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A new fractional Jacobi elliptic equation method for solving fractional partial differential equations

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Abstract

In this paper, we propose a new fractional Jacobi elliptic equation method to seek exact solutions of fractional partial differential equations. Based on a traveling wave transformation, certain fractional partial differential equation can be turned into another fractional ordinary differential equation. Then the fractional Jacobi elliptic equation is used as the auxiliary sub-equation to solve the fractional ordinary differential equation. As for applications of this method, we apply it to seek exact solutions for the space-time fractional Kortweg-de Vries (KdV) equation, the space-time fractional Benjamin-Bona-Mahony (BBM) equation, and the space-time fractional $(2 + 1)$-dimensional breaking soliton equations. With the aid of symbolic computation program, a series of exact solutions expressed in the Jacobi elliptic functions for the two equations are successfully found.

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Keywords: auxiliary sub-equation; fractional partial differential equation; exact solution; space-time fractional Kortweg-de Vries equation; space-time fractional Benjamin-Bona-Mahony equation; space-time fractional $(2 + 1)$-dimensional breaking soliton equations

1 Introduction

In recent decades, fractional differential equations have been paid an increasing attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, control theory, systems identification, biology, and other areas. In particular, fractional derivative is useful in describing the memory and hereditary properties of materials and processes. Among the investigations for fractional differential equations, finding numerical and exact solutions to fractional differential equations is a hot topic. Many efficient methods have been proposed so far to obtain numerical solutions and exact solutions of fractional differential equations. For example, these methods include the fractional sub-equation method [1–7], the first integral method [8], the (G'/G) method [9–13], the variational iterative method [14–16], the Exp method [17], the simplest equation method [18], the Adomian decomposition method [19, 20], the homotopy perturbation method [21–23], the shifted Jacobi-Gauss-Lobatto collocation method [24], the shifted Legendre spectral method [25], the generalized Laguerre spectral algorithms [26], the modified generalized Laguerre tau method combining with a new operational matrix [27] and so on.

The Jacobi elliptic function method is an effective method for solving some fractional differential equations, which have been investigated in detail in [28, 29]. In this paper, we propose a new fractional Jacobi elliptic equation method to seek exact solutions of fractional partial differential equations in the sense of the modified Riemann-Liouville derivative. First based on a traveling wave transformation, certain fractional partial differential equation can be turned into another fractional ordinary differential equation, the exact solutions of the latter are assumed to be expressed in a polynomial in $(\frac{D^\alpha G}{G})$, where D^α denotes the modified Riemann-Liouville derivative of α order, and $G = G(\xi)$ satisfies the following fractional Jacobi elliptic equation:

$$\left(D_\xi^\alpha G(\xi)\right)^2 = e_2 G^4(\xi) + e_1 G^2(\xi) + e_0, \quad 0 < \alpha \le 1, \tag{1}$$

where e_0, e_1, e_2 are arbitrary constants. The degree of the polynomial can be determined by the homogeneous balancing principle. By use of a fractional complex transformation, the general solutions of (1) can be obtained, with which the exact solutions for the original fractional partial differential equation can be deduced subsequently.

The definition and some important properties of the modified Riemann-Liouville derivative [1–7, 30] are listed as follows:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\xi)^{-\alpha} (f(\xi) - f(0))\, d\xi, & 0 < \alpha < 1, \\ (f^{(\alpha-n)}(t))^{(n)}, & n \le \alpha < n+1, n \ge 1, \end{cases} \tag{2}$$

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \tag{3}$$

$$D_t^\alpha \bigl(f(t)g(t)\bigr) = g(t) D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \tag{4}$$

$$D_t^\alpha f\bigl[g(t)\bigr] = f_g'\bigl[g(t)\bigr] D_t^\alpha g(t), \tag{5}$$

$$D_t^\alpha f\bigl[g(t)\bigr] = D_g^\alpha f\bigl[g(t)\bigr] \bigl(g'(t)\bigr)^\alpha. \tag{6}$$

The rest of this paper is organized as follows. In Section 2, we give the description of the fractional Jacobi elliptic equation method for solving fractional partial differential equations. Then in Section 3 we apply this method to seek exact solutions for the space-time fractional KdV equation, the space-time fractional BBM equation, and the space-time fractional (2 + 1)-dimensional breaking soliton equations. Some concluding comments are presented at the end of this paper.

2 Summary of the fractional Jacobi elliptic equation method

In this section we give the description of the fractional Jacobi elliptic equation method for solving fractional partial differential equations.

Suppose that a fractional partial differential equation, say in the independent variables $t, x_1, x_2, \ldots, x_n$, is given by

$$\begin{aligned} &P\bigl(u_1, \ldots u_k, D_t^\alpha u_1, \ldots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \ldots, D_{x_1}^\alpha u_k, \ldots, D_{x_n}^\alpha u_1, \ldots, \\ &\quad D_{x_n}^\alpha u_k, D_t^{2\alpha} u_1, \ldots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \ldots\bigr) = 0, \end{aligned} \tag{7}$$

where $u_i = u_i(t, x_1, x_2, \ldots, x_n)$, $i = 1, \ldots, k$ are unknown functions, P is a polynomial in u_i and their various partial derivatives including fractional derivatives.

Step 1. Suppose

$$u_i(t,x_1,x_2,\dots,x_n) = U_i(\xi), \quad i = 1,\dots,k,$$

and a traveling wave transformation

$$\xi = ct + k_1x_1 + k_2x_2 + \cdots + k_nx_n + \xi_0. \tag{8}$$

Then by the property (6), (7) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned}&\widetilde{P}\big(U_1,\dots,U_k,c^\alpha D_\xi^\alpha U_1,\dots,c^\alpha D_\xi^\alpha U_k,k_1^\alpha D_\xi^\alpha U_1,\dots,k_1^\alpha D_\xi^\alpha U_k,\dots,k_n^\alpha D_\xi^\alpha U_1,\dots,\\&\quad k_n^\alpha D_\xi^\alpha U_k,c^{2\alpha}D_\xi^{2\alpha}U_1,\dots,c^{2\alpha}D_\xi^{2\alpha}U_k,k_1^{2\alpha}D_\xi^{2\alpha}U_1,\dots\big) = 0.\end{aligned} \tag{9}$$

Step 2. Suppose that the solution of (9) can be expressed by a polynomial in $(\frac{D_\xi^\alpha G}{G})$ as follows:

$$U_j(\xi) = \sum_{i=-m_j}^{m_j} a_{j,i}\left(\frac{D_\xi^\alpha G}{G}\right)^i, \quad j = 1,2,\dots,k, \tag{10}$$

where $a_{j,i}$, $i = -m_j,\dots,0,1,\dots,m_j$, $j = 1,2,\dots,k$ are constants to be determined later, the positive integer m_j can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in (9), $G = G(\xi)$ satisfies the fractional Jacobi elliptic equation denoted by (1).

Step 3. Substituting (10) into (9) and using (1), collecting all terms with the same order of $(D_\xi^\alpha G)^i G^j$ together, the left-hand side of (9) is converted into another polynomial in $(D_\xi^\alpha G)^i G^j$. Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_{j,i}$, $i = -m_j,\dots,0,1,\dots,m_j$, $j = 1,2,\dots,k$.

Step 4. Solving the equations system in Step 3, and using the general solutions of (1), we can construct a variety of exact solutions for (7).

In order to obtain the general expressions for $(\frac{D_\xi^\alpha G}{G})$ in (1), we suppose $G(\xi) = H(\eta)$, and a nonlinear fractional complex transformation $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$. Then by the properties (3) and (5), we have $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta)D_\xi^\alpha \eta = H'(\eta)$. So (1) can be turned into the following ordinary differential equation:

$$\big(H'(\eta)\big)^2 = e_2H^4(\eta) + e_1H^2(\eta) + e_0. \tag{11}$$

By the general solutions of (11), one has

$$\left(\frac{H'(\eta)}{H(\eta)}\right) = \begin{cases} cn(\eta)ds(\eta), & e_2 = m^2, e_1 = -(1+m^2), e_0 = 0,\\ -sn(\eta)dc(\eta), & e_2 = -m^2, e_1 = 2m^2-1, e_0 = 1-m^2,\\ -m^2sn(\eta)cd(\eta), & e_2 = -1, e_1 = 2-m^2, e_0 = m^2-1,\\ -dc(\eta)ns(\eta), & e_2 = 1, e_1 = 2-m^2, e_0 = 1-m^2,\\ cs(\eta)nd(\eta), & e_2 = m^2(m^2-1), e_1 = 2m^2-1, e_0 = 1,\\ (1-m^2)sd(\eta)nc(\eta), & e_2 = 1, e_1 = -(m^2+1), e_0 = m^2, \end{cases} \tag{12}$$

where $sn(\eta), cn(\eta), dn(\eta)$ denote the Jacobi elliptic sine function, Jacobi elliptic cosine function, and the Jacobi elliptic function of the third kind, respectively, m is the modulus, and

$$\begin{aligned}
&cs(\eta) = \frac{cn(\eta)}{sn(\eta)}, \qquad sd(\eta) = \frac{sn(\eta)}{dn(\eta)}, \qquad dc(\eta) = \frac{dn(\eta)}{cn(\eta)}, \\
&sc(\eta) = \frac{1}{cs(\eta)}, \qquad ds(\eta) = \frac{1}{sd(\eta)}, \qquad cd(\eta) = \frac{1}{dc(\eta)}, \\
&nd(\eta) = \frac{1}{dn(\eta)}, \qquad ns(\eta) = \frac{1}{sn(\eta)}, \qquad nc(\eta) = \frac{1}{cn(\eta)}.
\end{aligned}$$

Furthermore, one can obtain the following expressions for $\frac{D_\xi^\alpha G(\xi)}{G(\xi)}$:

$$\begin{aligned}
&\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right) \\
&\quad = \begin{cases}
cn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ds(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = m^2, e_1 = -(1+m^2), e_0 = 0, \\
-sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = -m^2, e_1 = 2m^2 - 1, e_0 = 1 - m^2, \\
-m^2 sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})cd(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = -1, e_1 = 2 - m^2, e_0 = m^2 - 1, \\
-dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ns(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = 1, e_1 = 2 - m^2, e_0 = 1 - m^2, \\
cs(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nd(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = m^2(m^2 - 1), e_1 = 2m^2 - 1, e_0 = 1, \\
(1-m^2)sd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nc(\frac{\xi^\alpha}{\Gamma(1+\alpha)}), & e_2 = 1, e_1 = -(m^2+1), e_0 = m^2.
\end{cases}
\end{aligned} \tag{13}$$

Remark 1 For the sake of simplicity, other expressions for $(\frac{D_\xi^\alpha G(\xi)}{G(\xi)})$ with e_2, e_1, e_0 taken different values are omitted here.

3 Application of the fractional Jacobi elliptic equation method to some fractional partial differential equations

3.1 Space-time fractional KdV equation

Consider the following space-time fractional KdV equation:

$$D_t^\alpha u + 30u^2 D_x^\alpha u + 20\left(D_x^\alpha u\right)\left(D_x^{2\alpha} u\right) + 10u D_x^{3\alpha} u + D_x^{5\alpha} u = 0, \quad 0 < \alpha \leq 1, \tag{14}$$

which is a variation of the following KdV equation [31]:

$$u_t + 30u^2 u_x + 20u_x u_{xx} + 10u u_{xxx} + u_{xxxxx} = 0. \tag{15}$$

In order to apply the fractional auxiliary sub-equation method described in Section 2, suppose $u(x,t) = U(\xi)$, where $\xi = ct + kx + \xi_0$, k, c, ξ_0 are all constants with $k, c \neq 0$. By use of (3) and (6), one has

$$\begin{cases}
D_t^\alpha u = D_t^\alpha U(\xi) = D_\xi^\alpha U(\xi)(\xi_t'(t))^\alpha = c^\alpha D_\xi^\alpha U(\xi), \\
D_x^\alpha u = D_x^\alpha U(\xi) = D_\xi^\alpha U(\xi)(\xi_x'(x))^\alpha = k^\alpha D_\xi^\alpha U(\xi),
\end{cases} \tag{16}$$

and then (14) can be turned into the following form:

$$c^\alpha D_\xi^\alpha U + 30k^\alpha U^2 D_\xi^\alpha U + 20k^{3\alpha}\left(D_\xi^\alpha U\right)\left(D_\xi^{2\alpha} U\right) + 10k^{3\alpha} U D_\xi^{3\alpha} U + k^{5\alpha} D_\xi^{5\alpha} U = 0. \tag{17}$$

Suppose that the solution of (17) can be expressed by

$$U(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{D_\xi^\alpha G}{G} \right)^i, \tag{18}$$

where $G = G(\xi)$ satisfies (1). By Balancing the order between the highest order derivative term and nonlinear term in (17), we can obtain $m = 2$. So we have

$$U(\xi) = a_0 + a_1 \left(\frac{D_\xi^\alpha G}{G} \right) + a_2 \left(\frac{D_\xi^\alpha G}{G} \right)^2 + a_{-1} \left(\frac{D_\xi^\alpha G}{G} \right)^{-1} + a_{-2} \left(\frac{D_\xi^\alpha G}{G} \right)^{-2}. \tag{19}$$

Substituting (19) into (17), using (1), and collecting all the terms with the same power of $(D_\xi^\alpha G)^i G^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations with the aid of a symbolic computation program, yields the following results.

Case 1:

$$a_{-2} = 2k^{2\alpha}\left(-e_1^2 + 4e_0 e_2\right), \qquad a_{-1} = 0, \qquad a_1 = 0, \qquad a_2 = 0,$$
$$a_0 = \frac{40k^{3\alpha} e_1 \pm \sqrt{-80k^{6\alpha} e_1^2 - 30k^\alpha c^\alpha - 960k^{6\alpha} e_2 e_0}}{30k^\alpha}.$$

Case 2:

$$a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_1 = 0, \qquad a_2 = -2k^{2\alpha},$$
$$a_0 = \frac{40k^{3\alpha} e_1 \pm \sqrt{-80k^{6\alpha} e_1^2 - 30k^\alpha c^\alpha - 960k^{6\alpha} e_2 e_0}}{30k^\alpha}.$$

Substituting the results above into (19), and combining with (13) we can obtain a series of exact solutions in the forms of the Jacobi elliptic functions for (14). For example, from Case 1 we get the following exact solutions.

Family 1: when $e_2 = m^2$, $e_1 = -(1 + m^2)$, $e_0 = 0$,

$$\begin{aligned} u_1(x,t) &= \frac{-40k^{3\alpha}(1+m^2) \pm \sqrt{-80k^{6\alpha}(1+m^2)^2 - 30k^\alpha c^\alpha}}{30k^\alpha} \\ &\quad - 2k^{2\alpha}\left(1+m^2\right)^2 \left[cn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)} \right) ds\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)} \right) \right]^{-2}, \end{aligned} \tag{20}$$

where $\xi = ct + kx + \xi_0$.

Family 2: when $e_2 = -m^2$, $e_1 = 2m^2 - 1$, $e_0 = 1 - m^2$,

$$\begin{aligned} u_2(x,t) &= \frac{40k^{3\alpha}(2m^2-1) \pm \sqrt{-80k^{6\alpha}(2m^2-1)^2 - 30k^\alpha c^\alpha + 960k^{6\alpha} m^2(1-m^2)}}{30k^\alpha} \\ &\quad - 2k^{2\alpha} \left[sn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)} \right) dc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)} \right) \right]^{-2}, \end{aligned} \tag{21}$$

where $\xi = ct + kx + \xi_0$.

Family 3: when $e_2 = -1$, $e_1 = 2 - m^2$, $e_0 = m^2 - 1$,

$$u_3(x,t) = \frac{40k^{3\alpha}(2-m^2) \pm \sqrt{-80k^{6\alpha}(2-m^2)^2 - 30k^\alpha c^\alpha + 960k^{6\alpha}(m^2-1)}}{30k^\alpha} - 2k^{2\alpha}\left[sn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)cd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{22}$$

where $\xi = ct + kx + \xi_0$.

Family 4: when $e_2 = 1$, $e_1 = 2 - m^2$, $e_0 = 1 - m^2$,

$$u_4(x,t) = \frac{40k^{3\alpha}(2-m^2) \pm \sqrt{-80k^{6\alpha}(2-m^2)^2 - 30k^\alpha c^\alpha + 960k^{6\alpha}(m^2-1)}}{30k^\alpha} - 2k^{2\alpha}m^4\left[dc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)ns\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{23}$$

where $\xi = ct + kx + \xi_0$.

Family 5: when $e_2 = m^2(m^2-1)$, $e_1 = 2m^2 - 1$, $e_0 = 1$,

$$u_5(x,t) = \frac{40k^{3\alpha}(2m^2-1) \pm \sqrt{-80k^{6\alpha}(2m^2-1)^2 - 30k^\alpha c^\alpha - 960k^{6\alpha}m^2(m^2-1)}}{30k^\alpha} - 2k^{2\alpha}\left[cs\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)nd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{24}$$

where $\xi = ct + kx + \xi_0$.

Family 6: when $e_2 = 1$, $e_1 = -(m^2+1)$, $e_0 = m^2$,

$$u_6(x,t) = \frac{-40k^{3\alpha}(m^2+1) \pm \sqrt{-80k^{6\alpha}(m^2+1)^2 - 30k^\alpha c^\alpha - 960k^{6\alpha}m^2}}{30k^\alpha} - 2k^{2\alpha}\left[sd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)nc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{25}$$

where $\xi = ct + kx + \xi_0$.

From Case 2 we can also get some exact solutions expressed in the forms of the Jacobi elliptic functions for (14), which are omitted here.

3.2 Space-time fractional BBM equation

Consider the space-time fractional BBM equation

$$D_t^\alpha u + uD_x^\alpha u + D_x^\alpha u - \mu D_t^\alpha\left(D_x^{2\alpha}u\right) = 0, \quad 0 < \alpha \le 1, \tag{26}$$

which is a variation of the following BBM equation of integer order [32]:

$$u_t + uu_x + u_x - \mu u_{xxt} = 0. \tag{27}$$

Suppose $u(x,t) = U(\xi)$, where $\xi = ct + kx + \xi_0$, k, c, ξ_0 are all constants with $k, c \neq 0$. Then similar to the process of (16)-(17), (26) can be turned into the following form:

$$c^\alpha D_\xi^\alpha U + k^\alpha U D_\xi^\alpha U + k^\alpha D_\xi^\alpha U - \mu c^\alpha k^{2\alpha} D_\xi^{3\alpha} U = 0. \tag{28}$$

Suppose that the solution of (28) can be expressed by

$$U(\xi) = \sum_{i=-m}^{m} a_i \left(\frac{D_\xi^\alpha G}{G}\right)^i, \tag{29}$$

where $G = G(\xi)$ satisfies (1). By Balancing the order between the highest order derivative term and nonlinear term in (28), we can obtain $m = 2$. So we have

$$U(\xi) = a_0 + a_1\left(\frac{D_\xi^\alpha G}{G}\right) + a_2\left(\frac{D_\xi^\alpha G}{G}\right)^2 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} + a_{-2}\left(\frac{D_\xi^\alpha G}{G}\right)^{-2}. \tag{30}$$

Substituting (30) into (28), using (1), and collecting all the terms with the same power of $(D_\xi^\alpha G)^i G^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields the following two groups of values.

Case 1:

$$a_{-2} = 12\mu c^\alpha k^\alpha \left(e_1^2 - 4e_2e_0\right), \qquad a_{-1} = 0,$$
$$a_1 = 0, \qquad a_2 = 0, \qquad a_0 = -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha} e_1 + c^\alpha}{k^\alpha}.$$

Case 2:

$$a_{-2} = 0, \qquad a_{-1} = 0, \qquad a_1 = 0, \qquad a_2 = 12\mu c^\alpha k^\alpha, \qquad a_0 = -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha} e_1 + c^\alpha}{k^\alpha}.$$

Substituting the results above into (30), and combining with (13) we can obtain a series of exact solutions in the forms of the Jacobi elliptic functions for (26).

From Case 1 we get the following exact solutions.

Family 1: when $e_2 = m^2$, $e_1 = -(1 + m^2)$, $e_0 = 0$,

$$\begin{aligned} u_1(x,t) &= -\frac{k^\alpha - 8\mu c^\alpha k^{2\alpha}(1+m^2) + c^\alpha}{k^\alpha} \\ &\quad + 12\mu c^\alpha k^\alpha \left(1+m^2\right)^2 \left[cn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) ds\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \end{aligned} \tag{31}$$

where $\xi = ct + kx + \xi_0$.

Family 2: when $e_2 = -m^2$, $e_1 = 2m^2 - 1$, $e_0 = 1 - m^2$,

$$\begin{aligned} u_2(x,t) &= -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha}(2m^2-1) + c^\alpha}{k^\alpha} \\ &\quad + 12\mu c^\alpha k^\alpha \left[sn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) dc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \end{aligned} \tag{32}$$

where $\xi = ct + kx + \xi_0$.

Family 3: when $e_2 = -1$, $e_1 = 2 - m^2$, $e_0 = m^2 - 1$,

$$\begin{aligned} u_3(x,t) &= -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha}(2-m^2) + c^\alpha}{k^\alpha} \\ &\quad + 12\mu c^\alpha k^\alpha \left[sn\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) cd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \end{aligned} \tag{33}$$

where $\xi = ct + kx + \xi_0$.

Family 4: when $e_2 = 1$, $e_1 = 2 - m^2$, $e_0 = 1 - m^2$,

$$u_4(x,t) = -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha}(2-m^2) + c^\alpha}{k^\alpha} + 12\mu c^\alpha k^\alpha m^4 \left[dc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) ns\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{34}$$

where $\xi = ct + kx + \xi_0$.

Family 5: when $e_2 = m^2(m^2-1)$, $e_1 = 2m^2 - 1$, $e_0 = 1$,

$$u_5(x,t) = -\frac{k^\alpha + 8\mu c^\alpha k^{2\alpha}(2m^2-1) + c^\alpha}{k^\alpha} + 12\mu c^\alpha k^\alpha \left[cs\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) nd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{35}$$

where $\xi = ct + kx + \xi_0$.

Family 6: when $e_2 = 1$, $e_1 = -(m^2+1)$, $e_0 = m^2$,

$$u_6(x,t) = -\frac{k^\alpha - 8\mu c^\alpha k^{2\alpha}(m^2+1) + c^\alpha}{k^\alpha} + 12\mu c^\alpha k^\alpha \left[sd\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right) nc\left(\frac{\xi^\alpha}{\Gamma(1+\alpha)}\right)\right]^{-2}, \tag{36}$$

where $\xi = ct + kx + \xi_0$.

From Case 2 we can also get some Jacobi elliptic function solutions for (26), which are omitted here.

Remark 2 The Jacobi elliptic function solutions (20)-(25) and (31)-(36) are new exact solutions for the space-time fractional KdV equation and the space-time fractional BBM equation respectively to the best of our knowledge.

3.3 Space-time fractional (2 + 1)-dimensional breaking soliton equations

Consider the following space-time fractional (2 + 1)-dimensional breaking soliton equations:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + a\frac{\partial^{3\alpha} u}{\partial x^{2\alpha} y^\alpha} + 4au\frac{\partial^\alpha v}{\partial x^\alpha} + 4a\frac{\partial^\alpha u}{\partial x^\alpha} v = 0, \\ \frac{\partial^\alpha u}{\partial y^\alpha} = \frac{\partial^\alpha v}{\partial x^\alpha}, \end{cases} \quad 0 < \alpha \le 1, a \neq 0, \tag{37}$$

where the contained fractional derivative is the modified Riemann-Liouville derivative.

The corresponding integer order equation to (37) can be found in [33, 34]. Now we will apply the described method in Section 2 to solve (37). To begin with, we suppose $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$, where $\xi = ct + k_1 x + k_2 y + \xi_0$, k_1, k_2, c, ξ_0 are all constants with $k_1, k_2, c \neq 0$. Then similar to the process of (16)-(17), (37) can be turned into the following form:

$$\begin{cases} c^\alpha D_\xi^\alpha U + ak_1^{2\alpha} k_2^\alpha D_\xi^{3\alpha} U + 4ak_1^\alpha U D_\xi^\alpha V + 4ak_1^\alpha V D_\xi^\alpha U = 0, \\ k_2^\alpha D_\xi^\alpha U = k_1^\alpha D_\xi^\alpha V. \end{cases} \tag{38}$$

Suppose that the solution of (38) can be expressed by

$$\begin{cases} U(\xi) = \sum_{i=-m_1}^{m_1} a_i(\frac{G'}{G})^i, \\ V(\xi) = \sum_{i=-m_2}^{m_2} b_i(\frac{G'}{G})^i, \end{cases} \tag{39}$$

where $G = G(\xi)$ satisfies (1). By balancing the order between the highest order derivative term and nonlinear term in (38), we can obtain $m_1 = m_2 = 2$. So we have

$$\begin{cases} U(\xi) = a_0 + a_1(\frac{D_\xi^\alpha G}{G}) + a_2(\frac{D_\xi^\alpha G}{G})^2 + a_{-1}(\frac{D_\xi^\alpha G}{G})^{-1} + a_{-2}(\frac{D_\xi^\alpha G}{G})^{-2}, \\ V(\xi) = b_0 + b_1(\frac{G'}{G}) + b_2(\frac{G'}{G})^2 + b_{-1}(\frac{D_\xi^\alpha G}{G})^{-1} + b_{-2}(\frac{D_\xi^\alpha G}{G})^{-2}. \end{cases} \tag{40}$$

Substituting (40) into (38), using (1), and collecting all the terms with the same power of $(D_\xi^\alpha G)^i G^j$ together, equating each coefficient to zero, yields a set of algebraic equations. Solving these equations yields

$$a_0 = a_0, \qquad a_1 = 0, \qquad a_2 = -\frac{3}{2}k_1^{2\alpha}, \qquad a_{-1} = 0, \qquad a_{-2} = 0,$$
$$b_0 = -\frac{-8ak_1^{2\alpha}k_2^\alpha e_1 + c^\alpha + 4aa_0k_2^\alpha}{4ak_1^\alpha}, \qquad b_1 = 0,$$
$$b_2 = -\frac{3}{2}k_1^\alpha k_2^\alpha, \qquad b_{-1} = 0, \qquad b_{-2} = 0,$$

where a_0 is an arbitrary constant.

Substituting the results above into (40), and combining with (13) we can obtain a series of exact solutions in the forms of the Jacobi elliptic functions for (37).

Family 1: when $e_2 = m^2$, $e_1 = -(1 + m^2)$, $e_0 = 0$,

$$\begin{cases} u_1(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}[cn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ds(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_1(x,y,t) = -\frac{8ak_1^{2\alpha}k_2^\alpha(1+m^2)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha[cn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ds(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{41}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Family 2: when $e_2 = -m^2$, $e_1 = 2m^2 - 1$, $e_0 = 1 - m^2$,

$$\begin{cases} u_2(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}[sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_2(x,y,t) = -\frac{-8ak_1^{2\alpha}k_2^\alpha(2m^2-1)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha[sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{42}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Family 3: when $e_2 = -1$, $e_1 = 2 - m^2$, $e_0 = m^2 - 1$,

$$\begin{cases} u_3(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}m^4[sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})cd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_3(x,y,t) = -\frac{-8ak_1^{2\alpha}k_2^\alpha(2-m^2)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha m^4[sn(\frac{\xi^\alpha}{\Gamma(1+\alpha)})cd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{43}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Family 4: when $e_2 = 1$, $e_1 = 2 - m^2$, $e_0 = 1 - m^2$,

$$\begin{cases} u_4(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}[dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ns(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_4(x,y,t) = -\frac{-8ak_1^{2\alpha}k_2^\alpha(2-m^2)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha[dc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})ns(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{44}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Family 5: when $e_2 = m^2(m^2 - 1)$, $e_1 = 2m^2 - 1$, $e_0 = 1$,

$$\begin{cases} u_1(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}[cs(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_1(x,y,t) = -\frac{-8ak_1^{2\alpha}k_2^\alpha(2m^2-1)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha[cs(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{45}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Family 6: when $e_2 = 1$, $e_1 = -(m^2 + 1)$, $e_0 = m^2$,

$$\begin{cases} u_1(x,y,t) = a_0 - \frac{3}{2}k_1^{2\alpha}(1-m^2)^2[sd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \\ v_1(x,y,t) = -\frac{8ak_1^{2\alpha}k_2^\alpha(m^2+1)+c^\alpha+4aa_0k_2^\alpha}{4ak_1^\alpha} - \frac{3}{2}k_1^\alpha k_2^\alpha(1-m^2)^2[sd(\frac{\xi^\alpha}{\Gamma(1+\alpha)})nc(\frac{\xi^\alpha}{\Gamma(1+\alpha)})]^2, \end{cases} \tag{46}$$

where $\xi = ct + k_1x + k_2y + \xi_0$.

Remark 3 We note that the Jacobi elliptic function solutions established in (41)-(46) for the space-time fractional (2 + 1)-dimensional breaking soliton equations are new exact solutions so far in the literature.

Remark 4 Combining with other general solutions of the Jacobi elliptic equation (1) where e_2, e_1, e_0 taken different values, one can obtain abundant different exact solutions from those listed above for the space-time fractional KdV equation, the space-time fractional BBM equation, and the space-time fractional (2 + 1)-dimensional breaking soliton equations, which are omitted here for the sake of simplicity.

4 Conclusions

In this paper, we have proposed a new approach for seek exact solutions of fractional partial differential equations in the sense of the modified Riemann-Liouville derivative. Based the fractional Jacobi elliptic equation, exact traveling wave solutions in the forms of the Jacobi elliptic functions can be obtained for fractional partial differential equations. For illustrating the validity of this method, we apply it to seek exact solutions for two fractional equations: the space-time fractional KdV equation, the space-time fractional BBM equation, and the space-time fractional (2 + 1)-dimensional breaking soliton equations. As a result, a series of explicit solutions expressed in the Jacobi elliptic functions for them are successfully found with the aid of symbolic computation program. Being concise and powerful, we note that this method can also be applied to solve many other fractional partial differential equations arising in mathematical physics.

Competing interests

The author declares that they have no competing interests.

Author's contributions

BZ carried out the research of this article and read and approved the final manuscript.

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