Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence and uniqueness of a positive periodic solution for Rayleigh type ϕ-Laplacian equation

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Abstract
By using the Manásevich-Mawhin continuation theorem and some analysis skills, we establish some sufficient condition for the existence and uniqueness of positive T-periodic solutions for a generalized Rayleigh type ϕ-Laplacian operator equation. The results of this paper are new and they complement previous known results.

MSC: 34K13; 34C25

Keywords: positive periodic solution; uniqueness; ϕ-Laplacian; Rayleigh equation

1 Introduction

During the past few years, many researchers have discussed the periodic solutions of a Rayleigh type differential equation (see [1–10]). For example, in 2009, Xiao and Liu [7] studied the Rayleigh type p-Laplacian equation with a deviating argument of the form

$$\big(\phi_p\big(x'(t)\big)\big)' + f\big(t,x'(t)\big) + g\big(t,x\big(t-\tau(t)\big)\big) = e(t).$$

By using the coincidence degree theory, we establish new results on the existence of periodic solutions for the above equation. Afterward, Xiong and Shao [9] used the coincidence degree theory to establish new results on the existence and uniqueness of positive T-periodic solutions for the Rayleigh type p-Laplacian equation of the form

$$\big(\phi_p\big(x'(t)\big)\big)' + f\big(t,x'(t)\big) + g\big(t,x(t)\big) = e(t).$$

In this paper, we consider the following Rayleigh type ϕ-Laplacian operator equation:

$$\big(\phi\big(x'(t)\big)\big)' + f\big(t,x'(t)\big) + g\big(t,x(t)\big) = e(t), \tag{1.1}$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and $\phi(0) = 0$. $f, g \in \mathrm{Car}(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is an L^p-Carathéodory function and $p = \frac{m}{m-1}$, $m \geq 2$, which means it is measurable in the first variable and continuous in the second variable. For every $0 < r < s$, there exists $h_{r,s} \in L^p[0,T]$ such that $|g(t,x(t))| \leq h_{r,s}$ for all $x \in [r,s]$ and a.e. $t \in [0,T]$; and f, g is a T-periodic function about t and $f(t,0) = 0$. $e \in L^p([0,T],\mathbb{R})$ and is T-periodic.

Here $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous function and $\phi(0) = 0$, which satisfies

(A_1) $(\phi(x_1) - \phi(x_2))(x_1 - x_2) > 0$ for $\forall x_1 \neq x_2$, $x_1, x_2 \in \mathbb{R}$;

(A_2) there exists a function $\alpha : [0, +\infty] \to [0, +\infty]$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that $\phi(x) \cdot x \geq \alpha(|x|)|x|$ for $\forall x \in \mathbb{R}$.

It is easy to see that ϕ represents a large class of nonlinear operators, including $\phi_p : \mathbb{R} \to \mathbb{R}$ is a p-Laplacian, *i.e.*, $\phi_p(x) = |x|^{p-2}x$ for $x \in \mathbb{R}$.

We know that the study on ϕ-Laplacian is relatively infrequent, the main difficulty lies in the fact that the ϕ-Laplacian operator typically possesses more uncertainty than the p-Laplacian operator. For example, the key step for ϕ_p to get *a priori* solutions, $\int_0^T (\phi_p'(x(t)))'x(t)\,dt = -\int_0^T |x'(t)|^p\,dt$, is no longer available for general ϕ-Laplacian. So, we need to find a new method to get over it.

By using the Manásevich-Mawhin continuation theorem and some analysis skills, we establish some sufficient condition for the existence of positive T-periodic solutions of (1.1). The results of this paper are new and they complement previous known results.

2 Main results

For convenience, define

$$C_T^1 = \{x \in C^1(\mathbb{R}, \mathbb{R}) : x \text{ is } T\text{-periodic}\},$$

which is a Banach space endowed with the norm $\|\cdot\|$; define $\|x\| = \max\{|x|_0, |x'|_0\}$ for all x, and

$$|x|_0 = \max_{t\in[0,T]} |x(t)|, \qquad |x'|_0 = \max_{t\in[0,T]} |x'(t)|.$$

For the T-periodic boundary value problem

$$(\phi(x'(t)))' = \tilde{f}(t, x, x'), \tag{2.1}$$

here $\tilde{f} : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is assumed to be Carathéodory.

Lemma 2.1 (Manásevich-Mawhin [11]) *Let Ω be an open bounded set in C_T^1. If*

(i) *for each $\lambda \in (0, 1)$, the problem*

$$(\phi(x'))' = \lambda\tilde{f}(t, x, x'), \qquad x(0) = x(T), \qquad x'(0) = x'(T)$$

has no solution on $\partial\Omega$;

(ii) *the equation*

$$F(a) := \frac{1}{T}\int_0^T \tilde{f}(t, x, x')\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}$;

(iii) *the Brouwer degree of F*

$$\deg\{F, \Omega \cap \mathbb{R}, 0\} \neq 0.$$

Then the periodic boundary value problem (2.1) *has at least one periodic solution on $\overline{\Omega}$.*

Lemma 2.2 *If $\phi(x)$ is bounded, then x is also bounded.*

Proof Since $\phi(x)$ is bounded, then there exists a positive constant N such that $|\phi(x)| \le N$. From (A_2), we have $\alpha(|x|)|x| \le \phi(x) \cdot x \le |\phi(x)| \cdot |x| \le N|x|$. Hence, we can get $\alpha(|x|) \le N$ for all $x \in \mathbb{R}$. If x is not bounded, then from the definition of α, we get $\alpha(|x|) > N$ for some $x \in \mathbb{R}$, which is a contradiction. So x is also bounded. □

Lemma 2.3 *Suppose that the following condition holds*:

(A_3) $(x_1 - x_2)(g(t,x_1) - g(t,x_2)) < 0$ *for all* $t, x_1, x_2 \in \mathbb{R}$, $x_1 \neq x_2$.

Then (1.1) *has at most one* T*-periodic solution in* C_T^1.

Proof Assume that $x_1(t)$ and $x_2(t)$ are two T-periodic solutions of (1.1). Then we obtain

$$\bigl(\phi\bigl(x_1'(t)\bigr) - \phi\bigl(x_2'(t)\bigr)\bigr)' + f\bigl(t, x_1'(t)\bigr) - f\bigl(t, x_2'(t)\bigr) + g\bigl(t, x_1(t)\bigr) - g\bigl(t, x_2(t)\bigr) = 0. \tag{2.2}$$

Set $u(t) = x_1(t) - x_2(t)$. Now, we claim that

$$u(t) \le 0 \quad \text{for all } t \in \mathbb{R}.$$

In contrast, in view of $x_1, x_2 \in C^1[0,T]$, for $t \in \mathbb{R}$, we obtain

$$\max_{t\in\mathbb{R}} u(t) > 0.$$

Then there must exist $t^* \in \mathbb{R}$ (for convenience, we can choose $t^* \in (0,T)$) such that

$$u\bigl(t^*\bigr) = \max_{t\in[0,T]} u(t) = \max_{t\in\mathbb{R}} u(t) > 0,$$

which implies that

$$u'\bigl(t^*\bigr) = x_1'\bigl(t^*\bigr) - x_2'\bigl(t^*\bigr) = 0$$

and

$$x_1\bigl(t^*\bigr) - x_2\bigl(t^*\bigr) > 0.$$

By hypothesis (A_3) and (2.2), we have

$$\begin{aligned}\bigl(\phi\bigl(x_1'\bigl(t^*\bigr)\bigr) - \phi\bigl(x_2'\bigl(t^*\bigr)\bigr)\bigr)' &= -\bigl[f\bigl(t^*, x_1'\bigl(t^*\bigr)\bigr) - f\bigl(t^*, x_2'\bigl(t^*\bigr)\bigr)\bigr] \\ &\quad - \bigl[g\bigl(t^*, x_1\bigl(t^*\bigr)\bigr) - g\bigl(t^*, x_2\bigl(t^*\bigr)\bigr)\bigr] \\ &= -\bigl[g\bigl(t^*, x_1\bigl(t^*\bigr)\bigr) - g\bigl(t^*, x_2\bigl(t^*\bigr)\bigr)\bigr] > 0,\end{aligned}$$

and there exists $\varepsilon > 0$ such that $(\phi(x_1'(t)) - \phi(x_2'(t)))' > 0$ for all $t \in (t^* - \varepsilon, t^*]$. Therefore, $\phi(x_1'(t)) - \phi(x_2'(t))$ is strictly increasing for $t \in (t^* - \varepsilon, t^*]$, which implies that

$$\phi\bigl(x_1'(t)\bigr) - \phi\bigl(x_2'(t)\bigr) < \phi\bigl(x_1'\bigl(t^*\bigr)\bigr) - \phi\bigl(x_2'\bigl(t^*\bigr)\bigr) = 0 \quad \text{for all } t \in \bigl(t^* - \varepsilon, t^*\bigr).$$

From (A_1) we get

$$u'(t) = x_1'(t) - x_2'(t) < 0 \quad \text{for all } t \in \bigl(t^* - \varepsilon, t^*\bigr).$$

This contradicts the definition of t^*. Thus,

$$u(t) = x_1(t) - x_2(t) \leq 0 \quad \text{for all } t \in \mathbb{R}.$$

By using a similar argument, we can also show that

$$x_2(t) - x_1(t) \leq 0.$$

Therefore, we obtain

$$x_1(t) \equiv x_2(t) \quad \text{for all } t \in \mathbb{R}.$$

Hence, (1.1) has at most one T-periodic solution in C_T^1. The proof of Lemma 2.3 is now complete. □

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

(H_1) there exists a positive constant D such that $g(t,x) - e(t) < 0$ for $x > D$ and $t \in \mathbb{R}$, $g(t,x) - e(t) > 0$ for $x \leq 0$ and $t \in \mathbb{R}$;
(H_2) there exist constants $\sigma > 0$ and $m \geq 2$ such that $f(t,u)u \geq \sigma |u|^m$ for $(t,u) \in [0,T] \times \mathbb{R}$;
(H_3) there exist positive constants ρ and γ such that $|f(t,u)| \leq \rho |u|^{m-1} + \gamma$ for $(t,u) \in [0,T] \times \mathbb{R}$;
(H_4) there exist positive constants α, β, B such that

$$\left|g(t,x)\right| \leq \alpha |x|^{m-1} + \beta \quad \text{for } |x| \geq B \text{ and } t \in \mathbb{R}.$$

By using Lemmas 2.1-2.3, we obtain our main results.

Theorem 2.1 *Assume that conditions* (H_1)-(H_4) *and* (A_3) *hold. Then* (1.1) *has a unique positive T-periodic solution if* $\sigma - \frac{\alpha T^{m-1}}{2^{m-1}} > 0$.

Proof Consider the homotopic equation of (1.1) as follows:

$$\bigl(\phi\bigl(x'(t)\bigr)\bigr)' + \lambda f\bigl(t,x'(t)\bigr) + \lambda g\bigl(t,x(t)\bigr) = \lambda e(t). \tag{2.3}$$

By Lemma 2.3, it is easy to see that (1.1) has at most one T-periodic solution in C_T^1. Thus, to prove Theorem 2.1, it suffices to show that (1.1) has at least one T-periodic solution in C_T^1. To do this, we are going to apply Lemmas 2.1 and 2.2. Firstly, we will claim that the set of all possible T-periodic solutions of (2.3) is bounded. Let $x(t) \in C_T^1$ be an arbitrary solution of (2.3) with period T. As $x(0) = x(T)$, there exists $t_0 \in [0,T]$ such that $x'(t_0) = 0$, while $\phi(0) = 0$, we see

$$\begin{aligned} \bigl|\phi\bigl(x'(t)\bigr)\bigr| &= \left|\int_{t_0}^{t} \bigl(\phi\bigl(x'(s)\bigr)\bigr)'\,ds\right| \\ &\leq \lambda \int_0^T \bigl|f\bigl(t,x'(t)\bigr)\bigr|\,dt + \lambda \int_0^T \bigl|g\bigl(t,x(t)\bigr)\bigr|\,dt + \lambda \int_0^T \bigl|e(t)\bigr|\,dt, \end{aligned} \tag{2.4}$$

where $t \in [t_0, t_0 + T]$.

We claim that there is a constant $\xi \in \mathbb{R}$ such that

$$|x(\xi)| \le D. \tag{2.5}$$

Let $\bar{t}$, $\underline{t}$ be, respectively, the global maximum point and the global minimum point of $x(t)$ on $[0, T]$; then $x'(\bar{t}) = 0$, and we claim that

$$\bigl(\phi\bigl(x'(\bar{t})\bigr)\bigr)' \le 0. \tag{2.6}$$

Assume, by way of contradiction, that (2.6) does not hold. Then $(\phi(x'(\bar{t})))' > 0$ and there exists $\varepsilon > 0$ such that $(\phi(x'(t)))' > 0$ for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. Therefore $\phi(x'(t))$ is strictly increasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. From (A_1) we know that $x'(t)$ is strictly increasing for $t \in (\bar{t}-\varepsilon, \bar{t}+\varepsilon)$. This contradicts the definition of $\bar{t}$. Thus, (2.6) is true. From $f(t, 0) = 0$, (2.3) and (2.6), we have

$$g\bigl(\bar{t}, x(\bar{t})\bigr) - e(\bar{t}) \ge 0. \tag{2.7}$$

Similarly, we get

$$g\bigl(\underline{t}, x(\underline{t})\bigr) - e(\underline{t}) \le 0. \tag{2.8}$$

In view of (H_1), (2.7) and (2.8) imply that

$$x(\bar{t}) \le D, \qquad x(\underline{t}) > 0.$$

Case (1): If $x(\underline{t}) \in (0, D)$, define $\xi = \bar{t}$, obviously, $|x(\xi)| \le D$.

Case (2): If $x(\underline{t}) \ge D$, from $x(\bar{t}) \le D$, we know $x(\bar{t}) = x(\underline{t})$. Define $\xi = \bar{t}$, we have $|x(\xi)| = D$. This proves (2.5).

Then we have

$$|x(t)| = \left|x(\xi) + \int_\xi^t x'(s)\,ds\right| \le D + \int_\xi^t |x'(s)|\,ds, \quad t \in [\xi, \xi + T]$$

and

$$|x(t)| = |x(t-T)| = \left|x(\xi) - \int_{t-T}^{\xi} x'(s)\,ds\right| \le D + \int_{t-T}^{\xi} |x'(s)|\,ds, \quad t \in [\xi, \xi + T].$$

Combining the above two inequalities, we obtain

$$\begin{aligned} |x|_0 &= \max_{t\in[0,T]} |x(t)| = \max_{t\in[\xi,\xi+T]} |x(t)| \\ &\le \max_{t\in[\xi,\xi+T]} \left\{ D + \frac{1}{2}\left(\int_\xi^t |x'(s)|\,ds + \int_{t-T}^{\xi} |x'(s)|\,ds\right)\right\} \\ &\le D + \frac{1}{2}\int_0^T |x'(s)|\,ds. \end{aligned} \tag{2.9}$$

Since $x'(t)$ is T-periodic, multiplying $x'(t)$ and (2.3) and then integrating it from 0 to T, we have

$$\begin{aligned} 0 &= \int_0^T \big(\phi\big(x'(t)\big)\big)' x'(t)\,dt \\ &= -\lambda \int_0^T f\big(t, x'(t)\big)x'(t)\,dt - \lambda \int_0^T g\big(t, x(t)\big)x'(t)\,dt + \lambda \int_0^T e(t)x'(t)\,dt. \end{aligned} \tag{2.10}$$

In view of (2.10), we have

$$\left|\int_0^T f\big(t, x'(t)\big)x'(t)\,dt\right| = \left|-\int_0^T g\big(t, x(t)\big)x'(t)\,dt + \int_0^T e(t)x'(t)\,dt\right|.$$

From (H_2), we know

$$\left|\int_0^T f\big(t, x'(t)\big)x'(t)\,dt\right| \geq \sigma \int_0^T \big|x'(t)\big|^m\,dt.$$

Set

$$E_1 = \big\{t \in [0, T] \mid \big|x(t)\big| \leq B\big\}, \qquad E_2 = \big\{t \in [0, T] \mid \big|x(t)\big| \geq B\big\}.$$

From (H_4), we have

$$\begin{aligned} &\sigma \int_0^T \big|x'(t)\big|^m\,dt \\ &\quad\leq \int_{E_1+E_2} \big|g\big(t, x(t)\big)\big|\big|x'(t)\big|\,dt + \int_0^T \big|e(t)\big|\big|x'(t)\big|\,dt \\ &\quad\leq \left(\int_{E_1} \big|g\big(t, x(t)\big)\big|^{\frac{m}{m-1}}\,dt\right)^{\frac{m-1}{m}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} + \alpha \int_0^T \big|x(t)\big|^{m-1}\big|x'(t)\big|\,dt \\ &\qquad + \beta \int_0^T \big|x'(t)\big|\,dt + \int_0^T \big|e(t)\big|\big|x'(t)\big|\,dt \\ &\quad\leq |g_B|_{\frac{m}{m-1}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} + \alpha \left(D + \frac{1}{2}\int_0^T \big|x'(t)\big|\,dt\right)^{m-1} \int_0^T \big|x'(t)\big|\,dt \\ &\qquad + \beta T^{\frac{m-1}{m}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} + \left(\int_0^T \big|e(t)\big|^{\frac{m}{m-1}}\right)^{\frac{m-1}{m}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} \\ &\quad= |g_B|_{\frac{m}{m-1}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} + \frac{\alpha}{2^{m-1}} \left(\frac{2D}{\int_0^T |x'(t)|\,dt} + 1\right)^{m-1} \left(\int_0^T \big|x'(t)\big|\,dt\right)^m \\ &\qquad + \beta T^{\frac{m-1}{m}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}} + |e|_{\frac{m}{m-1}} \left(\int_0^T \big|x'(t)\big|^m\,dt\right)^{\frac{1}{m}}, \end{aligned} \tag{2.11}$$

where $g_B = \max_{|x|\leq B} |g(t, x(t))|$, $|g_B|_{\frac{m}{m-1}} = (\int_0^T |g_B|^{\frac{m}{m-1}}\,dt)^{\frac{m-1}{m}}$.

For the constant $\delta > 0$, which is only dependent on $k > 0$, we have

$$(1 + x)^k \leq 1 + (1 + k)x \quad \text{for } x \in [0, \delta].$$

So, from (2.11), we have

$$\begin{aligned}
&\sigma \int_0^T \left|x'(t)\right|^m dt \\
&\quad \le |g_B|_{\frac{m}{m-1}} \left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + \frac{\alpha}{2^{m-1}}\left(1 + \frac{2Dm}{\int_0^T |x'(t)|\, dt}\right)\left(\int_0^T \left|x'(t)\right| dt\right)^m \\
&\qquad + \beta T^{\frac{m-1}{m}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + |e|_{\frac{m}{m-1}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} \\
&\quad = |g_B|_{\frac{m}{m-1}} \left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + \frac{\alpha}{2^{m-1}}\left(\int_0^T \left|x'(t)\right| dt\right)^m + \frac{\alpha Dm}{2^{m-2}}\left(\int_0^T \left|x'(t)\right| dt\right)^{m-1} \\
&\qquad + \beta T^{\frac{m-1}{m}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + |e|_{\frac{m}{m-1}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} \\
&\quad \le |g_B|_{\frac{m}{m-1}} \left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + \frac{\alpha T^{m-1}}{2^{m-1}}\int_0^T \left|x'(t)\right|^m dt \\
&\qquad + \frac{\alpha Dm T^{\frac{(m-1)^2}{m}}}{2^{m-2}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{m-1}{m}} \\
&\qquad + \beta T^{\frac{m-1}{m}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} + |e|_{\frac{m}{m-1}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} \\
&\quad = \frac{\alpha T^{m-1}}{2^{m-1}}\int_0^T \left|x'(t)\right|^m dt + \frac{\alpha Dm T^{\frac{(m-1)^2}{m}}}{2^{m-2}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{m-1}{m}} \\
&\qquad + \left(|g_B|_{\frac{m}{m-1}} + \beta T^{\frac{m-1}{m}} + |e|_{\frac{m}{m-1}}\right)\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}}.
\end{aligned}$$

Since $\sigma - \frac{\alpha T^{m-1}}{2^{m-1}} > 0$, so it is easy to see that there is a constant $M_1' > 0$ (independent of λ) such that

$$\int_0^T \left|x'(t)\right|^m dt \le M_1'.$$

By applying Hölder's inequality and (2.9), we have

$$|x|_0 \le D + \frac{1}{2}\int_0^T \left|x'(s)\right| ds \le D + \frac{1}{2}T^{\frac{m-1}{m}}\left(\int_0^T \left|x'(t)\right|^m dt\right)^{\frac{1}{m}} \le D + \frac{1}{2}T^{\frac{m-1}{m}}\left(M_1'\right)^{\frac{1}{m}} := M_1.$$

In view of (2.4) and (H_3), we have

$$\begin{aligned}
\left|\phi(x')\right|_0 &= \max_{t\in[0,T]}\left\{\left|\phi\left(x'(t)\right)\right|\right\} \\
&= \max_{t\in[t_0,t_0+T]}\left\{\left|\int_{t_0}^t \left(\phi\left(x'(s)\right)\right)' ds\right|\right\} \\
&\le \int_0^T \left|f\left(t,x'(t)\right)\right| dt + \int_0^T \left|g\left(t,x(t)\right)\right| dt + \int_0^T \left|e(t)\right| dt \\
&\le \rho\int_0^T \left|x'(t)\right|^{m-1} dt + \gamma T + T^{\frac{1}{m}}\left(\int_0^T \left|g\left(t,x(t)\right)\right|^{\frac{m}{m-1}} dt\right)^{\frac{m-1}{m}}
\end{aligned}$$

$$
\begin{aligned}
&\quad + T^{\frac{1}{m}} \left(\int_0^T \left| e(t) \right|^{\frac{m}{m-1}} dt \right)^{\frac{m-1}{m}} \\
&\le \rho T^{\frac{1}{m}} \left(\int_0^T \left| x'(t) \right|^m dt \right)^{\frac{m-1}{m}} + \gamma T + T^{\frac{1}{m}} \left(\int_0^T \left| g\bigl(t, x(t)\bigr) \right|^{\frac{m}{m-1}} dt \right)^{\frac{m-1}{m}} \\
&\quad + T^{\frac{1}{m}} \left(\int_0^T \left| e(t) \right|^{\frac{m}{m-1}} dt \right)^{\frac{m-1}{m}} \\
&\le \rho T^{\frac{1}{m}} \bigl(M_1' \bigr)^{\frac{m-1}{m}} + \gamma T + T^{\frac{1}{m}} |g_{M_1}|_{\frac{m-1}{m}} + T^{\frac{1}{m}} |e|_{\frac{m-1}{m}} := M_2',
\end{aligned}
$$

where $|g_{M_1}| = \max_{|x(t)| \le M_1} |g(t, x(t))|$.

Thus, from Lemma 2.2, we know that there exists some positive constant $M_2 > M_2' + 1$ such that, for all $t \in \mathbb{R}$,

$$
\left| x'(t) \right| \le M_2.
$$

Set $M = \sqrt{M_1^2 + M_2^2} + 1$, we have

$$
\Omega = \left\{ x \in C_T^1(\mathbb{R}, \mathbb{R}) \mid |x|_0 \le M + 1, \left| x' \right|_0 \le M + 1 \right\},
$$

we know that (2.4) has no solution on $\partial\Omega$ as $\lambda \in (0,1)$ and when $x(t) \in \partial\Omega \cap \mathbb{R}$, $x(t) = M + 1$ or $x(t) = -M - 1$, from (2.11) we know that $M + 1 > D$. So, from (H_1) we see that

$$
\begin{aligned}
&\frac{1}{T} \int_0^T \bigl\{ g(t, M+1) - e(t) \bigr\} \, dt < 0, \\
&\frac{1}{T} \int_0^T \bigl\{ g(t, -M-1) - e(t) \bigr\} \, dt > 0.
\end{aligned}
$$

So condition (ii) is also satisfied. Set

$$
H(x, \mu) = \mu x - (1 - \mu) \frac{1}{T} \int_0^T \bigl\{ g(t, x) - e(t) \bigr\} \, dt,
$$

where $x \in \partial\Omega \cap \mathbb{R}$, $\mu \in [0, 1]$, we have

$$
xH(x, \mu) = \mu x^2 - (1 - \mu) x \frac{1}{T} \int_0^T \bigl\{ g(t, x) - e(t) \bigr\} \, dt > 0,
$$

and thus $H(x, \mu)$ is a homotopic transformation and

$$
\begin{aligned}
\deg\{F, \Omega \cap \mathbb{R}, 0\} &= \deg\left\{ -\frac{1}{T} \int_0^T \bigl\{ g(t, x) - e(t) \bigr\} \, dt, \Omega \cap \mathbb{R}, 0 \right\} \\
&= \deg\{x, \Omega \cap \mathbb{R}, 0\} \ne 0.
\end{aligned}
$$

So condition (iii) is satisfied. In view of Lemma 2.1, there exists at least one solution with period T.

Suppose that $x(t)$ is the T-periodic solution of (1.1). We can easily show that (2.8) also holds. Thus,

$$
x(t) \ge \min_{t \in [0,T]} x(t) = x(\underline{t}) > 0 \quad \text{for all } t \in \mathbb{R},
$$

which implies that (1.1) has a unique positive solution with period T. This completes the proof. □

We illustrate our results with some examples.

Example 2.1 Consider the following second-order p-Laplacian-like Rayleigh equation:

$$\big(\phi_p\big(x'(t)\big)\big)' + \big(10 + 5\sin^2 t\big)x'(t) - \big(5x(t) + \sin^2 t - 8\big) = e^{\cos^2 t}, \qquad (2.12)$$

where $\phi_p(u) = |u|^{p-2}u$.

Comparing (2.12) to (1.1), we see that $g(t,x) = -5x(t) - \sin^2 t + 8$, $f(t,u) = (10 + 5\sin^2 t)u$, $e(t) = e^{\cos^2 t}$, $T = \pi$. Obviously, we know that ϕ_p is a homeomorphism from $\mathbb{R}$ to $\mathbb{R}$ satisfying (A_1) and (A_2). Consider $(x_1 - x_2)(g(t,x_1) - g(t,x_2)) = -5(x_1 - x_2)^2 < 0$ for $x_1 \neq x_2$, then (A_3) holds. Moreover, it is easily seen that there exists a constant $D = 2$ such that (H_1) holds. Consider $f(t,u)u = (10 + 5\sin^2 t)u^2 \geq 10u^2$, here $\sigma = 10$, $m = 2$, and $|f(t,u)| = |(10 + 5\sin^2 t)u| \leq 15|u| + 1$, here $\rho = 15$, $\gamma = 1$. So, we can get that conditions (H_2) and (H_3) hold. Choose $B > 0$, we have $|g(t,x)| \leq 5|x| + 9$, here $\alpha = 5$, $\beta = 9$, then (H_4) holds and $\sigma - \frac{\alpha T}{2} = 10 - \frac{5\pi}{2} > 0$. So, by Theorem 2.1, we can get that (2.12) has a unique positive periodic solution.

Example 2.2 Consider the following second-order p-Laplacian-like Rayleigh equation:

$$\big(\phi\big(x'(t)\big)\big)' + \big(200 + 16\cos^2 t\big)\big(x'(t)\big)^3 - \big(20x^3(t) + 10\cos^2(t) - 15\big) = e^{\sin^2 t}, \qquad (2.13)$$

where $\phi(u) = ue^{|u|^2}$.

Comparing (2.13) to (1.1), we see that $g(t,x) = -20x^3 - 10\cos^2 t + 15$, $f(t,v) = (200 + 16\cos^2 t)v^3$, $e(t) = e^{\sin^2 t}$, $T = \pi$. Obviously, we get

$$\big(xe^{|x|^2} - ye^{|y|^2}\big)(x - y) \geq \big(|x|e^{|x|^2} - |y|e^{|y|^2}\big)\big(|x| - |y|\big) \geq 0$$

and

$$\phi(x) \cdot x = |x|^2 e^{|x|^2}.$$

So, we know that (A_1) and (A_2) hold. Consider $(x_1 - x_2)(g(t,x_1) - g(t,x_2)) = -20(x_1 - x_2)^2(x_1^2 + x_1x_2 + x_2^2) < 0$ for $x_1 \neq x_2$, then (A_3) holds. Moreover, it is easily seen that there exists a constant $D = 1$ such that (H_1) holds. Consider $f(t,v)v = (200 + 16\cos^2 t)v^4 \geq 200v^4$, here $\sigma = 200$, $m = 4$, and $|f(t,v)| = |(200 + 16\cos^2 t)v^3| \leq 216|v|^3 + 5$, here $\rho = 216$, $\gamma = 5$. So, we can get that conditions (H_2) and (H_3) hold. Choose $B > 0$, we have $|g(t,x)| \leq 20|x|^3 + 25$, here $\alpha = 20$, $\beta = 25$, then (H_4) holds and $\sigma - \frac{\alpha T^{m-1}}{2^{m-1}} = 200 - \frac{20\times\pi^3}{2^3} > 0$. Therefore, by Theorem 2.1, we know that (2.13) has a unique positive periodic solution.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

YX and ZBC worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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Acknowledgements
Research is supported by the National Natural Science Foundation of China (Nos. 11326124, 11271339).

Received: 31 March 2014 Accepted: 30 June 2014 Published: 15 Aug 2014

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10.1186/1687-1847-2014-225
Cite this article as: Xin and Cheng: Existence and uniqueness of a positive periodic solution for Rayleigh type ϕ-Laplacian equation. *Advances in Difference Equations* 2014, 2014:225