{R}. \qquad (3.4)
$$

By $(\mathcal{F}4)'$, there exist $a_3 > C(p')$ and $a_4 > C(q')$ such that

$$
\begin{aligned}
&\lim_{|x_1|+|x_2|\to\infty} \frac{F(t,x_1,x_2)}{w_1^{p'}(|x_1|) + w_2^{q'}(|x_2|)} \\
&\quad < -\max\left\{\frac{[a_3M_{11}c_{10}]^{p'}}{p'}\left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right],\right. \\
&\quad \left.\frac{[a_4M_{21}c_{20}]^{q'}}{q'}\left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right]\right\}. \qquad (3.5)
\end{aligned}
$$

It follows from $(\mathcal{F}3)$, Lemma 2.1, and Young's inequality that, for all $(u_1, u_2) \in \mathcal{H}$,

$$
\begin{aligned}
&\left|\sum_{t=1}^{T}\big(\nabla_{x_1}F\big(t,\hat{u}_1(t),\hat{u}_2(t)\big),\tilde{u}_1(t)\big)+\sum_{t=1}^{T}\big(\nabla_{x_2}F\big(t,\hat{u}_1(t),\hat{u}_2(t)\big),\tilde{u}_2(t)\big)\right| \\
&\quad\le\left|\sum_{t=1}^{T}\big(\nabla_{x_1}F\big(t,P_1\bar{u}_1+Q_1\bar{u}_1+\tilde{u}_1(t),\hat{u}_2(t)\big),\tilde{u}_1(t)\big)\right| \\
&\qquad+\left|\sum_{t=1}^{T}\big(\nabla_{x_2}F\big(t,\hat{u}_1(t),P_2\bar{u}_2+Q_2\bar{u}_2+\tilde{u}_2(t)\big),\tilde{u}_2(t)\big)\right| \\
&\quad\le\sum_{t=1}^{T}f_1(t)w_1\big(\big|P_1\bar{u}_1+Q_1\bar{u}_1+\tilde{u}_1(t)\big|\big)\big|\tilde{u}_1(t)\big|+\sum_{t=1}^{T}g_1(t)\big|\tilde{u}_1(t)\big| \\
&\qquad+\sum_{t=1}^{T}f_2(t)w_2\big(\big|P_2\bar{u}_2+Q_2\bar{u}_2+\tilde{u}_2(t)\big|\big)\big|\tilde{u}_2(t)\big|+\sum_{t=1}^{T}g_2(t)\big|\tilde{u}_2(t)\big| \\
&\quad\le\|\tilde{u}_1\|_\infty c_{10}\big(w_1\big(|P_1\bar{u}_1|\big)+w_1\big(|Q_1\bar{u}_1+\tilde{u}_1|\big)\big)\sum_{t=1}^{T}f_1(t)+\|\tilde{u}_1\|_\infty\sum_{t=1}^{T}g_1(t) \\
&\qquad+\|\tilde{u}_2\|_\infty c_{20}\big(w_2\big(|P_2\bar{u}_2|\big)+w_2\big(|Q_2\bar{u}_2+\tilde{u}_2|\big)\big)\sum_{t=1}^{T}f_2(t)+\|\tilde{u}_2\|_\infty\sum_{t=1}^{T}g_2(t) \\
&\quad\le\frac{1}{pa_3^p}\|\tilde{u}_1\|_\infty^p+\frac{[a_3M_{11}c_{10}]^{p'}}{p'}w_1^{p'}\big(|P_1\bar{u}_1|\big) \\
&\qquad+\|\tilde{u}_1\|_\infty c_{10}\big(k_{11}\big|Q_1\bar{u}_1+\tilde{u}_1(t)\big|^{\alpha_1}+k_{12}\big)\sum_{t=1}^{T}f_1(t) \\
&\qquad+\frac{1}{qa_4^q}\|\tilde{u}_2\|_\infty^q+\frac{[a_4M_{21}c_{20}]^{q'}}{q'}w_2^{q'}\big(|P_2\bar{u}_2|\big) \\
&\qquad+\|\tilde{u}_2\|_\infty c_{20}\big(k_{21}\big|Q_2\bar{u}_2+\tilde{u}_2(t)\big|^{\alpha_2}+k_{22}\big)\sum_{t=1}^{T}f_2(t) \\
&\qquad+\|\tilde{u}_1\|_\infty\sum_{t=1}^{T}g_1(t)+\|\tilde{u}_2\|_\infty\sum_{t=1}^{T}g_2(t) \\
&\quad\le\frac{1}{pa_3^p}\|\tilde{u}_1\|_\infty^p+\frac{[a_3M_{11}c_{10}]^{p'}}{p'}w_1^{p'}\big(|P_1\bar{u}_1|\big) \\
&\qquad+M_{11}c_{10}k_{11}(1+\varepsilon_1)|Q_1\bar{u}_1|^{\alpha_1}\|\tilde{u}_1\|_\infty+\|\tilde{u}_1\|_\infty^{\alpha_1+1}M_{11}c_{10}k_{11}B_1(\varepsilon_1) \\
&\qquad+\|\tilde{u}_1\|_\infty M_{11}c_{10}k_{12} \\
&\qquad+\frac{1}{qa_4^q}\|\tilde{u}_2\|_\infty^q+\frac{[a_4M_{21}c_{20}]^{q'}}{q'}w_2^{q'}\big(|P_2\bar{u}_2|\big) \\
&\qquad+M_{21}c_{20}k_{21}(1+\varepsilon_2)|Q_2\bar{u}_2|^{\alpha_2}\|\tilde{u}_2\|_\infty+\|\tilde{u}_2\|_\infty^{\alpha_2+1}M_{21}c_{20}k_{21}B_2(\varepsilon_2) \\
&\qquad+\|\tilde{u}_2\|_\infty M_{21}c_{20}k_{22} \\
&\quad\le\frac{[C(p')]^p}{pa_3^p}\sum_{t=1}^{T}\big|\Delta u_1(t)\big|^p+\frac{[a_3M_{11}c_{10}]^{p'}}{p'}w_1^{p'}\big(|P_1\bar{u}_1|\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ C_{15}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{1/p} + C_{16}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{(\alpha_1+1)/p} \\
&+ \frac{[C(q')]^q}{qa_4^q}\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q + \frac{[a_4M_{21}c_{20}]^{q'}}{q'}w_2^{q'}\left(|P_2\bar{u}_2|\right) \\
&+ C_{25}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{1/q} + C_{26}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{(\alpha_2+1)/q}.
\end{aligned}
\tag{3.6}
$$

Hence, by ($\mathcal{A}2$) and (3.6), we have

$$
\begin{aligned}
&D_0\left(\left\|\tilde{u}_1^{[n]}\right\|_p + \left\|\tilde{u}_2^{[n]}\right\|_q\right) \\
&\quad\geq \left|\left\langle\varphi'\left(u_1^{[n]}, u_2^{[n]}\right), \left(\tilde{u}_1^{[n]}, \tilde{u}_2^{[n]}\right)\right\rangle\right| \\
&\quad= \left|\sum_{t=1}^{T}\left[\left(\phi_1\left(\Delta u_1^{[n]}(t)\right), \Delta u_1^{[n]}(t)\right) + \left(\phi_2\left(\Delta u_2^{[n]}(t)\right), \Delta u_2^{[n]}(t)\right)\right.\right. \\
&\qquad + \left(\nabla_{x_1}F\left(t, u_1^{[n]}(t), u_2^{[n]}(t)\right), \tilde{u}_1^{[n]}(t)\right) + \left(\nabla_{x_2}F\left(t, u_1^{[n]}(t), u_2^{[n]}(t)\right), \tilde{u}_2^{[n]}(t)\right) \\
&\qquad \left.\left. + \left(h_1(t), \tilde{u}_1^{[n]}(t)\right) + \left(h_2(t), \tilde{u}_2^{[n]}(t)\right)\right]\right| \\
&\quad\geq d_1\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p - d_2\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^{\beta_3} + d_3\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^p \\
&\qquad - d_4\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^{\beta_4} - \frac{[C(p')]^p}{pa_3^p}\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p - \frac{[a_3M_{11}c_{10}]^{p'}}{p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&\qquad - C_{15}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - C_{16}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{(\alpha_1+1)/p} \\
&\qquad - \frac{[C(q')]^q}{qa_4^q}\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q - \frac{[a_4M_{21}c_{20}]^{q'}}{q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&\qquad - C_{25}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} - C_{26}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{(\alpha_2+1)/q} \\
&\qquad - M_{14}C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - M_{24}C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} \\
&\quad\geq \left(d_1 - \frac{[C(p')]^p}{pa_3^p}\right)\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p - d_2T^{1-\frac{\beta_3}{p}}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\beta_3/p} \\
&\qquad + \left(d_3 - \frac{[C(q')]^q}{qa_4^q}\right)\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^p - d_4T^{1-\frac{\beta_4}{q}}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\beta_4/q} \\
&\qquad - \frac{[a_3M_{11}c_{10}]^{p'}}{p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&\qquad - C_{15}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - C_{16}(\varepsilon_1)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{(\alpha_1+1)/p}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{[a_4 M_{21} c_{20}]^{q'}}{q'} w_2^{q'}\left(\left|P_2 \bar{u}_2^{[n]}\right|\right) \\
&- C_{25}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} - C_{26}(\varepsilon_2)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{(\alpha_2+1)/q} \\
&- M_{14} C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - M_{24} C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q}
\end{aligned} \tag{3.7}
$$

for all $n \in \mathbb{R}$. Moreover, by Lemma 2.1, we have

$$
\begin{aligned}
&D_0\left(\left\|\tilde{u}_1^{[n]}\right\|_p + \left\|\tilde{u}_2^{[n]}\right\|_q\right) \\
&\quad \le D_0\left(C\left(p, p'\right) + 1\right)^{1/p}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} \\
&\qquad + D_0\left(C\left(q, q'\right) + 1\right)^{1/q}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q}.
\end{aligned} \tag{3.8}
$$

Then (3.7) and (3.8) imply that

$$
\begin{aligned}
&\frac{[a_3 M_{11} c_{10}]^{p'}}{p'} w_1^{p'}\left(\left|P_1 \bar{u}_1^{[n]}\right|\right) + \frac{[a_4 M_{21} c_{20}]^{q'}}{q'} w_2^{q'}\left(\left|P_2 \bar{u}_2^{[n]}\right|\right) \\
&\quad \ge \left(d_1 - \frac{1}{p}\right)\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p + \left(d_3 - \frac{1}{q}\right)\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q + D_1,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
D_1 = {} & \min_{s\in[0,+\infty)}\left\{\left(\frac{1}{p} - \frac{[C(p')]^p}{p a_3^p}\right)s^p - d_2 T^{1-\frac{\beta_3}{p}} s^{\beta_3} - C_{16}(\varepsilon_1) s^{\alpha_1+1}\right. \\
&\left. - \left(C_{15}(\varepsilon_1) + M_{14} C\left(p'\right)\right)s\right\} \\
&+ \min_{s\in[0,+\infty)}\left\{\left(\frac{1}{q} - \frac{[C(q')]^q}{q a_4^q}\right)s^p - d_4 T^{1-\frac{\beta_4}{q}} s^{\beta_4} - C_{26}(\varepsilon_2) s^{\alpha_2+1}\right. \\
&\left. - \left(C_{25}(\varepsilon_2) + M_{24} C\left(q'\right)\right)s\right\}.
\end{aligned}
$$

Note that $a_3 > C(p')$, $a_4 > C(q')$, $\alpha_1, \beta_3 \in [0, p)$, and $\alpha_2, \beta_3 \in [0, q)$. Hence (3.9) implies that there exist positive constants D_2 and D_3 such that

$$
\begin{aligned}
\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p \le {} & \frac{p[a_3 M_{11} c_{10}]^{p'}}{(d_1 p - 1)p'} w_1^{p'}\left(\left|P_1 \bar{u}_1^{[n]}\right|\right) \\
&+ \frac{p[a_4 M_{21} c_{20}]^{q'}}{(d_1 p - 1)q'} w_2^{q'}\left(\left|P_2 \bar{u}_2^{[n]}\right|\right) + D_2,
\end{aligned} \tag{3.10}
$$

$$\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q \leq \frac{q[a_3M_{11}c_{10}]^{p'}}{(d_3q-1)p'}w_1^{p'}\left(P_1\bar{u}_1^{[n]}\right) + \frac{q[a_4M_{21}c_{20}]^{q'}}{(d_3q-1)q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + D_3. \tag{3.11}$$

Then it is easy to see that $-\infty < D_1 < 0$. By (3.2), we know that

$$\begin{aligned}
&\sum_{t=1}^{T}\left|F\left(t,u_1^{[n]}(t),u_2^{[n]}(t)\right) - F\left(t,P_1\bar{u}_1^{[n]},P_2\bar{u}_2^{[n]}\right)\right| \\
&\quad\leq \frac{C(p')}{pa_3}\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p + \frac{M_{11}^{p'}c_{10}^{p'}a_3^{p'/p}C(p')}{p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{14} \\
&\qquad + C_{11}(\varepsilon_1)\left[C\left(p'\right)\right]^{\alpha_1+1}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\alpha_1+1}{p}} + M_{11}M_{13}c_{10}w_1\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&\qquad + C_{12}(\varepsilon_1)\left[C\left(p'\right)\right]^{\alpha_1}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\alpha_1}{p}} + C_{13}(\varepsilon_1)C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{1}{p}} \\
&\qquad + \frac{C(q')}{qa_4}\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q + \frac{M_{21}^{q'}c_{20}^{q'}a_4^{q'/q}C(q')}{q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + C_{24} \\
&\qquad + M_{21}M_{23}c_{20}w_2\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + C_{21}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2+1}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2+1}{q}} \\
&\qquad + C_{23}(\varepsilon_2)C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{1}{q}} \\
&\qquad + C_{22}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2}{q}}.
\end{aligned} \tag{3.12}$$

By $(\mathcal{A}1)'$, (3.10), (3.11), (3.12), and Lemma 2.1, we have

$$\begin{aligned}
&\varphi\left(u^{[n]}\right) \\
&\quad = \varphi\left(u_1^{[n]},u_2^{[n]}\right) \\
&\quad = \sum_{t=1}^{T}\left[\Phi_1\left(\Delta u_1^{[n]}(t)\right) + \Phi_2\left(\Delta u_2^{[n]}(t)\right) + F\left(t,u_1^{[n]}(t),u_2^{[n]}(t)\right)\right. \\
&\qquad \left. + \left(h_1(t),u_1^{[n]}(t)\right) + \left(h_2(t),u_2^{[n]}(t)\right)\right] \\
&\quad = \sum_{t=1}^{T}\left[\Phi_1\left(\Delta u_1^{[n]}(t)\right) + \Phi_2\left(\Delta u_2^{[n]}(t)\right) + F\left(t,u_1^{[n]}(t),u_2^{[n]}(t)\right) - F\left(t,P_1\bar{u}_1^{[n]},P_2\bar{u}_2^{[n]}\right)\right. \\
&\qquad \left. + F\left(t,P_1\bar{u}_1^{[n]},P_2\bar{u}_2^{[n]}\right) + \left(h_1(t),u_1^{[n]}(t)\right) + \left(h_2(t),u_2^{[n]}(t)\right)\right] \\
&\quad \leq \sum_{t=1}^{T}\left(\zeta_1\left|\Delta u_1^{[n]}(t)\right|^p + \zeta_3\left|\Delta u_2^{[n]}(t)\right|^q + \zeta_2\left|\Delta u_1^{[n]}(t)\right|^{\theta_1} + \zeta_4\left|\Delta u_2^{[n]}(t)\right|^{\theta_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{C(p')}{pa_3}\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p+\frac{M_{11}^{p'}c_{10}^{p'}a_3^{p'/p}C(p')}{p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right)\\
&+M_{11}M_{13}c_{10}w_1\left(\left|P_1\bar{u}_1^{[n]}\right|\right)+C_{11}(\varepsilon_1)\left[C\left(p'\right)\right]^{\alpha_1+1}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\alpha_1+1}{p}}\\
&+C_{12}(\varepsilon_1)\left[C\left(p'\right)\right]^{\alpha_1}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\alpha_1}{p}}+C_{13}(\varepsilon_1)C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{1}{p}}\\
&+C_{14}+\left\|\tilde{u}_1^{[n]}\right\|_\infty\sum_{t=1}^{T}\left|h_1(t)\right|\\
&+\frac{C(q')}{qa_4}\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q+\frac{M_{21}^{q'}c_{20}^{q'}a_4^{q'/q}C(q')}{q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right)\\
&+M_{21}M_{23}c_{20}w_2\left(\left|P_2\bar{u}_2^{[n]}\right|\right)+C_{21}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2+1}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2+1}{q}}\\
&+C_{22}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2}{q}}+C_{23}(\varepsilon_2)C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{1}{q}}\\
&+C_{24}+\left\|\tilde{u}_2^{[n]}\right\|_\infty\sum_{t=1}^{T}\left|h_2(t)\right|+\sum_{t=1}^{T}F\left(t,P_1\bar{u}_1^{[n]},P_2\bar{u}_2^{[n]}\right)\\
\le{}&\left(\frac{C(p')}{pa_3}+\zeta_1\right)\left(\frac{p[a_3M_{11}c_{10}]^{p'}}{(d_1p-1)p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right)+\frac{p[a_4M_{21}c_{20}]^{q'}}{(d_1p-1)q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right)\right)\\
&+\frac{M_{11}^{p'}c_{10}^{p'}a_3^{p'/p}C(p')}{p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right)+M_{11}M_{13}c_{10}w_1\left(\left|P_1\bar{u}_1^{[n]}\right|\right)\\
&+M_{21}M_{23}c_{20}w_2\left(\left|P_2\bar{u}_2^{[n]}\right|\right)+C_{11}(\varepsilon_1)\left[C\left(p'\right)\right]^{\alpha_1+1}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\alpha_1+1}{p}}\\
&+C_{13}(\varepsilon_1)C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{1}{p}}+\zeta_2T^{1-\frac{\theta_1}{p}}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{\theta_1}{p}}\\
&+D_2\left(\frac{C(p')}{pa_3}+\zeta_1\right)+C_{14}\\
&+\left(\frac{C(q')}{qa_4}+\zeta_3\right)\left(\frac{q[a_3M_{11}c_{10}]^{p'}}{(d_3q-1)p'}w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right)+\frac{q[a_4M_{21}c_{20}]^{q'}}{(d_3q-1)q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right)\right)\\
&+\frac{M_{21}^{q'}c_{20}^{q'}a_4^{q'/q}C(q')}{q'}w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right)+C_{22}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2}{q}}\\
&+C_{21}(\varepsilon_2)\left[C\left(q'\right)\right]^{\alpha_2+1}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\alpha_2+1}{q}}+M_{21}M_{23}c_{20}w_2\left(\left|P_2\bar{u}_2^{[n]}\right|\right)\\
&+C_{23}(\varepsilon_2)C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{1}{q}}+\zeta_4T^{1-\frac{\theta_2}{p}}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{\theta_2}{q}}+C_{24}
\end{aligned}
$$

$$
\begin{aligned}
&+ C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{\frac{1}{p}}\sum_{t=1}^{T}\left|h_1(t)\right| + C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{\frac{1}{q}}\sum_{t=1}^{T}\left|h_2(t)\right| \\
&+ \sum_{t=1}^{T}F\left(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\right) + D_3\left(\frac{C(q')}{qa_4} + \zeta_3\right) \\
\le{}& \frac{[a_3M_{11}c_{10}]^{p'}}{p'} \\
&\times\left[\frac{C(p')}{(d_1p-1)a_3} + \frac{p\zeta_1}{d_1p-1} + \frac{C(p')}{a_3} + \frac{C(q')}{(d_3q-1)a_4} + \frac{q\zeta_3}{d_3q-1}\right]w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&+ \frac{[a_4M_{21}c_{20}]^{q'}}{q'} \\
&\times\left[\frac{C(p')}{(d_1p-1)a_3} + \frac{p\zeta_1}{d_1p-1} + \frac{C(q')}{a_4} + \frac{C(q')}{(d_3q-1)a_4} + \frac{q\zeta_3}{d_3q-1}\right]w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ C_{17}(\varepsilon)w_1^{\frac{p'(\alpha_1+1)}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{27}(\varepsilon)w_2^{\frac{q'(\alpha_2+1)}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ C_{18}(\varepsilon)w_1^{\frac{p'\alpha_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{28}(\varepsilon)w_2^{\frac{q'\alpha_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ C_{19}w_1^{\frac{p'\theta_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{29}w_2^{\frac{q'\theta_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ D_{10}w_1^{\frac{p'}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + D_{20}w_2^{\frac{q'}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + \sum_{t=1}^{T}F\left(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\right) \\
\le{}& \frac{[a_3M_{11}c_{10}]^{p'}}{p'}\left[\frac{1}{d_1p-1} + \frac{p\zeta_1}{d_1p-1} + 1 + \frac{1}{d_3q-1} + \frac{q\zeta_3}{d_3q-1}\right]w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&+ \frac{[a_4M_{21}c_{20}]^{q'}}{q'}\left[\frac{1}{d_1p-1} + \frac{p\zeta_1}{d_1p-1} + 1 + \frac{1}{d_3q-1} + \frac{q\zeta_3}{d_3q-1}\right]w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ C_{17}(\varepsilon)w_1^{\frac{p'(\alpha_1+1)}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{27}(\varepsilon)w_2^{\frac{q'(\alpha_2+1)}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ C_{18}(\varepsilon)w_1^{\frac{p'\alpha_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{28}(\varepsilon)w_2^{\frac{q'\alpha_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + C_{19}w_1^{\frac{p'\theta_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&+ C_{29}w_2^{\frac{q'\theta_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + D_{10}w_1^{\frac{p'}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + D_{20}w_2^{\frac{q'}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ \sum_{t=1}^{T}F\left(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\right) \\
\le{}& \left(w_1^{p'}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + w_2^{q'}\left(\left|P_2\bar{u}_2^{[n]}\right|\right)\right)\Bigg[\max\Bigg\{\frac{[a_3M_{11}c_{10}]^{p'}}{p'}\left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right], \\
&\frac{[a_4M_{21}c_{20}]^{q'}}{q'}\left[\frac{1+p\zeta_1}{d_1p-1} + \frac{1+q\zeta_3}{d_3q-1} + 1\right]\Bigg\} + \frac{\sum_{t=1}^{T}F(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]})}{w_1^{p'}(|P_1\bar{u}_1^{[n]}|) + w_2^{q'}(|P_2\bar{u}_2^{[n]}|)}\Bigg] \\
&+ C_{17}(\varepsilon)w_1^{\frac{p'(\alpha_1+1)}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{18}(\varepsilon)w_1^{\frac{p'\alpha_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + C_{19}w_1^{\frac{p'\theta_1}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) \\
&+ C_{27}(\varepsilon)w_2^{\frac{q'(\alpha_2+1)}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + C_{28}(\varepsilon)w_2^{\frac{q'\alpha_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) + C_{29}w_2^{\frac{q'\theta_2}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right) \\
&+ D_{10}w_1^{\frac{p'}{p}}\left(\left|P_1\bar{u}_1^{[n]}\right|\right) + D_{20}w_2^{\frac{q'}{q}}\left(\left|P_2\bar{u}_2^{[n]}\right|\right).
\end{aligned}
\tag{3.13}
$$

Then (3.5) and ($\mathcal{A}3$) imply that $\{P_1\bar{u}_1^{[n]}\}$, $\{P_2\bar{u}_2^{[n]}\}$, $\{w_1(P_1\bar{u}_2^{[n]})\}$, and $\{w_2(P_2\bar{u}_2^{[n]})\}$ are bounded. Furthermore, (3.10), (3.11), and (3.8) imply that $\{\tilde{u}_1^{[n]}\}$ and $\{\tilde{u}_2^{[n]}\}$ are bounded. Then $\{u^{[n]}\}$ is bounded in $\mathcal{H}$. Since $\dim\mathcal{H} < \infty$, $\{u^{[n]}\}$ has a convergent subsequence. Hence, Ψ satisfies the (PS) condition.

In order to use Lemma 2.3, next we prove the following conclusions:

(i) $\inf\{\Psi((z,v))|(z,v) \in Z \times T^{r_1+r_2}\} > -\infty$;

(ii) $\Psi((y,v)) \to -\infty$ uniformly for $(y,v) \in Y \times T^{r_1+r_2}$ as $|y| \to \infty$.

For $(z,v) \in Z \times T^{r_1+r_2}$, set $u = u(t) = z(t) + v = (z_1(t) + v_1, z_2(t) + v_2)^\tau$. Then $z_m(t) = \tilde{u}_m(t)$, $v_m = Q_m\bar{u}_m$, and $u_m(t) = z_m(t) + v_m$, $m = 1, 2$. By ($\mathcal{F}3$) and Lemma 2.1, we have

$$
\begin{aligned}
&\left|\sum_{t=1}^{T}\bigl[F\bigl(t,u_1(t),u_2(t)\bigr) - F(t,0,0)\bigr]\right| \\
&\quad\le \sum_{t=1}^{T}\bigl|F\bigl(t,u_1(t),u_2(t)\bigr) - F\bigl(t,0,u_2(t)\bigr)\bigr| + \sum_{t=1}^{T}\bigl|F\bigl(t,0,u_2(t)\bigr) - F(t,0,0)\bigr| \\
&\quad\le \sum_{t=1}^{T}\left|\int_0^1 \bigl(\nabla_{x_1}F\bigl(t,s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr),u_2(t)\bigr), Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr)\,ds\right| \\
&\qquad + \sum_{t=1}^{T}\left|\int_0^1 \bigl(\nabla_{x_2}F\bigl(t,0,s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr), Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\,ds\right| \\
&\quad\le \bigl(|Q_1\bar{u}_1| + \|\tilde{u}_1\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl|\nabla_{x_1}F\bigl(t,s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr),u_2(t)\bigr)\bigr|\,ds \\
&\qquad + \bigl(|Q_2\bar{u}_2| + \|\tilde{u}_2\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl|\nabla_{x_2}F\bigl(t,0,s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr)\bigr|\,ds \\
&\quad\le \bigl(M_{13} + \|\tilde{u}_1\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl[f_1(t)w_1\bigl(\bigl|s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr)\bigr|\bigr) + g_1(t)\bigr]\,ds \\
&\qquad + \bigl(M_{23} + \|\tilde{u}_2\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl[f_2(t)w_2\bigl(\bigl|s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr|\bigr) + g_2(t)\bigr]\,ds \\
&\quad\le \bigl(M_{13} + \|\tilde{u}_1\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl[k_{11}f_1(t)\bigl|s\bigl(Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr)\bigr|^{\alpha_1} + k_{12}f_1(t) + g_1(t)\bigr]\,ds \\
&\qquad + \bigl(M_{23} + \|\tilde{u}_2\|_\infty\bigr)\sum_{t=1}^{T}\int_0^1 \bigl[k_{21}f_2(t)\bigl|s\bigl(Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr)\bigr|^{\alpha_2} + k_{22}f_2(t) + g_2(t)\bigr]\,ds \\
&\quad\le \bigl(M_{13} + \|\tilde{u}_1\|_\infty\bigr)\sum_{t=1}^{T}\bigl[k_{11}f_1(t)\bigl|Q_1\bar{u}_1 + \tilde{u}_1(t)\bigr|^{\alpha_1} + k_{12}f_1(t) + g_1(t)\bigr] \\
&\qquad + \bigl(M_{23} + \|\tilde{u}_2\|_\infty\bigr)\sum_{t=1}^{T}\bigl[k_{21}f_2(t)\bigl|Q_2\bar{u}_2 + \tilde{u}_2(t)\bigr|^{\alpha_2} + k_{22}f_2(t) + g_2(t)\bigr] \\
&\quad\le D_{11}\|\tilde{u}_1\|_\infty^{\alpha_1+1} + D_{21}\|\tilde{u}_2\|_\infty^{\alpha_2+1} + D_{12}\|\tilde{u}_1\|_\infty^{\alpha_1} + D_{22}\|\tilde{u}_2\|_\infty^{\alpha_2} \\
&\qquad + D_{13}\|\tilde{u}_1\|_\infty + D_{23}\|\tilde{u}_2\|_\infty + D_4
\end{aligned}
$$

$$\begin{aligned}
&\le D_{11}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{\alpha_1+1}{p}} + D_{21}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{\alpha_2+1}{q}} \\
&\quad + D_{12}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{\alpha_1}{p}} + D_{22}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{\alpha_2}{q}} \\
&\quad + D_{13}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{1}{p}} + D_{23}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{1}{q}} + D_4. \qquad (3.14)
\end{aligned}$$

Then

$$\begin{aligned}
\Psi\big((z,v)\big) &= \varphi(u_1,u_2) \\
&= \sum_{t=1}^{T}\big[\Phi_1\big(\Delta u_1(t)\big) + \Phi_2\big(\Delta u_2(t)\big) + F\big(t,u_1(t),u_2(t)\big) \\
&\quad + \big(h_1(t),u_1(t)\big) + \big(h_2(t),u_2(t)\big)\big] \\
&= \sum_{t=1}^{T}\big[\Phi_1\big(\Delta u_1(t)\big) + \Phi_2\big(\Delta u_2(t)\big) + F\big(t,u_1(t),u_2(t)\big) - F(t,0,0) \\
&\quad + F(t,0,0) + \big(h_1(t),u_1(t)\big) + \big(h_2(t),u_2(t)\big)\big] \\
&\ge \sum_{t=1}^{T}\big(\gamma_1\left|\Delta u_1(t)\right|^p + \gamma_3\left|\Delta u_2(t)\right|^q - \gamma_2\left|\Delta u_1(t)\right|^{\beta_1} - \gamma_4\left|\Delta u_2(t)\right|^{\beta_2}\big) \\
&\quad - D_{11}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{\alpha_1+1}{p}} - D_{21}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{\alpha_2+1}{q}} \\
&\quad - D_{12}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{\alpha_1}{p}} - D_{22}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{\alpha_2}{q}} \\
&\quad - D_{13}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{1}{p}} - D_{23}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{1}{q}} - D_4 \\
&\quad - M_{14}C(p')\left(\sum_{s=1}^{T}\left|\Delta u_1(s)\right|^p\right)^{\frac{1}{p}} - M_{24}C(q')\left(\sum_{s=1}^{T}\left|\Delta u_2(s)\right|^q\right)^{\frac{1}{q}} \\
&\quad + \sum_{t=1}^{T}F(t,0,0). \qquad (3.15)
\end{aligned}$$

It is easy to see that conclusion (i) holds from (3.15).

For any $(y,v) \in Y \times T^{r_1+r_2}$, it follows from (1.2) and (2.11) that

$$\begin{aligned}
&\Psi\big((y,v)\big) \\
&\quad = \varphi(y_1+v_1, y_2+v_2) \\
&\quad = \sum_{t=1}^{T}F(t, y_1+v_1, y_2+v_2)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{T} F(t, y_1 + v_1, y_2 + v_2) - \sum_{t=1}^{T} F(t, y_1, y_2 + v_2) \\
&\quad + \sum_{t=1}^{T} F(t, y_1, y_2 + v_2) - \sum_{t=1}^{T} F(t, y_1, y_2) + \sum_{t=1}^{T} F(t, y_1, y_2) \\
&= \sum_{t=1}^{T} F(t, y_1, y_2) + \sum_{t=1}^{T} \int_0^1 \bigl(\nabla F(t, y_1 + sv_1, y_2 + v_2), v_1\bigr)\, ds \\
&\quad + \sum_{t=1}^{T} \int_0^1 \bigl(\nabla F(t, y_1, y_2 + sv_2), v_2\bigr)\, ds \\
&\le \sum_{t=1}^{T} F(t, y_1, y_2) + |v_1| \sum_{t=1}^{T} \int_0^1 f_1(t) w_1\bigl(|y_1 + sv_1|\bigr)\, ds + |v_1| \sum_{t=1}^{T} g_1(t) \\
&\quad + |v_2| \sum_{n=1}^{T} \int_0^1 f_2(t) w_2\bigl(|y_2 + sv_2|\bigr)\, ds + |v_2| \sum_{t=1}^{T} g_2(t) \\
&\le \sum_{t=1}^{T} F(t, y_1, y_2) + |v_1| \sum_{t=1}^{T} f_1(t) w_1\bigl(|y_1|\bigr) + |v_1| \sum_{t=1}^{T} \int_0^1 w_1\bigl(|sv_1|\bigr)\, ds \\
&\quad + |v_1| \sum_{t=1}^{T} g_1(t) + |v_2| \sum_{t=1}^{T} f_2(t) w_2\bigl(|y_2|\bigr) \\
&\quad + |v_2| \sum_{t=1}^{T} \int_0^1 w_2\bigl(|sv_2|\bigr)\, ds + |v_2| \sum_{t=1}^{T} g_2(t) \\
&\le \sum_{t=1}^{T} F(t, y_1, y_2) + D_5 w_1\bigl(|y_1|\bigr) + D_6 w_2\bigl(|y_2|\bigr) + D_7 \\
&= \bigl[w_1^{p'\alpha_1}\bigl(|y_1|\bigr) + w_2^{q'\alpha_2}\bigl(|y_2|\bigr)\bigr] \left(\bigl[w_1^{p'\alpha_1}\bigl(|y_1|\bigr) + w_2^{q'\alpha_2}\bigl(|y_2|\bigr)\bigr]^{-1} \sum_{t=1}^{T} F(t, y_1, y_2) \right) \\
&\quad + D_5 w_1\bigl(|y_1|\bigr) + D_6 w_2\bigl(|y_2|\bigr) + D_7
\end{aligned}
$$

for positive constants D_5, D_6, and D_7. Hence, the above inequality, (3.5) and ($\mathcal{A}3$) imply that conclusion (ii) holds. It follows from Lemma 2.6 that Ψ has at least $r_1 + r_2 + 1$ critical points. Hence φ has at least $r_1 + r_2 + 1$ geometrically distinct critical points. Therefore, system (1.1) has at least $r_1 + r_2 + 1$ geometrically distinct solutions in $\mathcal{H}$. The proof is complete. □

Proof of Theorem 1.3 Note that Φ_m are coercive, $m = 1, 2$. Then by Remark 1.1, we know that (1.2) holds. Hence, it follows from (1.2), ($\mathcal{F}5$), ($\mathcal{F}6$), and ($\mathcal{E}$) that

$$
\begin{aligned}
&\left| \sum_{t=1}^{T} \bigl[F\bigl(t, u_1(t), u_2(t)\bigr) - F(t, P_1\bar{u}_1, P_2\bar{u}_2)\bigr] \right| \\
&\quad \le \sum_{t=1}^{T} \bigl| F\bigl(t, u_1(t), u_2(t)\bigr) - F\bigl(t, P_1\bar{u}_1, u_2(t)\bigr) \bigr|
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{t=1}^{T}\bigl|F(t,P_1\bar{u}_1,u_2(t)-F(t,P_1\bar{u}_1,P_2\bar{u}_2)\bigr| \\
&\leq \sum_{t=1}^{T}\left|\int_0^1\bigl(\nabla_{x_1}F\bigl(t,P_1\bar{u}_1+s\bigl(Q_1\bar{u}_1+\tilde{u}_1(t)\bigr),u_2(t)\bigr),Q_1\bar{u}_1+\tilde{u}_1(t)\bigr)\,ds\right| \\
&\quad+\sum_{t=1}^{T}\left|\int_0^1\bigl(\nabla_{x_2}F\bigl(t,P_1\bar{u}_1,P_2\bar{u}_2+s\bigl(Q_2\bar{u}_2+\tilde{u}_2(t)\bigr)\bigr),Q_2\bar{u}_2+\tilde{u}_2(t)\bigr)\,ds\right| \\
&\leq \sum_{t=1}^{T}b_1(t)\bigl|Q_1\bar{u}_1+\tilde{u}_1(t)\bigr|+\sum_{t=1}^{T}b_2(t)\bigl|Q_2\bar{u}_2+\tilde{u}_2(t)\bigr| \\
&\leq \bigl(|Q_1\bar{u}_1|+\|\tilde{u}_1\|_\infty\bigr)\sum_{t=1}^{T}b_1(t)+\bigl(|Q_2\bar{u}_2|+\|\tilde{u}_2\|_\infty\bigr)\sum_{t=1}^{T}b_2(t) \\
&\leq M_{13}M_{15}+C\bigl(p'\bigr)M_{15}\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{1/p}+M_{23}M_{25} \\
&\quad+C\bigl(q'\bigr)M_{25}\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{1/q}.
\end{aligned}
\tag{3.16}
$$

Then

$$
\begin{aligned}
\varphi(u) &= \varphi(u_1,u_2) \\
&= \sum_{t=1}^{T}\bigl[\Phi_1\bigl(\Delta u_1(t)\bigr)+\Phi_2\bigl(\Delta u_2(t)\bigr)+F\bigl(t,u_1(t),u_2(t)\bigr)-F(t,P_1\bar{u}_1,P_2\bar{u}_2) \\
&\quad+F(t,P_1\bar{u}_1,P_2\bar{u}_2)+\bigl(h_1(t),u_1(t)\bigr)+\bigl(h_2(t),u_2(t)\bigr)\bigr] \\
&\geq \delta_1\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|+\delta_2\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|-C\bigl(p'\bigr)M_{15}\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{1/p} \\
&\quad-C\bigl(q'\bigr)M_{25}\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{1/q}+\sum_{t=1}^{T}F(t,P_1\bar{u}_1,P_2\bar{u}_2) \\
&\quad-C\bigl(p'\bigr)M_{14}\left(\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|^p\right)^{1/p}-C\bigl(q'\bigr)M_{24}\left(\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|^q\right)^{1/q} \\
&\quad-M_{13}M_{15}-M_{23}M_{25}-(\delta_1+\delta_2)T \\
&\geq \delta_1\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|+\delta_2\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|-M_{13}M_{15}-C\bigl(p'\bigr)M_{15}\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr| \\
&\quad-M_{23}M_{25}-C\bigl(q'\bigr)M_{25}\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|+\sum_{t=1}^{T}F(t,P_1\bar{u}_1,P_2\bar{u}_2)-(\delta_1+\delta_2)T \\
&\quad-C\bigl(p'\bigr)M_{14}\sum_{t=1}^{T}\bigl|\Delta u_1(t)\bigr|-C\bigl(q'\bigr)M_{24}\sum_{t=1}^{T}\bigl|\Delta u_2(t)\bigr|.
\end{aligned}
\tag{3.17}
$$

The features ($\mathcal{F}6$) and ($\mathcal{F}7$) imply that φ is bounded from below. Similar to the proof of Theorem 1.1, we can prove that φ is G-invariant and satisfies the $(\mathrm{PS})_G$ condition. Then by Lemma 2.5, we obtain the conclusion. □

Proof of Theorem 1.4 First, we prove that Ψ defined by (2.11) satisfies the (PS) condition. Assume that $\{(y^{[n]}+z^{[n]},v^{[n]})\}_{n=1}^{\infty}\subset X\times T^{r_1+r_2}$ is a (PS) sequence for Ψ, that is, $\{\Psi((y^{[n]}+z^{[n]},v^{[n]}))\}$ is bounded and $\Psi'((y^{[n]}+z^{[n]},v^{[n]}))\to 0$, where $y^{[n]}=(y_1^{[n]},y_2^{[n]})^\tau\in Y$, $z^{[n]}=z^{[n]}(t)=(z_1^{[n]}(t),z_2^{[n]}(t))^\tau\in Z$, $v^{[n]}=(v_1^{[n]},v_2^{[n]})^\tau\in T^{r_1+r_2}$ for $n=1,2,\ldots$. Let

$$u^{[n]}=y^{[n]}+v^{[n]}+z^{[n]}=\left(y_1^{[n]}+v_1^{[n]}+z_1^{[n]},y_2^{[n]}+v_2^{[n]}+z_2^{[n]}\right)^\tau,\quad n=1,2,\ldots.$$

Then it is easy to see that

$$y_m^{[n]}=P_m\bar{u}_m^{[n]},\qquad v_m^{[n]}=Q_m\bar{u}_m^{[n]},\qquad z_m^{[n]}(t)=\tilde{u}_m^{[n]}(t),\quad m=1,2,n=1,2,\ldots.$$

By (2.12) and (2.13), we find that $\{\varphi(u_1^{[n]},u_2^{[n]})\}$ is bounded and $\varphi'(u_1^{[n]},u_2^{[n]})\to 0$. Then there exists a positive constant G_0 such that

$$\left|\varphi\left(u_1^{[n]},u_2^{[n]}\right)\right|\le G_0,\qquad \left\|\varphi'\left(u_1^{[n]},u_2^{[n]}\right)\right\|\le G_0,\quad \forall n\in\mathbb{N}. \tag{3.18}$$

It follows from ($\mathcal{F}3$), Lemma 2.1, and Young's inequality that, for all $(u_1,u_2)\in\mathcal{H}$,

$$\begin{aligned}
&\left|\sum_{t=1}^{T}\left(\nabla_{x_1}F\left(t,\hat{u}_1(t),\hat{u}_2(t)\right),\tilde{u}_1(t)\right)+\sum_{t=1}^{T}\left(\nabla_{x_2}F\left(t,\hat{u}_1(t),\hat{u}_2(t)\right),\tilde{u}_2(t)\right)\right| \\
&\quad\le\left|\sum_{t=1}^{T}\left(\nabla_{x_1}F\left(t,P_1\bar{u}_1+Q_1\bar{u}_1+\tilde{u}_1(t),\hat{u}_2(t)\right),\tilde{u}_1(t)\right)\right| \\
&\qquad+\left|\sum_{t=1}^{T}\left(\nabla_{x_2}F\left(t,\hat{u}_1(t),P_2\bar{u}_2+Q_2\bar{u}_2+\tilde{u}_2(t)\right),\tilde{u}_2(t)\right)\right| \\
&\quad\le\sum_{t=1}^{T}b_1(t)\left|\tilde{u}_1(t)\right|+\sum_{t=1}^{T}b_2(t)\left|\tilde{u}_2(t)\right| \\
&\quad\le\|\tilde{u}_1\|_\infty\sum_{t=1}^{T}b_1(t)+\|\tilde{u}_2\|_\infty\sum_{t=1}^{T}b_2(t) \\
&\quad\le C\left(p'\right)\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{1/p}\sum_{t=1}^{T}b_1(t) \\
&\qquad+C\left(q'\right)\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{1/q}\sum_{t=1}^{T}b_2(t).
\end{aligned}\tag{3.19}$$

Hence we have

$$\begin{aligned}
&\left\|\tilde{u}_1^{[n]}\right\|_p+\left\|\tilde{u}_2^{[n]}\right\|_q \\
&\quad\ge\left|\left\langle\varphi'\left(u_1^{[n]},u_2^{[n]}\right),\left(\tilde{u}_1^{[n]},\tilde{u}_2^{[n]}\right)\right\rangle\right| \\
&\quad=\left|\sum_{t=1}^{T}\left[\left(\phi_1\left(\Delta u_1^{[n]}(t)\right),\Delta u_1^{[n]}(t)\right)+\left(\phi_2\left(\Delta u_2^{[n]}(t)\right),\Delta u_2^{[n]}(t)\right)\right.\right. \\
&\qquad+\left(\nabla_{x_1}F\left(t,u_1^{[n]}(t),u_2^{[n]}(t)\right),\tilde{u}_1^{[n]}(t)\right)+\left(\nabla_{x_2}F\left(t,u_1^{[n]}(t),u_2^{[n]}(t)\right),\tilde{u}_2^{[n]}(t)\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \left(h_1(t), \tilde{u}_1^{[n]}(t)\right) + \left(h_2(t), \tilde{u}_2^{[n]}(t)\right)\Big]\Bigg| \\
&\geq \sum_{t=1}^{T} \delta_1\left(\left|\Delta u_1^{[n]}(t)\right| - 1\right) + \sum_{t=1}^{T} \delta_2\left(\left|\Delta u_2^{[n]}(t)\right| - 1\right) \\
&\quad - C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} \sum_{t=1}^{T} b_1(t) - C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} \sum_{t=1}^{T} b_2(t) \\
&\quad - M_{14}C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - M_{15}C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} \\
&\geq \delta_1\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} + \delta_2\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} - (\delta_1 + \delta_2)T \\
&\quad - C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} \sum_{t=1}^{T} b_1(t) - C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} \sum_{t=1}^{T} b_2(t) \\
&\quad - M_{14}C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} - M_{15}C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q}
\end{aligned}
\tag{3.20}
$$

for large $n \in \mathbb{N}$ by the fact $\varphi'(u_1^{[n]}, u_2^{[n]}) \to 0$ as $n \to \infty$. Moreover, by Lemma 2.1, we have

$$
\begin{aligned}
&\left\|\tilde{u}_1^{[n]}\right\|_p + \left\|\tilde{u}_2^{[n]}\right\|_q \\
&\quad \leq \left(C(p,p') + 1\right)^{1/p}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} \\
&\qquad + \left(C(q,q') + 1\right)^{1/q}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q}.
\end{aligned}
\tag{3.21}
$$

Then $(\mathcal{F}7)'$, (3.20), and (3.21) imply that there exists a positive constant G_1 such that

$$
\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p \leq G_1, \qquad \sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q \leq G_1, \quad \forall n \in \mathbb{N}.
\tag{3.22}
$$

By (3.16) and the above inequality, we know that there exists a positive constant G_2 such that

$$
\begin{aligned}
&\sum_{t=1}^{T}\left|F\left(t, u_1^{[n]}(t), u_2^{[n]}(t)\right) - F\left(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\right)\right| \\
&\quad \leq M_{13}M_{15} + C(p')M_{15}\left(\sum_{t=1}^{T}\left|\Delta u_1^{[n]}(t)\right|^p\right)^{1/p} \\
&\qquad + M_{23}M_{25} + C(q')M_{25}\left(\sum_{t=1}^{T}\left|\Delta u_2^{[n]}(t)\right|^q\right)^{1/q} \\
&\quad \leq G_2.
\end{aligned}
\tag{3.23}
$$

By ($\mathcal{A}0$) and (3.22), there exists a positive constant G_3 such that

$$\Phi_1\big(\Delta u_1^{[n]}(t)\big) \le G_3, \qquad \Phi_2\big(\Delta u_2^{[n]}(t)\big) \le G_3. \tag{3.24}$$

Then it follows from (3.18), (3.22), (3.23), (3.24), and Lemma 2.1 that

$$\begin{aligned} -G_0 &\le \varphi\big(u^{[n]}\big) \\ &= \varphi\big(u_1^{[n]}, u_2^{[n]}\big) \\ &= \sum_{t=1}^{T}\big[\Phi_1\big(\Delta u_1^{[n]}(t)\big) + \Phi_2\big(\Delta u_2^{[n]}(t)\big) + F\big(t, u_1^{[n]}(t), u_2^{[n]}(t)\big) - F\big(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\big) \\ &\quad + F\big(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\big) + \big(h_1(t), u_1^{[n]}(t)\big) + \big(h_2(t), u_2^{[n]}(t)\big)\big] \\ &\le 2G_3T + G_2 + \sum_{t=1}^{T} F\big(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\big) + M_{14}\|\tilde{u}_1\|_\infty + M_{24}\|\tilde{u}_1\|_\infty \\ &\le 2G_3T + G_2 + \sum_{t=1}^{T} F\big(t, P_1\bar{u}_1^{[n]}, P_2\bar{u}_2^{[n]}\big) + M_{14}C\big(p'\big)G_1^{1/p} + M_{24}C\big(q'\big)G_1^{1/q}. \end{aligned} \tag{3.25}$$

Then $(\mathcal{F}6)'$ implies that $\{P_1\bar{u}_1^{[n]}\}$ and $\{P_2\bar{u}_2^{[n]}\}$ are bounded. Then (3.22) implies that $\{u^{[n]}\}$ is bounded in $\mathcal{H}$. Since $\dim\mathcal{H} < \infty$, $\{u^{[n]}\}$ has a convergent subsequence. Hence, Ψ satisfies the (PS) condition.

Next we prove the following conclusions:

(i) $\inf\{\Psi((z,v)) | (z,v) \in Z \times T^{r_1+r_2}\} > -\infty$;

(ii) $\Psi((y,v)) \to -\infty$ uniformly for $(y,v) \in Y \times T^{r_1+r_2}$ as $|y| \to \infty$.

For $(z,v) \in Z \times T^{r_1+r_2}$, set $u = u(t) = z(t) + v = (z_1(t) + v_1, z_2(t) + v_2)^\tau$. Then $z_m(t) = \tilde{u}_m(t)$, $v_m = Q_m\bar{u}_m$, and $u_m(t) = z_m(t) + v_m$, $m = 1, 2$. By ($\mathcal{F}3$) and Lemma 2.1, we have

$$\begin{aligned} &\left|\sum_{t=1}^{T}\big[F\big(t, u_1(t), u_2(t)\big) - F(t,0,0)\big]\right| \\ &\quad\le \sum_{t=1}^{T}\big|F\big(t, u_1(t), u_2(t)\big) - F\big(t, 0, u_2(t)\big)\big| + \sum_{t=1}^{T}\big|F\big(t, 0, u_2(t)\big) - F(t,0,0)\big| \\ &\quad\le \sum_{t=1}^{T}\left|\int_0^1 \big(\nabla_{x_1}F\big(t, s\big(Q_1\bar{u}_1 + \tilde{u}_1(t)\big), u_2(t)\big), Q_1\bar{u}_1 + \tilde{u}_1(t)\big)\,ds\right| \\ &\qquad + \sum_{t=1}^{T}\left|\int_0^1 \big(\nabla_{x_2}F\big(t, 0, s\big(Q_2\bar{u}_2 + \tilde{u}_2(t)\big)\big), Q_2\bar{u}_2 + \tilde{u}_2(t)\big)\,ds\right| \\ &\quad\le \big(|Q_1\bar{u}_1| + \|\tilde{u}_1\|_\infty\big)\sum_{t=1}^{T}\int_0^1 \big|\nabla_{x_1}F\big(t, s\big(Q_1\bar{u}_1 + \tilde{u}_1(t)\big), \hat{u}_2(t)\big)\big|\,ds \\ &\qquad + \big(|Q_2\bar{u}_2| + \|\tilde{u}_2\|_\infty\big)\sum_{t=1}^{T}\int_0^1 \big|\nabla_{x_2}F\big(t, 0, s\big(Q_2\bar{u}_2 + \tilde{u}_2(t)\big)\big)\big|\,ds \\ &\quad\le \big(M_{13} + \|\tilde{u}_1\|_\infty\big)\sum_{t=1}^{T} b_1(t) + \big(M_{23} + \|\tilde{u}_2\|_\infty\big)\sum_{t=1}^{T} b_2(t) \end{aligned}$$

$$\begin{aligned}&\leq C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{1/p}\sum_{t=1}^{T}b_1(t)+C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{1/q}\sum_{t=1}^{T}b_2(t)\\&\quad+M_{13}\sum_{t=1}^{T}b_1(t)+M_{23}\sum_{t=1}^{T}b_2(t).\end{aligned}\tag{3.26}$$

Then

$$\begin{aligned}\Psi\bigl((z,v)\bigr)&=\varphi(u_1,u_2)\\&=\sum_{t=1}^{T}\bigl[\Phi_1\bigl(\Delta u_1(t)\bigr)+\Phi_2\bigl(\Delta u_2(t)\bigr)+F\bigl(t,u_1(t),u_2(t)\bigr)\\&\quad+\bigl(h_1(t),u_1(t)\bigr)+\bigl(h_2(t),u_2(t)\bigr)\bigr]\\&=\sum_{t=1}^{T}\bigl[\Phi_1\bigl(\Delta u_1(t)\bigr)+\Phi_2\bigl(\Delta u_2(t)\bigr)+F\bigl(t,u_1(t),u_2(t)\bigr)-F(t,0,0)\\&\quad+F(t,0,0)+\bigl(h_1(t),u_1(t)\bigr)+\bigl(h_2(t),u_2(t)\bigr)\bigr]\\&\geq\sum_{t=1}^{T}\bigl[\delta_1\bigl(\left|\Delta u_1(t)\right|-1\bigr)+\delta_2\bigl(\left|\Delta u_2(t)\right|-1\bigr)\bigr]+\sum_{t=1}^{T}F(t,0,0)\\&\quad-C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{1/p}\sum_{t=1}^{T}b_1(t)-C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{1/q}\sum_{t=1}^{T}b_2(t)\\&\quad-C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|^p\right)^{1/p}\sum_{t=1}^{T}\left|h_1(t)\right|-C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|^q\right)^{1/q}\sum_{t=1}^{T}\left|h_2(t)\right|\\&\quad-M_{13}\sum_{t=1}^{T}b_1(t)-M_{23}\sum_{t=1}^{T}b_2(t)\\&\geq\sum_{t=1}^{T}\bigl[\delta_1\bigl(\left|\Delta u_1(t)\right|-1\bigr)+\delta_2\bigl(\left|\Delta u_2(t)\right|-1\bigr)\bigr]+\sum_{t=1}^{T}F(t,0,0)-M_{13}\sum_{t=1}^{T}b_1(t)\\&\quad-C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|\right)\sum_{t=1}^{T}b_1(t)-C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|\right)\sum_{t=1}^{T}b_2(t)\\&\quad-C(p')\left(\sum_{t=1}^{T}\left|\Delta u_1(t)\right|\right)\sum_{t=1}^{T}\left|h_1(t)\right|-C(q')\left(\sum_{t=1}^{T}\left|\Delta u_2(t)\right|\right)\sum_{t=1}^{T}\left|h_2(t)\right|\\&\quad-M_{23}\sum_{t=1}^{T}b_2(t).\end{aligned}\tag{3.27}$$

It is easy to see that conclusion (i) holds from $(\mathcal{F}7)'$.

For any $(y,v)\in Y\times T^{r_1+r_2}$, it follows from (1.2) and (2.11) that

$$\begin{aligned}\Psi\bigl((y,v)\bigr)&=\varphi(y_1+v_1,y_2+v_2)\\&=\sum_{t=1}^{T}F(t,y_1+v_1,y_2+v_2)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=1}^{T} F(t, y_1 + v_1, y_2 + v_2) - \sum_{t=1}^{T} F(t, y_1, y_2 + v_2) + \sum_{t=1}^{T} F(t, y_1, y_2 + v_2) \\
&\quad - \sum_{t=1}^{T} F(t, y_1, y_2) + \sum_{t=1}^{T} F(t, y_1, y_2) \\
&= \sum_{t=1}^{T} F(t, y_1, y_2) + \sum_{t=1}^{T} \int_0^1 \bigl(\nabla F(t, y_1 + s v_1, y_2 + v_2), v_1\bigr)\, ds \\
&\quad + \sum_{t=1}^{T} \int_0^1 \bigl(\nabla F(t, y_1, y_2 + s v_2), v_2\bigr)\, ds \\
&\le \sum_{t=1}^{T} F(t, y_1, y_2) + |v_1| \sum_{t=1}^{T} g_1(t) + |v_2| \sum_{t=1}^{T} g_2(t) \\
&\le \sum_{t=1}^{T} F(t, y_1, y_2) + \left(\sum_{k=1}^{r_1} T_{i_k}^2\right)^{1/2} \sum_{t=1}^{T} g_1(t) + \left(\sum_{s=1}^{r_2} T_{j_s}^2\right)^{1/2} \sum_{t=1}^{T} g_2(t).
\end{aligned}
$$

Hence, the above inequality and $(\mathcal{F}6)'$ imply that conclusion (ii) holds. It follows from Lemma 2.6 that Ψ has at least $r_1 + r_2 + 1$ critical points. Hence φ has at least $r_1 + r_2 + 1$ geometrically distinct critical points. Therefore, system (1.1) has at least $r_1 + r_2 + 1$ geometrically distinct solutions in $\mathcal{H}$. The proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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