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Controllability of nonlinear neutral fractional impulsive differential inclusions in Banach space

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Abstract
The paper is concerned with the controllability of nonlinear neutral fractional impulsive differential inclusions with infinite delay in a Banach space. Sufficient conditions for the controllability are obtained by using a fixed point theorem due to Dhage.

Keywords: mild solution; convex multivalued map; neutral impulsive differential inclusions; controllability; infinite delay

1 Introduction

Fractional differential equations have been proved to be one of the most effective tools in the modeling of many phenomena in various fields of physics, mechanics, chemistry, engineering, etc. For more details, see [1–5]. In order to describe various real-world problems in physical and engineering science subject to abrupt changes at certain instants during the evolution process, impulsive differential equations have been used to model the systems. The theory of impulsive differential equations is an important branch of differential equations, which has an extensive physical background [6–8].

Controllability is one of the important fundamental concepts in mathematical control theory and plays an important role in control systems. The problem of controllability is to show the existence of a control function, which steers the solution of the system from its initial state to a final state, where the initial and final states may vary over the entire space. A standard approach is to transform the controllability problem into a fixed point problem for an appropriate operator in a functional space. The problem of controllability and optimal controls for functional differential systems have been extensively studied in many papers [9–23]. For example, Wang JinRong and Zhou Yong [9] proved the existence and controllability results for fractional semilinear differential inclusions involving the Caputo derivative in Banach spaces by using operator semigroups and Bohnenblust-Karlin's fixed point theorem. Wang Jinrong *et al.* [11] established two sufficient conditions for nonlocal controllability for fractional evolution systems under some weak compactness conditions. Wang Jinrong *et al.* [13] considered the nonlinear control systems of fractional order and its optimal controls in Banach spaces and the sufficient condition is given for the existence and uniqueness of mild solutions for a broad class of fractional nonlinear infinite dimensional control systems. Wang Jinrong *et al.* [14] studied optimal feedback controls of a system governed by semilinear fractional evolution equations via a compact semigroup in Banach spaces. Wang Jinrong *et al.* [15] studied optimal relaxed controls and relaxation of nonlinear fractional impulsive evolution equations. Wang Jinrong *et al.* [16] investigated a class of Sobolev type semilinear fractional evolution systems in a separable Banach space. Applying a suitable fixed point theorem as well as condensing mapping, controllability results for two classes of control sets are established by means of the theory of propagation family and the technique of the measure of noncompactness. In [17], Chang YongKui *et al.* established a sufficient condition for the controllability of impulsive neutral functional differential inclusions in Banach space by using the Dhage fixed point theorem. M Benchohra *et al.* [24] discussed the controllability for first-order, second-order functional differential and integrodifferential inclusions in Banach space with finite delay. Jong Yeoul Park *et al.* [25] discussed the controllability for second-order neutral functional differential inclusions in Banach space with the help of some fixed point theorems. In [26, 27], Bing Liu investigated the controllability of neutral functional differential and integrodifferential inclusions with infinite delay. P Balasubramaniam and SK Ntouyas [28] obtained the controllability result of stochastic differential inclusions with infinite delay in abstract space. R Sakthivel *et al.* [18] considered a class of fractional neutral control systems governed by abstract nonlinear fractional neutral differential equations and established a new set of sufficient conditions for the controllability of nonlinear fractional systems by using a fixed point analysis approach. Using fixed point techniques, fractional calculations, stochastic analysis techniques and methods adopted directly from deterministic control problems, R Sakthivel *et al.* [20] gave a new set of sufficient conditions for approximate controllability of fractional stochastic differential equations. In [21], R Sakthivel and Y Ren investigated the complete controllability property of a nonlinear stochastic control system with jumps in a separable Hilbert space.

Since many systems arising from realistic models heavily depend on histories (*i.e.*, there is the effect of infinite delay on the state equations) [31], there is a real need to discuss partial functional differential systems with infinite delay. So, in the present paper, we will concentrate on the case with infinite delay and establish sufficient conditions for the controllability of systems (1.1) by relying on a fixed point theorem due to Dhage [29].

In this paper we will concentrate on the case with infinite delay and impulsive effect, and establish sufficient conditions for the controllability of the following fractional impulsive differential inclusions:

$$\begin{cases} {}^{C}D_t^{\alpha}[x(t) - g(t,x_t)] \in Ax(t) + F(t,x_t) + (Bu)(t), \\ \quad t \in J = [0,b], t \neq t_k, k = 1,2,\ldots,m, \\ \triangle x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1,2,\ldots,m, \\ x_0 = \phi \in \mathfrak{B}_h, \quad t \in J_0 = (-\infty,0], \end{cases} \tag{1.1}$$

where ${}^{C}D_t^{\alpha}$ is the Caputo fractional derivative of order $0 < \alpha < 1$; the state $x(\cdot)$ takes values in Banach space X with the norm $|\cdot|$, A is the infinitesimal generator of an analytic semigroup of the bounded linear operator $\{T(t), t \geq 0\}$ in X, the control function $u(\cdot)$ is given in $L^2(J,U)$, and we have a Banach space of admissible control functions with U as a Banach space. $F : J \times \mathfrak{B}_h \to \mathcal{P}(X)$ is a bounded, closed, convex-valued multivalued map, $g : J \times \mathfrak{B}_h \to X$ are given functions, where $\mathfrak{B}_h$ is a phase space defined in preliminaries. $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $I_k \in C(X,X)$ $(k = 1,2,\ldots,m)$ are bound functions.

$\triangle x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^-)$, $x(t_k^+)$ represent the left and right limit of $x(t)$ at $t = t_k$. $\mathcal{P}(X)$ denotes the class of all nonempty subsets of X. The histories $x_t : (-\infty, 0] \to X$, $x_t(s) = x(t+s)$, $s \leq 0$, belong to an abstract phase space $\mathfrak{B}_h$.

The structure of this paper is as follows. In Section 2 we briefly present some basic notations and preliminaries. The controllability result of system (1.1) is investigated by means of a fixed point theorem and operator theory in Section 3. A conclusion is given in Section 4.

2 Preliminaries

Let $(E, \|\cdot\|)$ be a Banach space. A multivalued map $\mathfrak{J} : E \to 2^E$ is convex (closed)-valued, if $\mathfrak{J}(x)$ is convex (closed) for all $x \in E$. $\mathfrak{J}$ is bounded on a bounded set if $\mathfrak{J}(B) = \bigcup_{x\in B} \mathfrak{J}(x)$ is bounded in E for any bounded set B of E; *i.e.*,

$$\sup_{x\in B} \sup\bigl\{\|y\| \in \mathfrak{J}(x)\bigr\} < \infty.$$

$\mathfrak{J}$ is called upper semicontinuous (u.s.c.) on E, if for each $x_* \in E$, the set $\mathfrak{J}(x_*)$ is a nonempty, closed subset of E, and if for each open set B of E containing $\mathfrak{J}(x_*)$, there exists an open neighborhood V of x_* such that $\mathfrak{J}(V) \subseteq B$.

$\mathfrak{J}$ is said to be completely continuous if $\mathfrak{J}(B)$ is relatively compact, for every bounded subset $B \subseteq E$.

If the multivalued map $\mathfrak{J}$ is completely continuous with nonempty compact values, then $\mathfrak{J}$ is u.s.c. if and only if $\mathfrak{J}$ has a closed graph (*i.e.*, $x_n = x_*$, $y_n = y_*$, $y_n \in \mathfrak{J}x_n$ imply $y_* \in \mathfrak{J}x_*$).

Let $BCC(E)$ denote the set of all the set of all nonempty, bounded, closed, and convex subsets of E. For more details of multivalued maps see the books of Deimling [32], and of Hu and Papageorgiou [33].

If T is an uniformly bounded and analytic semigroup with infinitesimal generator A such that $0 \in \rho(A)$ then it is possible to define the fractional power $(-A)^\alpha$, for $0 < \alpha \leq 1$, as a closed linear operator on its domain $D((-A)^\alpha)$. Furthermore, the subspace $D((-A)^\alpha)$ is dense in X and the expression

$$\|x\|_\alpha := \bigl\|(-A)^\alpha x\bigr\|, \quad x \in D\bigl((-A)^\alpha\bigr)$$

defines a norm on $D((-A)^\alpha)$. Hereafter we represent by X_α the space $D((-A)^\alpha)$ endowed with the norm $\|\cdot\|_\alpha$. We suppose that A is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t)$, $0 \in \rho(A)$, for $t \geq 0$, then there exist constants M such that $|T(t)| \leq M$. Then the following properties are well known [34].

Lemma 2.1 [34] *Suppose that the preceding conditions are satisfied.*

(a) *Let* $0 < \alpha \leq 1$. *Then* X_α *is a Banach space.*
(b) *If* $0 < \gamma < \alpha \leq 1$ *then* $X_\alpha \hookrightarrow X_\gamma$ *and the embedding is compact whenever the resolvent operator of* A *is compact.*
(c) *For every* $a > 0$, *there exists a positive constant* c_α *such that*

$$\bigl\|(-A)^\alpha T(t)\bigr\| \leq \frac{c_\alpha}{t^\alpha}, \quad 0 < t \leq a.$$

Definition 2.1 The fractional integral of order α with the lower limit 0 for a function f is defined as

$$I^\alpha f = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[0,\infty)$. Here $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Caputo derivative of order α with the lower limit 0 for a function f can be written as

$$^C D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^n(t), \quad t > 0, 0 \le n-1 < \alpha < n.$$

The key tool in our approach is the following fixed point theorem [24].

Theorem 2.2 ([29] Dhage's fixed point theorem) *Let X be a Banach space. $\Phi_1 : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $\Phi_2 : X \to \mathcal{P}_{cp,cv}(X)$ be two multivalued operators satisfying*:

(a) *Φ_1 is a contraction, and*
(b) *Φ_2 is completely continuous.*

Then either:

(i) *the operator inclusion $\lambda x \in \Phi_1 x + \Phi_2 x$ has a solution for $\lambda = 1$, or*
(ii) *the set $G = \{x \in X : \lambda x \in \Phi_1 x + \Phi_2 x, \lambda > 1\}$ is unbounded.*

We present the abstract phase space $\mathfrak{B}_h$, which has been used in [17, 27]. Assume that $h : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 h(t)\, dt < +\infty$. For any $a > 0$, we define $\mathfrak{B} = \{\psi : [-a, 0] \to X$ such that $\psi(t)$ is bounded and measurable$\}$ and equip the space $\mathfrak{B}$ with the norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} |\psi(s)|$, $\forall \psi \in \mathfrak{B}$. Let us define $\mathfrak{B}_h = \{\psi : (-\infty, 0] \to X$ such that, for any $c > 0$, $\psi|_{[-c,0]} \in \mathfrak{B}$ and $\int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}\, ds < +\infty\}$.

If $\mathfrak{B}_h$ is endowed with the norm $\|\psi\|_{\mathfrak{B}_h} = \int_{-\infty}^0 h(s)\|\psi\|_{[s,0]}\, ds$, $\forall \psi \in \mathfrak{B}$, then it is clear that $(\mathfrak{B}_h, \|\cdot\|_{\mathfrak{B}_h})$ is a Banach space [17, 27]. Now we consider the space $\mathfrak{B}_b = \{\psi : (-\infty, b] \to X$ such that $x_k \in C(J_k, X)$ and there exist $x(t_k^+)$ and $x(t_k^-)$ with $x(t_k) = x(t_k^-)$, $x_0 = \phi \in \mathfrak{B}_h$, $k = 0, 1, 2, \ldots, m\}$ where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, 2, \ldots, m$. Set $|\cdot|_b$ be a seminorm in $\mathfrak{B}_b$ defined by

$$\|x\|_b = \|x_0\|_{\mathfrak{B}_h} + \sup\bigl\{\bigl|x(s)\bigr| : s \in [0, b]\bigr\}, \quad x \in \mathfrak{B}_b.$$

3 Main result

In the following, we shall apply Theorem 2.2 to the study of the controllability of system (1.1).

Definition 3.1 A function $x : (-\infty, b] \to X$ is called a mild solution of system (1.1) if the following holds: $x_0 = \phi \in \mathfrak{B}_h$ on $(-\infty, 0]$, $\triangle x|_{t=t_k} = I_k(x(t_k^-))$, $k = 1, 2, \ldots, m$, the restriction of $x(\cdot)$ to the interval $[0,b) - \{t_1, t_2, \dots, t_m\}$ is continuous and the integral equation

$$\begin{cases} x(t) = S_\alpha(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s,x_s)\,ds \\ \qquad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\,ds + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)(Bu)(s)\,ds \\ \qquad + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(x(t_k^-)), \quad t \in J, \\ x_0 = \phi \in \mathfrak{B}_h, \quad t \in J_0, \end{cases} \tag{3.1}$$

is satisfied, where

$$f \in S_{F,x} = \{f \in L^1(J,X) : f(t) \in F(t,x_t), \text{ for a.e. } t \in J\},$$

$$S_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta)\,d\theta, \qquad T_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta)\,d\theta,$$

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-1-\frac{1}{\alpha}} \varpi_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\varpi_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \quad \theta \in (0,\infty).$$

ξ_α is a probability density function defined on $(0,\infty)$, that is, $\xi_\alpha(\theta) \geq 0$, $\theta \in (0,\infty)$, and $\int_0^\infty \xi_\alpha(\theta)\,d\theta = 1$.

Lemma 3.1 [30] *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*:

(1) *For any fixed $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are linear and bounded operators, i.e., for any $x \in X$,*

$$\|S_\alpha(t)x\| \leq M\|x\|, \qquad \|T_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(1+\alpha)}\|x\|.$$

(2) *The operators $\{S_\alpha(t)\}_{t\geq 0}$ and $\{T_\alpha(t)\}_{t\geq 0}$ are strongly continuous and compact.*

(3) *For any $x \in X$, $\beta \in (0,1)$ and $\theta \in (0,1]$, we have*

$$A T_\alpha(t) x = A^{1-\beta} T_\alpha(t) A^\beta x,$$

$$\|A^\theta T_\alpha(t)\| \leq \frac{\alpha c_\theta \Gamma(2-\theta)}{t^{\alpha\theta} \Gamma(1+\alpha(1-\theta))}, \quad t \in [0,b].$$

Definition 3.2 System (1.1) is said to be controllable on the interval J if for every continuous initial function $\phi \in \mathfrak{B}_h$, $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(t)$ of (1.1) satisfies $x(b) = x_1$.

To investigate the controllability of system (1.1), we use the following hypotheses:

(H_1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $T(t)$, $0 \in \rho(A)$, for $t \geq 0$, there exist constants M such that $|T(t)| \leq M$.

(H_2) The linear operator $W : L^2(J,U) \to X$ defined by

$$Wu = \int_0^b (b-s)^{\alpha-1} T(b-s) Bu(s)\,ds$$

has an induced inverse operator W^{-1}, which takes values in $L^2(J,U)/\ker W$ and there exist positive constants M_2, M_3 such that $|B| \leq M_2$ and $|W^{-1}| \leq M_3$.

(H_3) There exist constants $0 \leq \beta < 1$, c_0, c_1, c_2, L_g such that g is X_β-valued, $(-A)^\beta g$ is continuous, and

(i) $\|(-A)^\beta g(t,x)\| \leq c_1\|x\|_{\mathfrak{B}_h} + c_2$, $(t,x) \in J \times \mathfrak{B}_h$;

(ii) $\|(-A)^\beta g(t,x_1) - (-A)^\beta g(t,x_2)\| \leq L_g\|x_1 - x_2\|_{\mathfrak{B}_h}$, $(t,x_i) \in J \times \mathfrak{B}_h$, $i = 1,2$, with

$$C_0 = L_g l\left[\left\|(-A)^{-\beta}\right\| + \frac{c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right] < 1.$$

(H_4) There exists a constant d_k such that $\|I_k(x)\| \leq d_k$, $k = 1,2,\ldots,m$ for each $x \in X$.

(H_5) There exist an integrable function $p : J \to [0,+\infty)$ and a nondecreasing function $\psi : R_+ \to (0,+\infty)$ such that $\|F(t,x)\| = \sup\{|f| : f(t) \in F(t,x)\} \leq p(t)\psi(\|x\|_{\mathfrak{B}_h})$ for almost all $t \in J$ and all $x \in \mathfrak{B}_h$.

(H_6) There exists a positive constant r such that

$$\frac{r}{F_1 + F_2 r + F_3\psi(lr + \|\phi\|_{\mathfrak{B}_h} + lM|\phi(0)|)} > 1,$$

where

$$\begin{aligned}
F_1 &= K_1 + \frac{MM_2M_3b^\alpha}{\Gamma(1+\alpha)}\cdot\left\{|x_1| + M\left|\phi(0)\right| + K_1\right\},\\
F_2 &= K_2 + \frac{MM_2M_3b^\alpha}{\Gamma(1+\alpha)}\cdot K_2, \qquad F_3 = \left[1 + \frac{MM_2M_3b^\alpha}{\Gamma(1+\alpha)}\right]\frac{b^\alpha M}{\Gamma(1+\alpha)}\sup_{s\in J} p(s),\\
K_1 &= M\left[\left\|(-A)^{-\beta}\right\|\left(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\right)\right]\\
&\quad + \left[\left\|(-A)^{-\beta}\right\| + \frac{c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\cdot\frac{b^{\alpha\beta}}{\beta}\right]\left(c_1\|\phi\|_{\mathfrak{B}_h} + c_1 lM\left|\phi(0)\right| + c_2\right)\\
&\quad + M\sum_{k=1}^m d_k,\\
K_2 &= \left\|(-A)^{-\beta}\right\|c_1 l + \frac{c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\cdot\frac{b^{\alpha\beta}}{\beta}c_1 l.
\end{aligned}$$

Lemma 3.2 (Lasota and Opial [35]) *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying (H_5) and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$. Then the operator*

$$\Gamma \circ S_F : C(I,X) \to BCC\big(C(I,X)\big), \quad x \to (\Gamma \circ S_F)(x) = \Gamma(S_{F,x}),$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 3.3 [17, 27] *Suppose $x \in \mathfrak{B}_b$, then for $t \in J$, $x_t \in \mathfrak{B}_h$. Moreover,*

$$l\left|x(t)\right| \leq \|x_t\|_{\mathfrak{B}_h} \leq l\sup_{s\in[0,t]}\left|x(s)\right| + \|x_0\|_{\mathfrak{B}_h},$$

where $l = \int_{-\infty}^0 h(s)\,ds < +\infty$.

Now, consider the multivalued map $\mathfrak{L}:\mathfrak{B}_b\to 2^{\mathfrak{B}_b}$ defined by $\mathfrak{L}x$ the set of $\rho\in\mathfrak{B}_b$ such that

$$\rho(t)=\begin{cases}\phi(t), \quad t\in(-\infty,0],\\ S_\alpha(t)[\phi(0)-g(0,\phi)]+g(t,x_t)+\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,x_s)\,ds\\ \quad+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds+\sum_{0<t_k<t}S_\alpha(t-t_k)I_k(x(t_k^-))\\ \quad+\int_0^t(t-\eta)^{\alpha-1}T_\alpha(t-\eta)BW^{-1}[x_1-S_\alpha(b)[\phi(0)-g(0,\phi)]\\ \quad-g(b,y_b+\bar{\phi}_b)-\int_0^b(b-s)^{\alpha-1}AT_\alpha(b-s)g(s,x_s)\,ds\\ \quad-\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)f(s)\,ds\\ \quad-\sum_{k=1}^m S_\alpha(b-t_k)I_k(x(t_k^-))](\eta)\,d\eta, \quad t\in J,\end{cases} \tag{3.2}$$

where $f\in S_{F,x}$.

We shall show that the operator $\mathfrak{L}$ has fixed points, which are then a solution of system (1.1). For $\phi\in\mathfrak{B}_h$, we define $\bar{\phi}$ by

$$\bar{\phi}(t)=\begin{cases}\phi(t), & -\infty<t\le 0,\\ S_\alpha(t)\phi(0), & 0\le t\le b,\end{cases}$$

then $\bar{\phi}\in\mathfrak{B}_b$. Set

$$x(t)=y(t)+\bar{\phi}(t), \quad -\infty<t\le b.$$

It is clear that x satisfies (3.1) if and only if y satisfies $y_0=0$ and

$$\begin{aligned}y(t)={}&-S_\alpha(t)g(0,\phi)+g(t,y_t+\bar{\phi}_t)+\int_0^t(t-s)^{\alpha-1}AT_\alpha(t-s)g(s,y_s+\bar{\phi}_s)\,ds\\ &+\int_0^t(t-s)^{\alpha-1}T_\alpha(t-s)f(s)\,ds+\sum_{0<t_k<t}S_\alpha(t-t_k)I_k\bigl(y(t_k^-)+\bar{\phi}(t_k^-)\bigr)\\ &+\int_0^t(t-\eta)^{\alpha-1}T_\alpha(t-\eta)BW^{-1}\biggl[x_1-S_\alpha(b)\bigl[\phi(0)-g(0,\phi)\bigr]-g(b,y_b+\bar{\phi}_b)\\ &-\int_0^b(b-s)^{\alpha-1}AT_\alpha(b-s)g(s,y_s+\bar{\phi}_s)\,ds\\ &-\int_0^b(b-s)^{\alpha-1}T_\alpha(b-s)f(s)\,ds\\ &-\sum_{k=1}^m S_\alpha(b-t_k)I_k\bigl(y(t_k^-)+\bar{\phi}(t_k^-)\bigr)\biggr](\eta)\,d\eta, \quad t\in J.\end{aligned}$$

Let $\mathfrak{B}_b^0=\{y\in\mathfrak{B}_b:y_0=0\in\mathfrak{B}_h\}$. For any $y\in\mathfrak{B}_b^0$,

$$\|y\|_b=\|y_0\|_{\mathfrak{B}_h}+\sup\bigl\{|y(s)|:0\le s\le b\bigr\}=\sup\bigl\{|y(s)|:0\le s\le b\bigr\}.$$

Thus $(\mathfrak{B}_b^0, \|\cdot\|_b)$ is a Banach space. Set $\mathfrak{B}_q = \{y \in \mathfrak{B}_b^0 : \|y\|_b \le q\}$ for some $q \ge 0$, then $\mathfrak{B}_q \subseteq \mathfrak{B}_b^0$ is uniformly bounded, for any $y \in \mathfrak{B}_q$, and from Lemma 3.3, we have

$$\begin{aligned}
\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} &\le \|y_t\|_{\mathfrak{B}_h} + \|\bar{\phi}_t\|_{\mathfrak{B}_h} \\
&\le l \sup_{s\in[0,t]} |y(s)| + \|y_0\|_{\mathfrak{B}_h} + l \sup_{s\in[0,t]} |\bar{\phi}(s)| + \|\bar{\phi}_0\|_{\mathfrak{B}_h} \\
&\le l(q + M|\phi(0)|) + \|\phi\|_{\mathfrak{B}_h} = q'.
\end{aligned}$$

Define the multivalued map $\Phi : \mathfrak{B}_b^0 \to 2^{\mathfrak{B}_b^0}$ defined by Φy, the set of $\bar{\rho} \in \mathfrak{B}_b^0$ such that

$$\bar{\rho}(t) = \begin{cases}
0, \quad t \in (-\infty, 0], \\
-S_\alpha(t)g(0,\phi) + g(t, y_t + \bar{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s, y_s + \bar{\phi}_s)\, ds \\
\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(y(t_k^-) + \bar{\phi}(t_k^-)) \\
\quad + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1}[x_1 - S_\alpha(b)[\phi(0) - g(0,\phi)] \\
\quad - g(b, y_b + \bar{\phi}_b) - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\, ds \\
\quad - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\, ds \\
\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k(y(t_k^-) + \bar{\phi}(t_k^-))](\eta)\, d\eta, \quad t \in J.
\end{cases} \tag{3.3}$$

Now we decompose Φ as $\Phi_1 + \Phi_2$, where

$$\begin{aligned}
\Phi_1 y(t) &= -S_\alpha(t) g(0,\phi) + g(t, y_t + \bar{\phi}_t) + \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) g(s, y_s + \bar{\phi}_s)\, ds, \\
\Phi_2 y(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\, ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k(y(t_k^-) + \bar{\phi}(t_k^-)) \\
&\quad + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \Bigg[x_1 - S_\alpha(b)[\phi(0) - g(0,\phi)] - g(b, y_b + \bar{\phi}_b) \\
&\quad - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\, ds \\
&\quad - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\, ds \\
&\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k(y(t_k^-) + \bar{\phi}(t_k^-)) \Bigg](\eta)\, d\eta, \quad t \in J.
\end{aligned}$$

Theorem 3.4 *Assume that hypotheses* (H_1)-(H_6) *hold, then system* (1.1) *is controllable on* J.

Proof We divide the proof into several steps.

Step 1. We remark that Φ_1 for each $y \in \mathfrak{B}_b^0$ has closed, convex values on $\mathfrak{B}_b^0$. Next we show that Φ_1 has bounded values for bounded in $\mathfrak{B}_b^0$. To show this, let $\mathfrak{B}_q = \{y \in \mathfrak{B}_b^0 : \|y\|_b \le q\}$ for some $q > 0$. Then, for any $y \in \mathfrak{B}_q$, one has

$$\begin{aligned}
\|\Phi_1 y(t)\| &\le M\|g(0,\phi)\| + \|(-A)^{-\beta}\| [c_1 \|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} + c_2] \\
&\quad + \int_0^t \|(t-s)^{\alpha-1} A^{1-\beta} T_\alpha(t-s) A^\beta g(s, y_s + \bar{\phi}_s)\| \, ds
\end{aligned}$$

$$\begin{aligned}&\le M\|g(0,\phi)\| + \|(-A)^{-\beta}\|(c_1q' + c_2)\\&\quad + \frac{(c_1q' + c_2)\alpha c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\Gamma(1+\alpha\beta)\alpha\beta}.\end{aligned}$$

Hence Φ_1 is bounded.

Step 2. $\Phi_2 y$ is convex for each $y \in \mathfrak{B}_b^0$.

In fact, if $\bar{\rho}_1$, $\bar{\rho}_2$ belong to $\Phi_2 y$, then there exist $f_1, f_2 \in S_{F,y}$ such that, for each $t \in J$, we have

$$\begin{aligned}\bar{\rho}_i(t) &= \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f_i(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr)\\&\quad + \int_0^t (t-\eta)^{\alpha-1}T_\alpha(t-\eta)BW^{-1}\Biggl[x_1 - S_\alpha(b)\bigl[\phi(0) - g(0,\phi)\bigr] - g(b, y_b + \bar{\phi}_b)\\&\quad - \int_0^b (b-s)^{\alpha-1}AT_\alpha(b-s)g(s, y_s + \bar{\phi}_s)\,ds - \int_0^b (b-s)^{\alpha-1}T_\alpha(b-s)f_i(s)\,ds\\&\quad - \sum_{k=1}^m S_\alpha(b-t_k)I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr)\Biggr](\eta)\,d\eta, \quad i = 1,2.\end{aligned}$$

Let $\mu \in [0,1]$, since the operators B and W^{-1} are linear, we have

$$\begin{aligned}&\bigl(\mu\bar{\rho}_1 + (1-\mu)\bar{\rho}_2\bigr)(t)\\&\quad = \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)\bigl[\mu f_1(s) + (1-\mu)f_2(s)\bigr]\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k)I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr)\\&\quad\quad + \int_0^t (t-\eta)^{\alpha-1}T_\alpha(t-\eta)BW^{-1}\Biggl[x_1 - S_\alpha(b)\bigl[\phi(0) - g(0,\phi)\bigr] - g(b, y_b + \bar{\phi}_b)\\&\quad\quad - \int_0^b (b-s)^{\alpha-1}AT_\alpha(b-s)g(s, y_s + \bar{\phi}_s)\,ds\\&\quad\quad - \int_0^b (b-s)^{\alpha-1}T_\alpha(b-s)\bigl[\mu f_1(s) + (1-\mu)f_2(s)\bigr]\,ds\\&\quad\quad - \sum_{k=1}^m S_\alpha(b-t_k)I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr)\Biggr](\eta)\,d\eta.\end{aligned}$$

Since $S_{F,y}$ is convex (because F has convex values), we have $(\mu\bar{\rho}_1 + (1-\mu)\bar{\rho}_2) \in \Phi_2 y$.

Step 3. We will prove that the operator Φ_1 is a contraction operator on $\mathfrak{B}_b^0$. Let $u, v \in \mathfrak{B}_b^0$; we have

$$\begin{aligned}&\bigl\|\Phi_1 u(t) - \Phi_1 v(t)\bigr\|\\&\quad \le \bigl\|g(t, u_t + \bar{\phi}_t) - g(t, v_t + \bar{\phi}_t)\bigr\|\\&\quad\quad + \int_0^t \bigl\|(t-s)^{\alpha-1}AT_\alpha(t-s)\bigl[g(s, u_s + \bar{\phi}_s) - g(s, v_s + \bar{\phi}_s)\bigr]\bigr\|\,ds\\&\quad \le \bigl\|(-A)^{-\beta}\bigr\|L_g\|u_t - v_t\|_{\mathfrak{B}_h} + L_g\|u_t - v_t\|_{\mathfrak{B}_h}\int_0^t (t-s)^{\alpha-1}\bigl\|A^{1-\beta}T_\alpha(t-s)\bigr\|\,ds\end{aligned}$$

$$
\begin{aligned}
&\leq \|(-A)^{-\beta}\| L_g \|u_t - v_t\|_{\mathfrak{B}_h} + L_g \|u_t - v_t\|_{\mathfrak{B}_h} \int_0^t (t-s)^{\alpha-1} \frac{\alpha c_{1-\beta}\Gamma(1+\beta)}{(t-s)^{\alpha(1-\beta)}\Gamma(1+\alpha\beta)}\,ds \\
&\leq \|(-A)^{-\beta}\| L_g \|u_t - v_t\|_{\mathfrak{B}_h} + L_g \|u_t - v_t\|_{\mathfrak{B}_h} \frac{c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \\
&\leq \|(-A)^{-\beta}\| L_g \Big[l \sup_{s\in[0,t]} |u(s) - v(s)| + \|u_0 - v_0\|_{\mathfrak{B}_h} \Big] \\
&\quad + L_g \Big[l \sup_{s\in[0,t]} |u(s) - v(s)| + \|u_0 - v_0\|_{\mathfrak{B}_h} \Big] \frac{c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \\
&\leq L_g l \left[\|(-A)^{-\beta}\| + \frac{c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \right] \sup_{s\in[0,b]} |u(s) - v(s)|,
\end{aligned}
$$

since $\|u_0 - v_0\|_{\mathfrak{B}_h} = 0$, taking the supremum over t, $\|\Phi_1 u - \Phi_1 v\| \leq C_0 \|u - v\|$, where

$$
C_0 = L_g l \left[\|(-A)^{-\beta}\| + \frac{c_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \right] < 1.
$$

Thus Φ_1 is a contraction on $\mathfrak{B}_b^0$.

Step 4. Next we show that the operator Φ_2 is completely continuous. First, we prove that Φ_2 maps a bounded set into a bounded set in $\mathfrak{B}_b^0$. Indeed, it is enough to show that there exists a positive constant Λ such that, for each $\bar{\rho} \in \Phi_2 y$, $y \in \mathfrak{B}_q = \{y \in \mathfrak{B}_b^0 : \|y\|_b \leq q\}$, one has $\|\bar{\rho}\|_b \leq \Lambda$. If $\bar{\rho} \in \Phi_2 y$, then there exists $f \in S_{F,y}$, such that, for each $t \in J$,

$$
\begin{aligned}
\bar{\rho}(t) &= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big) \\
&\quad + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g(b, y_b + \bar{\phi}_b) \\
&\quad - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\,ds \\
&\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big) \bigg](\eta)\,d\eta, \quad t \in J.
\end{aligned}
$$

We have for $t \in J$

$$
\begin{aligned}
|\bar{\rho}(t)| &\leq \int_0^t (t-s)^{\alpha-1} |T_\alpha(t-s) f(s)|\,ds + \sum_{0<t_k<t} \big|S_\alpha(t-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big)\big| \\
&\quad + \int_0^t \bigg| (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g(b, y_b + \bar{\phi}_b) \\
&\quad - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\,ds \\
&\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big) \bigg] \bigg| (\eta)\,d\eta \\
&\leq \int_0^t (t-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} p(s) \psi\big(\|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h}\big)\,ds + M \sum_{k=1}^m d_k
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{\alpha M M_2 M_3}{\Gamma(1+\alpha)} \int_0^t (t-\eta)^{\alpha-1} \biggl| x_1 - S_\alpha(b)\bigl[\phi(0) - g(0,\phi)\bigr] - g(b, y_b + \bar{\phi}_b) \\
&- \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\,ds \\
&- \sum_{k=1}^{m} S_\alpha(b-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \bar{\phi}\bigl(t_k^-\bigr)\bigr) \biggr| (\eta)\,d\eta \\
\le{}& \sup_{s\in J} p(s) \frac{b^\alpha M}{\Gamma(1+\alpha)} \sup_{y\in[0,q']} \psi(y) + M \sum_{k=1}^{m} d_k \\
&+ \frac{\alpha M M_2 M_3}{\Gamma(1+\alpha)} \int_0^t (t-\eta)^{\alpha-1} \biggl[|x_1| + M\bigl(\phi(0) + \bigl\|(-A)^{-\beta}\bigr\| \bigl(c_1 \|\phi\|_{\mathfrak{B}_h} + c_2\bigr)\bigr) \\
&+ \bigl\|(-A)^{-\beta}\bigr\| \bigl(c_1 \|y_b + \bar{\phi}_b\|_{\mathfrak{B}_h} + c_2\bigr) \\
&+ \int_0^b \frac{\alpha c_{1-\beta} \Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} (b-s)^{\alpha\beta-1} \bigl(c_1 \|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h} + c_2\bigr)\,ds \\
&+ \int_0^b (b-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} p(s) \psi\bigl(\|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h}\bigr)\,ds + M \sum_{k=1}^{m} d_k \biggr] (\eta)\,d\eta \\
\le{}& \sup_{s\in J} p(s) \frac{b^\alpha M}{\Gamma(1+\alpha)} \sup_{y\in[0,q']} \psi(y) + M \sum_{k=1}^{m} d_k \\
&+ \frac{\alpha M M_2 M_3}{\Gamma(1+\alpha)} \biggl[|x_1| + M\bigl(\phi(0) + \bigl\|(-A)^{-\beta}\bigr\| \bigl(c_1 \|\phi\|_{\mathfrak{B}_h} + c_2\bigr)\bigr) \\
&+ \bigl\|(-A)^{-\beta}\bigr\| \bigl(c_1 q' + c_2\bigr) + \bigl(c_1 q' + c_2\bigr) \frac{c_{1-\beta} \Gamma(1+\beta) b^{\alpha\beta}}{\beta \Gamma(1+\alpha\beta)} \\
&+ \sup_{s\in J} p(s) \frac{b^\alpha M}{\Gamma(1+\alpha)} \sup_{y\in[0,q']} \psi(y) + M \sum_{k=1}^{m} d_k \biggr] \cdot \frac{b^\alpha}{\alpha} \\
={}& \Lambda,
\end{aligned}
$$

then, for each $\bar{\rho} \in \Phi_2(\mathfrak{B}_q)$, we have

$$\|\bar{\rho}\|_b \le \Lambda.$$

Step 5. Next, we show that Φ_2 maps bounded sets into equicontinuous sets of $\mathfrak{B}_b^0$.

Let $r_1, r_2 \in J$, $0 < r_1 < r_2 \le b$, for each $y \in \mathfrak{B}_q = \{y \in \mathfrak{B}_b^0 : \|y\|_b \le q\}$ and $\bar{\rho} \in \Phi_2 y$, then there exists $f \in S_{F,y}$, such that, for each $t \in J$,

$$
\begin{aligned}
\bar{\rho}(t) ={}& \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \bar{\phi}\bigl(t_k^-\bigr)\bigr) \\
&+ \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \biggl[x_1 - S_\alpha(b)\bigl[\phi(0) - g(0,\phi)\bigr] - g(b, y_b + \bar{\phi}_b) \\
&- \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds
\end{aligned}
$$

$$\begin{aligned}&- \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\, ds \\ &- \sum_{k=1}^m S_\alpha(b-t_k) I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr)\Bigg](\eta)\, d\eta, \quad t \in J.\end{aligned}$$

Let $r_1, r_2 \in J - \{t_1, t_2, \ldots, t_m\}$, we have

$$\begin{aligned}&\bigl\| \bar{\rho}(r_1) - \bar{\rho}(r_2) \bigr\| \\ &\quad\le \left| \int_0^{r_1} (r_1-s)^{\alpha-1} \bigl[T_\alpha(r_2-s) - T_\alpha(r_1-s)\bigr] f(s)\, ds \right| \\ &\qquad+ \left| \int_0^{r_1} \bigl[(r_2-s)^{\alpha-1} - (r_1-s)^{\alpha-1}\bigr] T_\alpha(r_2-s) f(s)\, ds \right| \\ &\qquad+ \left| \int_{r_1}^{r_2} (r_2-s)^{\alpha-1} T_\alpha(r_2-s) f(s)\, ds \right| \\ &\qquad+ \left| \sum_{0<t_k<r_1} \bigl[S_\alpha(r_2-t_k) - S_\alpha(r_1-t_k)\bigr] I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr) \right| \\ &\qquad+ \left| \sum_{r_1 \le t_k < r_2} S_\alpha(r_2-t_k) I_k\bigl(y(t_k^-) + \bar{\phi}(t_k^-)\bigr) \right| \\ &\qquad+ \left| \int_0^{r_1} (r_1-\eta)^{\alpha-1} \bigl[T_\alpha(r_2-\eta) - T_\alpha(r_1-\eta)\bigr] Bu(\eta)\, d\eta \right| \\ &\qquad+ \left| \int_0^{r_1} \bigl[(r_2-\eta)^{\alpha-1} - (r_1-\eta)^{\alpha-1}\bigr] T_\alpha(r_2-\eta) Bu(\eta)\, d\eta \right| \\ &\qquad+ \left| \int_{r_1}^{r_2} (r_2-\eta)^{\alpha-1} T_\alpha(r_2-\eta) Bu(\eta)\, d\eta \right| \\ &\quad\le \int_0^{r_1} (r_1-s)^{\alpha-1} \bigl\| T_\alpha(r_2-s) - T_\alpha(r_1-s) \bigr\| \bigl\| f(s) \bigr\|\, ds \\ &\qquad+ \int_0^{r_1} \bigl|(r_2-s)^{\alpha-1} - (r_1-s)^{\alpha-1}\bigr| \bigl|T_\alpha(r_2-s)\bigr| \bigl\| f(s) \bigr\|\, ds \\ &\qquad+ \int_{r_1}^{r_2} (r_2-s)^{\alpha-1} \bigl\| T_\alpha(r_2-s) \bigr\| \bigl\| f(s) \bigr\|\, ds \\ &\qquad+ \sum_{0<t_k<r_1} \bigl|S_\alpha(r_2-t_k) - S_\alpha(r_1-t_k)\bigr| d_k + \sum_{r_1 \le t_k < r_2} M d_k \\ &\qquad+ \int_0^{r_1} (r_1-\eta)^{\alpha-1} \bigl\| T_\alpha(r_2-\eta) - T_\alpha(r_1-\eta) \bigr\| \bigl\| Bu(\eta) \bigr\|\, d\eta \\ &\qquad+ \int_0^{r_1} \bigl[(r_2-\eta)^{\alpha-1} - (r_1-\eta)^{\alpha-1}\bigr] \bigl\| T_\alpha(r_2-\eta) \bigr\| \bigl\| Bu(\eta) \bigr\|\, d\eta \\ &\qquad+ \int_{r_1}^{r_2} (r_2-\eta)^{\alpha-1} \bigl\| T_\alpha(r_2-\eta) \bigr\| \bigl\| Bu(\eta) \bigr\|\, d\eta.\end{aligned}$$

As $r_2 \to r_1$, the right-hand side of the above inequality tends to zero, thus the set $\{\Phi_2 y : y \in \mathfrak{B}_q\}$ is equicontinuous. This proves the equicontinuity in the case where $t \neq t_i$, $i = 1, 2, \ldots, m$. Similarly one can prove that $t = t_i$. The equicontinuities for the other cases,

$r_1 < r_2 \leq 0$ or $r_1 \leq 0 \leq r_2 \leq b$, are very simple. As a consequence of the Arzela-Ascoli theorem, Φ_2 is completely continuous.

Step 6. Φ_2 has a closed graph.

Let $y^{(n)} \to y^*$, $\bar{\rho}_n \in \Phi_2(y^{(n)})$ and $\overline{\rho}_n \to \bar{\rho}_*$. We shall prove that $\bar{\rho}_* \in \Phi_2(y^*)$. Indeed, $\bar{\rho}_n \in \Phi_2(y^{(n)})$ means that there exists $f_n \in S_{F,y^{(n)}}$, such that

$$\begin{aligned}\bar{\rho}_n(t) = {}& \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_n(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\ & + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(n)} + \bar{\phi}_b\big) \\ & - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(n)} + \bar{\phi}_s\big)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f_n(s)\,ds \\ & - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \bigg](\eta)\,d\eta, \quad t \in J.\end{aligned}$$

We must prove that there exists $f_* \in S_{F,y^*}$ such that

$$\begin{aligned}\bar{\rho}_*(t) = {}& \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f_*(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\ & + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(*)} + \bar{\phi}_b\big) \\ & - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(*)} + \bar{\phi}_s\big)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f_*(s)\,ds \\ & - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \bigg](\eta)\,d\eta, \quad t \in J;\end{aligned}$$

since I_k, $k = 0,1,2,\ldots,m$ are continuous, we obtain

$$\begin{aligned}\Bigg\| \Bigg\{ & \bar{\rho}_n(t) - \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\ & - \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(n)} + \bar{\phi}_b\big) \\ & - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(n)} + \bar{\phi}_s\big)\,ds \\ & - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \bigg](\eta)\,d\eta \Bigg\} \\ & - \Bigg\{ \bar{\rho}_*(t) - \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\ & - \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) B W^{-1} \bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(*)} + \bar{\phi}_b\big)\end{aligned}$$

$$\begin{aligned}
&- \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(*)} + \bar{\phi}_s\big)\, ds \\
&- \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big)\Bigg](\eta)\, d\eta \Bigg\}\Bigg\|_b \\
&\longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned}$$

Consider the linear continuous operator

$$\begin{aligned}
&\Gamma : L^1(J,X) \longrightarrow C(J,X), \\
&f \longrightarrow \Gamma(f)(t) = \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\bigg[f(s) + BW^{-1}\int_0^b (b-\tau) T_\alpha(b-\tau) f(\tau)\, d\tau\bigg](s)\, ds.
\end{aligned}$$

From Lemma 3.1, it follows that $T \circ S_F$ is a closed graph operator. Moreover, we have

$$\begin{aligned}
&\bar{\rho}_n(t) - \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\
&\quad - \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) BW^{-1}\Bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(n)} + \bar{\phi}_b\big) \\
&\quad - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(n)} + \bar{\phi}_s\big)\, ds \\
&\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(n)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big)\Bigg](\eta)\, d\eta \in \Gamma(S_{F,y^{(n)}}).
\end{aligned}$$

Since $y^{(n)} \longrightarrow y^*$, it follows from Lemma 3.2 that

$$\begin{aligned}
&\bar{\rho}_*(t) - \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big) \\
&\quad - \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) BW^{-1}\Bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g\big(b, y_b^{(*)} + \bar{\phi}_b\big) \\
&\quad - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) g\big(s, y_s^{(*)} + \bar{\phi}_s\big)\, ds \\
&\quad - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y^{(*)}\big(t_k^-\big) + \bar{\phi}\big(t_k^-\big)\big)\Bigg](\eta)\, d\eta \\
&= \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\bigg[f_*(s) + BW^{-1}\int_0^b (b-\tau) T_\alpha(b-\tau) f_*(\tau)\, d\tau\bigg](s)\, ds
\end{aligned}$$

for some $f_* \in S_{F,y^*}$. Hence Φ_2 is a completely continuous multivalued map, u.s.c. with convex closed values.

Step 7. The operator inclusions $y \in \Phi_1 y + \Phi_2 y = \Phi y$ has a solution in $\mathfrak{B}_b^0$.

Let y be a possible solution of $y \in \lambda\Phi(y)$ for some $\lambda \in (0,1)$. Then there exists $f \in S_{F,y}$ such that, for $t \in J$, we have

$$\begin{aligned}
y(t) = \lambda\Bigg[& -S_\alpha(t)g(0,\phi) + g(t, y_t + \bar{\phi}_t) + \int_0^t (t-s)^{\alpha-1} AT_\alpha(t-s) g(s, y_s + \bar{\phi}_s)\,ds \\
& + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s)\,ds + \sum_{0<t_k<t} S_\alpha(t-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big) \\
& + \int_0^t (t-\eta)^{\alpha-1} T_\alpha(t-\eta) BW^{-1}\Bigg[x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g(b, y_b + \bar{\phi}_b) \\
& - \int_0^b (b-s)^{\alpha-1} AT_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\,ds \\
& - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big)\Bigg](\eta)\,d\eta\Bigg], \quad t \in J, \\
\big|y(t)\big| \le{} & M\big[\big\|(-A)^{-\beta}\big\|\big(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\big)\big] + \big\|(-A)^{-\beta}\big\|\big(c_1\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} + c_2\big) \\
& + \int_0^t \frac{\alpha c_{1-\beta}\Gamma(1+\beta)}{(t-s)^{\alpha(1-\beta)}\Gamma(1+\alpha\beta)} \cdot (t-s)^{\alpha-1}\big(c_1\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} + c_2\big)\,ds \\
& + \int_0^t (t-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} p(s)\psi\big(\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h}\big)\,ds + M\sum_{k=1}^m d_k \\
& + \frac{\alpha M M_2 M_3}{\Gamma(1+\alpha)} \int_0^t (t-\eta)^{\alpha-1}\bigg| x_1 - S_\alpha(b)\big[\phi(0) - g(0,\phi)\big] - g(b, y_b + \bar{\phi}_b) \\
& - \int_0^b (b-s)^{\alpha-1} AT_\alpha(b-s) g(s, y_s + \bar{\phi}_s)\,ds \\
& - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s)\,ds \\
& - \sum_{k=1}^m S_\alpha(b-t_k) I_k\big(y(t_k^-) + \bar{\phi}(t_k^-)\big)\bigg|(\eta)\,d\eta \\
\le{} & M\big[\big\|(-A)^{-\beta}\big\|\big(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\big)\big] + \big\|(-A)^{-\beta}\big\|\big(c_1\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} + c_2\big) \\
& + \int_0^t \frac{\alpha c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \cdot (t-s)^{\alpha\beta-1}\big(c_1\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} + c_2\big)\,ds \\
& + \int_0^t (t-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} p(s)\psi\big(\|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h}\big)\,ds + M\sum_{k=1}^m d_k \\
& + \frac{\alpha M M_2 M_3}{\Gamma(1+\alpha)} \int_0^t (t-\eta)^{\alpha-1}\bigg\{ |x_1| + M\big[\phi(0) + \big\|(-A)^{-\beta}\big\|\big(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\big)\big] \\
& + \big\|(-A)^{-\beta}\big\|\big(c_1\|y_b + \bar{\phi}_b\|_{\mathfrak{B}_h} + c_2\big) \\
& + \int_0^b \frac{\alpha c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \cdot (b-s)^{\alpha\beta-1}\big(c_1\|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h} + c_2\big)\,ds \\
& + \int_0^b (b-s)^{\alpha-1} \frac{\alpha M}{\Gamma(1+\alpha)} p(s)\psi\big(\|y_s + \bar{\phi}_s\|_{\mathfrak{B}_h}\big)\,ds + M\sum_{k=1}^m d_k\bigg\}(\eta)\,d\eta.
\end{aligned}$$

Since $\|y_t + \bar{\phi}_t\|_{\mathfrak{B}_h} \leq l \sup_{s\in[0,t]} |y(s)| + \|\phi\|_{\mathfrak{B}_h} + lM|\phi(0)|$,

$$
\begin{aligned}
\|y\|_b &= \sup_{t\in[0,b]} \left|y(t)\right| \\
&\leq M\Big[\big\|(-A)^{-\beta}\big\|\big(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\big)\Big] \\
&\quad + \big\|(-A)^{-\beta}\big\|\Big(c_1 l \sup_{t\in[0,b]}\left|y(t)\right| + c_1\|\phi\|_{\mathfrak{B}_h} + c_1 lM\left|\phi(0)\right| + c_2\Big) \\
&\quad + \frac{\alpha c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\cdot\frac{b^{\alpha\beta}}{\alpha\beta}\Big(c_1 l \sup_{t\in[0,b]}\left|y(t)\right| + c_1\|\phi\|_{\mathfrak{B}_h} + c_1 lM\left|\phi(0)\right| + c_2\Big) \\
&\quad + \sup_{s\in J} p(s)\frac{b^{\alpha}M}{\Gamma(1+\alpha)}\psi\Big(l \sup_{t\in[0,b]}\left|y(t)\right| + \|\phi\|_{\mathfrak{B}_h} + lM\left|\phi(0)\right|\Big) + M\sum_{k=1}^{m} d_k \\
&\quad + \frac{\alpha MM_2M_3}{\Gamma(1+\alpha)}\int_0^t (t-\eta)^{\alpha-1}\Bigg\{|x_1| + M\Big[\phi(0) + \big\|(-A)^{-\beta}\big\|\big(c_1\|\phi\|_{\mathfrak{B}_h} + c_2\big)\Big] \\
&\quad + \big\|(-A)^{-\beta}\big\|\Big(c_1 l \sup_{t\in[0,b]}\left|y(t)\right| + c_1\|\phi\|_{\mathfrak{B}_h} + c_1 lM\left|\phi(0)\right| + c_2\Big) \\
&\quad + \Big(c_1 l \sup_{t\in[0,b]}\left|y(t)\right| + c_1\|\phi\|_{\mathfrak{B}_h} + c_1 lM\left|\phi(0)\right| + c_2\Big)\frac{c_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\cdot\frac{b^{\alpha\beta}}{\beta} \\
&\quad + \sup_{s\in J} p(s)\frac{b^{\alpha}M}{\Gamma(1+\alpha)}\psi\Big(l \sup_{t\in[0,b]}\left|y(t)\right| + \|\phi\|_{\mathfrak{B}_h} + lM\left|\phi(0)\right|\Big) + M\sum_{k=1}^{m} d_k\Bigg\}(\eta)\,d\eta \\
&\leq F_1 + F_2\|y\|_b + F_3\psi\big(l\|y\|_b + \|\phi\|_{\mathfrak{B}_h} + lM\left|\phi(0)\right|\big),
\end{aligned}
$$

where F_1, F_2, F_3 are defined in (H_6).

So $\|y\|_b \leq F_1 + F_2\|y\|_b + F_3\psi(l\|y\|_b + \|\phi\|_{\mathfrak{B}_h} + lM|\phi(0)|)$, that is,

$$
\frac{\|y\|_b}{F_1 + F_2\|y\|_b + F_3\psi(l\|y\|_b + \|\phi\|_{\mathfrak{B}_h} + lM|\phi(0)|)} \leq 1.
$$

Then by (H_6) there exists r such that $\|y\|_b \neq r$. Hence, it follows from Theorem 2.2 that the operator Φ has a fixed point $y^* \in \mathfrak{B}_b^0$. Let $x(t) = y^*(t) + \bar{\phi}(t)$, $t \in (-\infty, b]$. Then x is a fixed point of the operator $\mathfrak{L}$ which is a mild solution of problem (1.1); then system (1.1) is controllable on J. □

4 Conclusion

In this paper, we have investigated the controllability of fractional impulsive neutral functional differential inclusions in Banach spaces. Based on a fixed point theorem, sufficient conditions for the controllability of the fractional impulsive neutral functional differential inclusions have been derived.

Competing interests
The author declares that he has no competing interests.

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