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Positive solutions for superlinear fractional boundary value problems

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Abstract

We establish the existence, uniqueness, and global behavior of a positive solution for the following superlinear fractional boundary value problem: $D^\alpha u(x) = u(x)\varphi(x, u(x))$, $x \in (0,1)$, $\lim_{x\to 0^+} D^{\alpha-1}u(x) = -a$, $u(1) = b$, where $1 < \alpha \le 2$, D^α is the standard Riemann-Liouville fractional derivative, a, b are nonnegative constants such that $a + b > 0$ and $\varphi(x,t)$ is a nonnegative continuous function in $(0,1) \times [0,\infty)$ that is required to satisfy some appropriate conditions related to a certain class of functions $\mathcal{K}_\alpha$. Our approach is based on estimates of the Green's function and on perturbation arguments.

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1 Introduction

Fractional differential equations are gaining much importance and attention since they can be applied in various fields of science and engineering. Many phenomena in viscoelasticity, electrochemistry, control, porous media, electromagnetic, *etc.*, can be modeled by fractional differential equations. They also serve as an excellent tool for the description of hereditary properties of various materials and processes. We refer the reader to [1–14] and references therein for details.

In [4], the authors considered the following fractional boundary value problem:

$$\begin{cases} -D^\alpha u(x) + q(x)u(x) = w(x)f(x,u(x)), & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$ and q is a continuous function on $[0,1]$.

Using spectral theory, they derived the Green's function for the following problem:

$$\begin{cases} -D^\alpha u(x) + q(x)u(x) = 0, & x \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1.2}$$

where the function q is required to satisfy the following growth condition: there exists $c > 0$ such that for each $x \in [0,1]$, we have

$$|q(x)| \le c < 4^{\alpha-1}\Gamma(\alpha), \tag{1.3}$$

where Γ is the Euler gamma function.

Exploiting this result, they proved the existence and uniqueness of a solution to problem (1.1), where the function w is required to be integrable on $[0,1]$ and f is continuous on $[0,1]\times\mathbb{R}$ and Lipschitz with respect to the second variable.

Motivated by the above mentioned work, we study in this paper the existence, uniqueness, and global behavior of a positive continuous solution for the following superlinear fractional boundary value problem:

$$\begin{cases} D^{\alpha}u(x)=u(x)\varphi(x,u(x)), \quad x\in(0,1),\\ \lim_{x\to 0^+}D^{\alpha-1}u(x)=-a, \qquad u(1)=b, \end{cases} \tag{1.4}$$

where $1<\alpha\le 2$, a, b are nonnegative constants such that $a+b>0$ and $\varphi(x,t)$ is a nonnegative continuous function in $(0,1)\times[0,\infty)$ that is required to satisfy some appropriate conditions related to the following class $\mathcal{K}_\alpha$.

Definition 1.1 Let $1<\alpha\le 2$. A Borel measurable function q in $(0,1)$ belongs to the class $\mathcal{K}_\alpha$ if q satisfies the following condition:

$$\int_0^1 r^{\alpha-2}(1-r)^{\alpha-1}\left|q(r)\right|dr<\infty. \tag{1.5}$$

More precisely, we will first prove that if q is a nonnegative sufficiently small function in $\mathcal{K}_\alpha\cap(C(0,1))$ and f is positive, then the following problem:

$$\begin{cases} -D^{\alpha}u(x)+q(x)u(x)=f(x),\\ \lim_{x\to 0^+}D^{\alpha-1}u(x)=0, \qquad u(1)=0, \end{cases} \tag{1.6}$$

has a positive solution. It turns out to prove that problem (1.6) admits a positive Green's function. Here the function q may be singular at $x=0$ and $x=1$ and therefore does not need to satisfy condition (1.3).

Based on the construction of this Green's function and by using perturbation arguments, we will answer the questions of existence, uniqueness and global behavior of a positive solution u in $C_{2-\alpha}([0,1])$ to problem (1.4), where $C_{2-\alpha}([0,1])$ is the set of all functions f such that $x\to x^{2-\alpha}f(x)$ is continuous on $[0,1]$.

Throughout this paper, we let

$$h_1(x)=\frac{1}{\Gamma(\alpha)}x^{\alpha-2}(1-x) \quad\text{and}\quad h_2(x)=x^{\alpha-2}, \quad x\in(0,1]. \tag{1.7}$$

Also we shall often refer to $\omega(x):=ah_1(x)+bh_2(x)$, the unique solution of the problem

$$\begin{cases} D^{\alpha}u(x)=0, \quad x\in(0,1),\\ \lim_{x\to 0^+}D^{\alpha-1}u(x)=-a, \qquad u(1)=b. \end{cases} \tag{1.8}$$

We denote by $G(x,t)$ the Green's function of the operator $u\to -D^{\alpha}u$, with boundary conditions $\lim_{x\to 0^+}D^{\alpha-1}u(x)=u(1)=0$, which can be explicitly given by

$$G(x,t)=\frac{1}{\Gamma(\alpha)}\left[x^{\alpha-2}(1-t)^{\alpha-1}-\left((x-t)^+\right)^{\alpha-1}\right], \tag{1.9}$$

where $x^+=\max(x,0)$.

The outline of the paper is as follows. In Section 2, we give some sharp estimates on the Green's function $G(x,t)$, including the following inequality: for each $x, r, t \in (0,1)$,

$$\frac{G(x,r)G(r,t)}{G(x,t)} \leq \frac{1}{(\alpha-1)\Gamma(\alpha)}\rho(r), \tag{1.10}$$

where $\rho(r) := r^{\alpha-2}(1-r)^{\alpha-1}$.

In particular, we deduce from this inequality that for each $q \in \mathcal{K}_\alpha$,

$$\alpha_q := \sup_{x,t\in(0,1)} \int_0^1 \frac{G(x,r)G(r,t)}{G(x,t)} \left|q(r)\right| dr < \infty. \tag{1.11}$$

In Section 3, our purpose is to study the superlinear fractional boundary value problem (1.4). To this end, as we have mentioned above, we will exploit the inequality (1.10) to prove that the inverse of fractional operators that are perturbed by a zero-order term, are positivity preserving. That is, if the function q is nonnegative and belongs to the class $\mathcal{K}_\alpha \cap (C(0,1))$ with $\alpha_q \leq \frac{1}{2}$, then problem (1.6) has a positive solution.

We require a combination of the following assumptions on the term φ.

(H_1) φ is a nonnegative continuous function in $(0,1) \times [0,\infty)$.
(H_2) There exists a nonnegative function $q \in \mathcal{K}_\alpha \cap C(0,1)$ with $\alpha_q \leq \frac{1}{2}$ such that for each $x \in (0,1)$, the map $t \to t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$.
(H_3) For each $x \in (0,1)$, the function $t \to t\varphi(x,t)$ is nondecreasing on $[0,\infty)$.

Our main results are the following.

Theorem 1.2 *Assume* (H_1)-(H_2), *then problem* (1.4) *has a positive solution u in* $C_{2-\alpha}([0,1])$ *satisfying*

$$c_0\omega(x) \leq u(x) \leq \omega(x), \tag{1.12}$$

where c_0 *is a constant in* $(0,1)$.

Moreover, if hypothesis (H_3) *is also satisfied, then the solution u to problem* (1.4) *satisfying* (1.12) *is unique*.

Corollary 1.3 *Let f be a nonnegative function in* $C^1([0,\infty))$ *such that the map* $t \to \theta(t) = tf(t)$ *is nondecreasing on* $[0,\infty)$. *Let p be a nonnegative continuous function on* $(0,1)$ *such that the function* $x \to p(x)\max_{0\leq\xi\leq\omega(x)} \theta'(\xi)$ *belongs to the class* $\mathcal{K}_\alpha$. *Then for sufficiently small positive constant* λ, *the following problem*:

$$\begin{cases} D^\alpha u(x) = \lambda p(x)u(x)f(u(x)), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = -a, & u(1) = b, \end{cases} \tag{1.13}$$

has a unique positive solution u in $C_{2-\alpha}([0,1])$ *satisfying*

$$c_0\omega(x) \leq u(x) \leq \omega(x),$$

where c_0 *is a constant in* $(0,1)$.

Observe that in Theorem 1.2 we obtain a positive u in $C_{2-\alpha}([0,1])$ to problem (1.4) which its behavior is not affected by the perturbed term. That is, it behaves like the solution ω of the homogeneous problem (1.8). Also note that for $1 < \alpha < 2$, the solution u blows up at $x = 0$.

As typical example of nonlinearity satisfying (H_1)-(H_3), we quote

$\varphi(x,t) = \lambda p(x)t^\sigma$, for $\sigma \geq 0$, λ is a positive constant sufficiently small and p is a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-2)(1+\sigma)}(1-r)^{\alpha-1}p(r)\,dr < \infty.$$

In order to simplify our statements, we introduce some convenient notations.

- $\mathcal{B}((0,1))$ denotes the set of Borel measurable functions in $(0,1)$ and $\mathcal{B}^+((0,1))$ the set of nonnegative ones.
- For $f \in \mathcal{B}^+((0,1))$ and $x \in (0,1]$, we put

$$Vf(x) := \int_0^1 G(x,t)f(t)\,dt.$$

- For $q \in \mathcal{B}^+((0,1))$, the kernel $V(q\cdot)$ is defined on $\mathcal{B}^+((0,1))$ by

$$V(q\cdot)(f) = V(qf).$$

2 Fractional calculus and estimates on the Green's function

For the convenience of the reader, we recall in this section some basic definitions on fractional calculus (see [7, 10, 12]) and we give some properties of the Green's function $G(x,t)$.

Definition 2.1 The Riemann-Liouville fractional integral of order $\beta > 0$ of a function $h : (0,1] \to \mathbb{R}$ is given by

$$I^\beta h(x) = \frac{1}{\Gamma(\beta)}\int_0^x (x-t)^{\beta-1}h(t)\,dt,$$

provided that the right-hand side is pointwise defined on $(0,1]$.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\beta > 0$ of a function $h : (0,1] \to \mathbb{R}$ is given by

$$D^\beta h(x) = \frac{1}{\Gamma(n-\beta)}\left(\frac{d}{dx}\right)^n \int_0^x (x-t)^{n-\beta-1}h(t)\,dt,$$

provided that the right-hand side is pointwise defined on $(0,1]$.

Here $n = [\beta] + 1$ and $[\beta]$ means the integer part of the number β.

So we have the following properties (see [7, 10, 12]).

Proposition 2.3

(i) *Let $\beta > 0$ and let $v \in L^1(0,1)$, then we have*

$$D^\beta I^\beta v(x) = v(x), \quad \textit{for a.e. } x \in [0,1].$$

(ii) *Let* $\beta > 0$, *then*

$$D^\beta v(x) = 0 \quad \textit{if and only if} \quad v(x) = \sum_{j=1}^{m} c_j x^{\beta-j},$$

where m *is the smallest integer greater than or equal to* β *and* $(c_1, c_2, \dots, c_m) \in \mathbb{R}^m$.

Next we give sharp estimates on the Green's function $G(x,t)$. To this end, we need the following lemma.

Lemma 2.4 *For* $\lambda, \mu \in (0,\infty)$, *and* $a, t \in [0,1]$, *we have*

$$\min\left(1, \frac{\mu}{\lambda}\right)\left(1 - at^\lambda\right) \le 1 - at^\mu \le \max\left(1, \frac{\mu}{\lambda}\right)\left(1 - at^\lambda\right).$$

Proposition 2.5 (see [9]) *For* $(x,t) \in (0,1] \times [0,1]$, *we have*

$$\begin{aligned} &\frac{\alpha-1}{\Gamma(\alpha)} x^{\alpha-2}(1-t)^{\alpha-2}\big(1 - \max(x,t)\big) \\ &\quad \le G(x,t) \le \frac{1}{\Gamma(\alpha)} x^{\alpha-2}(1-t)^{\alpha-2}\big(1 - \max(x,t)\big). \end{aligned} \tag{2.1}$$

In particular

$$\frac{\alpha-1}{\Gamma(\alpha)} x^{\alpha-2}(1-x)(1-t)^{\alpha-1} \le G(x,t) \le \frac{1}{\Gamma(\alpha)} x^{\alpha-2}(1-t)^{\alpha-2}\min(1-t, 1-x). \tag{2.2}$$

Proof From the explicit expression of the Green's function (1.9), we have for $x, t \in (0,1)$

$$G(x,t) = \frac{x^{\alpha-2}(1-t)^{\alpha-1}}{\Gamma(\alpha)}\left[1 - x\left(\frac{(x-t)^+}{x(1-t)}\right)^{\alpha-1}\right].$$

Since $\frac{(x-t)^+}{x(1-t)} \in (0,1]$ for $t \in [0,1)$, the required result follows from Lemma 2.4 with $\mu = \alpha - 1$ and $\lambda = 1$.

Inequality (2.2) follows from the fact that for $(x,t) \in (0,1] \times [0,1]$,

$$(1-x)(1-t) \le \big(1 - \max(x,t)\big) = \min(1-t, 1-x). \qquad \square$$

Using (2.2), we deduce the following.

Corollary 2.6 *Let* $f \in \mathcal{B}^+((0,1))$, *then the function* $x \to Vf(x)$ *is in* $C_{2-\alpha}([0,1])$ *if and only if the integral* $\int_0^1 (1-t)^{\alpha-1} f(t)\,dt$ *converges.*

Remark 2.7 (see [9]) Let $1 < \alpha \le 2$ and $f \in \mathcal{B}^+((0,1))$ such that the function $t \to (1-t)^{\alpha-1}f(t)$ is continuous and integrable on $(0,1)$, then Vf is the unique solution in $C_{2-\alpha}([0,1])$ of

$$\begin{cases} D^\alpha u(x) = -f(x), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = 0, & u(1) = 0. \end{cases}$$

Proposition 2.8 *For each $x, r, t \in (0,1)$, we have*

$$\frac{G(x,r)G(r,t)}{G(x,t)} \le \frac{1}{(\alpha-1)\Gamma(\alpha)}\rho(r), \tag{2.3}$$

where $\rho(r) = r^{\alpha-2}(1-r)^{\alpha-1}$.

Proof Using (2.1), for each $x, r, t \in (0,1)$, we have

$$\frac{G(x,r)G(r,t)}{G(x,t)} \le \frac{r^{\alpha-2}(1-r)^{\alpha-2}}{(\alpha-1)\Gamma(\alpha)} \frac{(1-\max(x,r))(1-\max(r,t))}{(1-\max(x,t))}. \tag{2.4}$$

We claim that

$$\frac{(1-\max(x,r))(1-\max(r,t))}{(1-\max(x,t))} \le 1-r. \tag{2.5}$$

Indeed, by symmetry, we may assume that $x \le t$. So we have the following three cases.

Case 1: $r \le x \le t$.

In this case, we have

$$\frac{(1-x)(1-t)}{(1-t)} = 1-x \le 1-r.$$

Case 2: $x \le r \le t$.

We obtain

$$\frac{(1-r)(1-t)}{(1-t)} = 1-r.$$

Case 3: $x \le t \le r$.

We have

$$\frac{(1-r)(1-r)}{(1-t)} \le 1-r.$$

This proves (2.5) and by using (2.4) we obtain the required result. □

In the sequel, for any $q \in \mathcal{B}((0,1))$, we recall that

$$\alpha_q := \sup_{x,t\in(0,1)} \int_0^1 \frac{G(x,r)G(r,t)}{G(x,t)} |q(r)|\, dr$$

and

$$h_1(x) = \frac{1}{\Gamma(\alpha)} x^{\alpha-2}(1-x) \quad \text{and} \quad h_2(x) = x^{\alpha-2}, \quad x \in (0,1]. \tag{2.6}$$

Proposition 2.9 *Let q be a function in $\mathcal{K}_\alpha$, then*:

(i)

$$\alpha_q \le \frac{1}{(\alpha-1)\Gamma(\alpha)} \int_0^1 \rho(r)|q(r)|\, dr < \infty. \tag{2.7}$$

(ii) *For* $x \in (0,1]$, *we have*

$$\int_0^1 G(x,t)h_1(t)\left|q(t)\right| dt \le \alpha_q h_1(x). \tag{2.8}$$

(iii) *For* $x \in (0,1]$, *we have*

$$\int_0^1 G(x,t)h_2(t)\left|q(t)\right| dt \le \alpha_q h_2(x). \tag{2.9}$$

In particular for $x \in (0,1]$, *we have*

$$\int_0^1 G(x,t)\omega(t)\left|q(t)\right| dt \le \alpha_q \omega(x). \tag{2.10}$$

Proof Let q be a function in $\mathcal{K}_\alpha$.

(i) The inequality (2.7) follows from (2.3).

(ii) Since for each $x,t \in (0,1)$, we have $\lim_{r\to 0} \frac{G(t,r)}{G(x,r)} = \frac{h_1(t)}{h_1(x)}$, then we deduce by Fatou's lemma and (1.11) that

$$\int_0^1 G(x,t)\frac{h_1(t)}{h_1(x)}\left|q(t)\right| dt \le \liminf_{r\to 0} \int_0^1 G(x,t)\frac{G(t,r)}{G(x,r)}\left|q(t)\right| dt \le \alpha_q,$$

which implies that for $x \in (0,1]$,

$$\int_0^1 G(x,t)h_1(t)\left|q(t)\right| dt \le \alpha_q h_1(x).$$

(iii) Similarly, we prove inequality (2.9) by observing that $\lim_{r\to 1} \frac{G(t,r)}{G(x,r)} = \frac{h_2(t)}{h_2(x)}$.

Inequality (2.10) follows from (2.8), (2.9), and the fact that $\omega(x) = ah_1(x) + bh_2(x)$.

This completes the proof. □

3 Proofs of main results

In this section, we aim at proving Theorem 1.2 and Corollary 1.3. To this end, we need the following preliminary results.

For a nonnegative function q in $\mathcal{K}_\alpha$ such that $\alpha_q < 1$, we define the function $\mathcal{G}(x,t)$ on $(0,1] \times [0,1]$, by

$$\mathcal{G}(x,t) = \sum_{n=0}^{\infty} (-1)^n G_n(x,t), \tag{3.1}$$

where $G_0(x,t) = G(x,t)$ and

$$G_n(x,t) = \int_0^1 G(x,r)G_{n-1}(r,t)q(r)\, dr, \quad n \ge 1. \tag{3.2}$$

Next, we establish some inequalities on $G_n(x,t)$. In particular, we deduce that $\mathcal{G}(x,t)$ is well defined.

Lemma 3.1 *Let* q *be a nonnegative function in* $\mathcal{K}_\alpha$ *such that* $\alpha_q < 1$, *then for each* $n \ge 0$ *and* $(x,t) \in (0,1] \times [0,1]$, *we have*:

(i) $G_n(x,t) \le \alpha_q^n G(x,t)$.
In particular, $\mathcal{G}(x,t)$ *is well defined in* $(0,1] \times [0,1]$.

(ii)

$$L_n x^{\alpha-2}(1-x)(1-t)^{\alpha-1} \le G_n(x,t) \le R_n x^{\alpha-2}(1-t)^{\alpha-2}\min(1-t,1-x), \tag{3.3}$$

where $L_n = \frac{(\alpha-1)^{n+1}}{(\Gamma(\alpha))^{n+1}}(\int_0^1 r^{\alpha-2}(1-r)^{\alpha}q(r)\,dr)^n$ *and*
$R_n = \frac{1}{(\Gamma(\alpha))^{n+1}}(\int_0^1 r^{\alpha-2}(1-r)^{\alpha-1}q(r)\,dr)^n$.

(iii) $G_{n+1}(x,t) = \int_0^1 G_n(x,r)G(r,t)q(r)\,dr$.

(iv) $\int_0^1 \mathcal{G}(x,r)G(r,t)q(r)\,dr = \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)\,dr$.

Proof (i) The assertion is clear for $n = 0$.

Assume that inequality in (i) holds for some $n \ge 0$, then by using (3.2) and (1.11), we obtain

$$G_{n+1}(x,t) \le \alpha_q^n \int_0^1 G(x,r)G(r,t)q(r)\,dr \le \alpha_q^{n+1}G(x,t).$$

Now, since $G_n(x,t) \le \alpha_q^n G(x,t)$, it follows that $\mathcal{G}(x,t)$ is well defined in $(0,1] \times [0,1]$.

(ii) Using (2.2) and (3.2), we obtain (3.3) by simple induction.

(iii) The equality is clear for $n = 0$.

Assume that for a given integer $n \ge 1$ and $(x,t) \in (0,1] \times [0,1]$, we have

$$G_n(x,t) = \int_0^1 G_{n-1}(x,r)G(r,t)q(r)\,dr. \tag{3.4}$$

Using (3.2) and Fubini-Tonelli's theorem, we obtain

$$\begin{aligned} G_{n+1}(x,t) &= \int_0^1 G(x,r)\left(\int_0^1 G_{n-1}(r,\xi)G(\xi,t)q(\xi)\,d\xi\right)q(r)\,dr \\ &= \int_0^1 \left(\int_0^1 G(x,r)G_{n-1}(r,\xi)q(r)\,dr\right)G(\xi,t)q(\xi)\,d\xi \\ &= \int_0^1 G_n(x,\xi)G(\xi,t)q(\xi)\,d\xi. \end{aligned}$$

(iv) Let $n \ge 0$ and $x,r,t \in (0,1]$. By Lemma 3.1(i) and (2.2), we have

$$0 \le G_n(x,r)G(r,t)q(r) \le \alpha_q^n G(x,r)G(r,t)q(r).$$

Hence the series $\sum_{n\ge 0}\int_0^1 G_n(x,r)G(r,t)q(r)\,dr$ converges.

So we deduce by the dominated convergence theorem and Lemma 3.1(iii) that

$$\begin{aligned} \int_0^1 \mathcal{G}(x,r)G(r,t)q(r)\,dr &= \sum_{n=0}^{\infty}\int_0^1 (-1)^n G_n(x,r)G(r,t)q(r)\,dr \\ &= \sum_{n=0}^{\infty}\int_0^1 (-1)^n G(x,r)G_n(r,t)q(r)\,dr \\ &= \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)\,dr. \end{aligned}$$

□

Proposition 3.2 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q < 1$. Then the function $(x,t) \to x^{2-\alpha}\mathcal{G}(x,t)$ is continuous on $[0,1] \times [0,1]$.*

Proof First, we claim that for $n \geq 0$, the function $(x,t) \to x^{2-\alpha}G_n(x,t)$ is continuous on $[0,1] \times [0,1]$.

The assertion is clear for $n = 0$.

Assume that for a given integer $n \geq 1$, the function $(x,t) \to x^{2-\alpha}G_{n-1}(x,t)$ is continuous on $[0,1] \times [0,1]$. By (3.2), we have

$$x^{2-\alpha}G_n(x,t) = \int_0^1 x^{2-\alpha}G(x,r)G_{n-1}(r,t)q(r)\,dr. \tag{3.5}$$

Note that for each $r \in (0,1)$, the function $(x,t) \to x^{2-\alpha}G(x,r)G_{n-1}(r,t)$ is continuous on $[0,1] \times [0,1]$.

On the other hand, by Lemma 3.1(i) and (2.2), we have for each $(x,t,r) \in [0,1] \times [0,1] \times (0,1)$,

$$\begin{aligned} x^{2-\alpha}G(x,r)G_{n-1}(r,t)q(r) &\leq \alpha_q^{n-1}x^{2-\alpha}G(x,r)G(r,t)q(r) \\ &\leq \frac{1}{(\Gamma(\alpha))^2}r^{\alpha-2}(1-r)^{\alpha-1}q(r). \end{aligned}$$

So we deduce by (3.5) and the dominated convergence theorem that the function $(x,t) \to x^{2-\alpha}G_n(x,t)$ is continuous on $[0,1] \times [0,1]$. This proves our claim.

Now by using again Lemma 3.1(i) and (2.2), we have for each $x,t \in [0,1]$,

$$x^{2-\alpha}G_n(x,t) \leq \alpha_q^n x^{2-\alpha}G(x,t) \leq \frac{1}{\Gamma(\alpha)}\alpha_q^n.$$

This implies that the series $\sum_{n\geq 0}(-1)^n x^{2-\alpha}G_n(x,t)$ is uniformly convergent on $[0,1] \times [0,1]$ and therefore the function $(x,t) \to x^{2-\alpha}\mathcal{G}(x,t)$ is continuous on $[0,1] \times [0,1]$. The proof is completed. □

Lemma 3.3 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$. Then for $(x,t) \in (0,1] \times [0,1]$, we have*

$$(1-\alpha_q)G(x,t) \leq \mathcal{G}(x,t) \leq G(x,t). \tag{3.6}$$

Proof Since $\alpha_q \leq \frac{1}{2}$, we deduce from Lemma 3.1(i) that

$$\left|\mathcal{G}(x,t)\right| \leq \sum_{n=0}^{\infty}(\alpha_q)^n G(x,t) = \frac{1}{1-\alpha_q}G(x,t). \tag{3.7}$$

On the other hand, from the expression of $\mathcal{G}$, we have

$$\mathcal{G}(x,t) = G(x,t) - \sum_{n=0}^{\infty}(-1)^n G_{n+1}(x,t). \tag{3.8}$$

Since the series $\sum_{n\geq 0}\int_0^1 G(x,r)G_n(r,t)q(r)\,dr$ is convergent, we deduce by (3.8) and (3.2) that

$$\begin{aligned}\mathcal{G}(x,t) &= G(x,t) - \sum_{n=0}^{\infty}(-1)^n \int_0^1 G(x,r)G_n(r,t)q(r)\,dr \\ &= G(x,t) - \int_0^1 G(x,r)\left(\sum_{n=0}^{\infty}(-1)^n G_n(r,t)\right)q(r)\,dr.\end{aligned}$$

That is

$$\mathcal{G}(x,t) = G(x,t) - V\big(q\mathcal{G}(\cdot,t)\big)(x). \tag{3.9}$$

Now from (3.7) and Lemma 3.1(i) (with $n = 1$), we obtain

$$\begin{aligned}V\big(q\mathcal{G}(\cdot,t)\big)(x) &\leq \frac{1}{1-\alpha_q}V\big(qG(\cdot,t)\big)(x) \\ &= \frac{1}{1-\alpha_q}G_1(x,t) \leq \frac{\alpha_q}{1-\alpha_q}G(x,t).\end{aligned} \tag{3.10}$$

This implies by (3.9) that

$$\mathcal{G}(x,t) \geq G(x,t) - \frac{\alpha_q}{1-\alpha_q}G(x,t) = \frac{1-2\alpha_q}{1-\alpha_q}G(x,t) \geq 0.$$

So it follows that $\mathcal{G}(x,t) \leq G(x,t)$ and by (3.9) and Lemma 3.1 (i) (with $n = 1$), we have

$$\mathcal{G}(x,t) \geq G(x,t) - V\big(qG(\cdot,t)\big)(x) \geq (1-\alpha_q)G(x,t). \qquad \square$$

In the sequel, for a given nonnegative function $q \in \mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$, we define the operator V_q on $\mathcal{B}^+((0,1))$ by

$$V_q f(x) = \int_0^1 \mathcal{G}(x,t)f(t)\,dt, \quad x \in (0,1].$$

Using Proposition 3.2, (3.6), and (2.2), we obtain the following.

Corollary 3.4 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$, then the following statements are equivalent*:

(i) *The function $x \to V_q f(x)$ is in $C_{2-\alpha}([0,1])$.*
(ii) *The integral $\int_0^1 (1-t)^{\alpha-1}f(t)\,dt$ converges.*

Next, we will prove that the kernel V_q satisfies the following resolvent equation.

Lemma 3.5 *Let q be a nonnegative function in $\mathcal{K}_\alpha$ such that $\alpha_q \leq \frac{1}{2}$ and $f \in \mathcal{B}^+((0,1))$, then $V_q f$ satisfies the following resolvent equation*:

$$Vf = V_q f + V_q(qVf) = V_q f + V(qV_q f). \tag{3.11}$$

In particular, if $V(qf) < \infty$, *we have*

$$\big(I - V_q(q\cdot)\big)\big(I + V(q\cdot)\big)f = \big(I + V(q\cdot)\big)\big(I - V_q(q\cdot)\big)f = f. \tag{3.12}$$

Proof Let $(x,t) \in (0,1] \times [0,1]$, then by (3.9), we have

$$G(x,t) = \mathcal{G}(x,t) + V\big(q\mathcal{G}(\cdot,t)\big)(x),$$

which implies by the Fubini-Tonelli theorem that for $f \in \mathcal{B}^+((0,1))$,

$$\begin{aligned} Vf(x) &= \int_0^1 \big(\mathcal{G}(x,t) + V\big(q\mathcal{G}(\cdot,t)\big)(x)\big)f(t)\,dt \\ &= V_q f(x) + V(qV_q f)(x). \end{aligned}$$

On the other hand, by Lemma 3.1(iii) and the Fubini-Tonelli theorem, we obtain for $f \in \mathcal{B}^+((0,1))$ and $x \in (0,1]$

$$\int_0^1 \int_0^1 \mathcal{G}(x,r)G(r,t)q(r)f(t)\,dr\,dt = \int_0^1 \int_0^1 G(x,r)\mathcal{G}(r,t)q(r)f(t)\,dr\,dt,$$

that is,

$$V_q(qVf)(x) = V(qV_q f)(x).$$

So we obtain

$$Vf = V_q f + V(qV_q f) = V_q f + V_q(qVf)(x).$$

This completes the proof. □

Proposition 3.6 *Let* q *be a nonnegative function in* $\mathcal{K}_\alpha \cap C(0,1)$ *such that* $\alpha_q \leq \frac{1}{2}$ *and* $f \in \mathcal{B}^+((0,1))$ *such that* $t \to (1-t)^{\alpha-1}f(t)$ *is continuous and integrable on* $(0,1)$. *Then* $V_q f$ *is the unique nonnegative solution in* $C_{2-\alpha}([0,1])$ *of the perturbed fractional problem* (1.6) *satisfying*

$$(1-\alpha_q)Vf \leq V_q f \leq Vf. \tag{3.13}$$

Proof Since by Corollary 3.4 the function $x \to V_q f(x)$ is in $C_{2-\alpha}([0,1])$, it follows that the function $x \to q(x)V_q f(x)$ is continuous on $(0,1)$.

Using (3.11) and (2.2), there exists a nonnegative constant c such that

$$V_q f(x) \leq Vf(x) \leq \frac{1}{\Gamma(\alpha)} \int_0^1 x^{\alpha-2}(1-t)^{\alpha-1}f(t)\,dt \leq cx^{\alpha-2} = ch_2(x). \tag{3.14}$$

So we deduce that

$$\int_0^1 (1-t)^{\alpha-1}q(t)V_q f(t)\,dt \leq c\int_0^1 t^{\alpha-2}(1-t)^{\alpha-1}q(t)\,dt < \infty.$$

Hence by using Remark 2.7, the function $u = V_q f = Vf - V(qV_q f)$ satisfies the equation

$$\begin{cases} D^\alpha u(x) = -f(x) + q(x)u(x), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = 0, & u(1) = 0. \end{cases}$$

and by the integration inequalities (3.6), we obtain (3.13).

It remains to prove the uniqueness. Assume that v is another nonnegative solution in $C_{2-\alpha}([0,1])$ of problem (1.6) satisfying (3.13).

Since the function $t \to q(t)v(t)$ is continuous on $(0,1)$ and by (3.13), (3.14), the function $t \to (1-t)^{\alpha-1}q(t)v(t)$ is integrable on $(0,1)$, it follows by Remark 2.7 that the function $\tilde{v} := v + V(qv)$ satisfies

$$\begin{cases} D^\alpha \tilde{v}(x) = -f(x), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}\tilde{v}(x) = 0, & \tilde{v}(1) = 0. \end{cases}$$

From the uniqueness in Remark 2.7, we deduce that

$$\tilde{v} := v + V(qv) = Vf.$$

Hence

$$\bigl(I + V(q\cdot)\bigr)(v - u) = 0.$$

Now since by (3.13), (3.14), and (2.9), we have

$$V\bigl(q|v-u|\bigr) \le 2cV(qh_2) \le 2c\alpha_q h_2 < \infty,$$

then by (3.12), we deduce that $u = v$. This completes the proof. □

Proof of Theorem 1.2 Let $a \ge 0$ and $b \ge 0$ with $a + b > 0$ and recall that

$$\omega(x) = \frac{a}{\Gamma(\alpha)}x^{\alpha-2}(1-x) + bx^{\alpha-2} = ah_1(x) + bh_2(x).$$

Since φ satisfies (H_2), there exists a positive function q in $\mathcal{K}_\alpha \cap C(0,1)$ such that $\alpha_q \le \frac{1}{2}$ and for each $x \in (0,1)$, the map $t \to t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$.

Let

$$\Lambda := \bigl\{u \in \mathcal{B}^+\bigl((0,1)\bigr) : (1-\alpha_q)\omega \le u \le \omega\bigr\}$$

and define the operator T on Λ by

$$Tu = \omega - V_q(q\omega) + V_q\bigl(\bigl(q - \varphi(\cdot, u)\bigr)u\bigr).$$

By (3.11) and (2.10), we have

$$V_q(q\omega) \le V(q\omega) \le \alpha_q\omega \le \omega \tag{3.15}$$

and by (H_2), we obtain

$$0 \le \varphi(\cdot,u) \le q, \quad \text{for all } u \in \Lambda. \tag{3.16}$$

So we claim that Λ is invariant under T. Indeed, using (3.16) and (3.15), we have for $u \in \Lambda$

$$Tu \le \omega - V_q(q\omega) + V_q(qu) \le \omega$$

and

$$\begin{aligned} Tu &\ge \omega - V_q(q\omega) \\ &\ge (1-\alpha_q)\omega. \end{aligned}$$

Next, we will prove that the operator T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ be such that $u \le v$. Since the map $t \to t(q(x) - \varphi(x, t\omega(x)))$ is nondecreasing on $[0,1]$, for $x \in (0,1)$, we obtain

$$Tv - Tu = V_q\big(\big[v\big(q - \varphi(\cdot,v)\big) - u\big(q - \varphi(\cdot,u)\big)\big]\big) \ge 0.$$

Now, we consider the sequence $\{u_n\}$ defined by $u_0 = (1-\alpha_q)\omega$ and $u_{n+1} = Tu_n$, for $n \in \mathbb{N}$. Since Λ is invariant under T, we have $u_1 = Tu_0 \ge u_0$ and by the monotonicity of T, we deduce that

$$(1-\alpha_q)\omega = u_0 \le u_1 \le \cdots \le u_n \le u_{n+1} \le \omega.$$

Hence by the dominated convergence theorem and (H_1)-(H_2), we conclude that the sequence $\{u_n\}$ converges to a function $u \in \Lambda$ satisfying

$$u = \big(I - V_q(q\cdot)\big)\omega + V_q\big(\big(q - \varphi(\cdot,u)\big)u\big).$$

That is,

$$\big(I - V_q(q\cdot)\big)u = \big(I - V_q(q\cdot)\big)\omega - V_q\big(u\varphi(\cdot,u)\big).$$

On the other hand, since by (3.15), we have $V(qu) \le V(q\omega) \le \omega < \infty$, then by applying the operator $(I + V(q\cdot))$ on both sides of the above equality and using (3.11) and (3.12), we conclude that u satisfies

$$u = \omega - V\big(u\varphi(\cdot,u)\big). \tag{3.17}$$

It remains to prove that u is the required solution.

To this end, we remark by (3.16) that

$$u(t)\varphi\big(t,u(t)\big) \le q(t)\omega(t) \le \max\left(\frac{a}{\Gamma(\alpha)}, b\right) t^{\alpha-2} q(t). \tag{3.18}$$

This implies by Corollary 2.6 that the function $x \to V(u\varphi(\cdot,u))(x)$ is in $C_{2-\alpha}([0,1])$ and so by (3.17), u is in $C_{2-\alpha}([0,1])$.

Now, since by (H_1) and (3.18), the function $t \to (1-t)^{\alpha-1}u(t)\varphi(t,u(t))$ is continuous and integrable on $(0,1)$, we conclude by Remark 2.7 that u is the required solution.

It remains to prove that under condition (H_3), u is the unique solution to problem (1.4) satisfying (1.12). Assume that v is another nonnegative solution in $C_{2-\alpha}([0,1])$ to problem (1.4) satisfying (1.12). Since $v \le \omega$, we deduce by (3.18) that

$$0 \le v(t)\varphi\big(t,v(t)\big) \le q(t)\omega(t) \le \max\left(\frac{a}{\Gamma(\alpha)},b\right)t^{\alpha-2}q(t).$$

So the function $t \to (1-t)^{\alpha-1}v(t)\varphi(t,v(t))$ is continuous and integrable on $(0,1)$ and by Remark 2.7, we conclude that the function $\tilde{v} := v + V(v\varphi(\cdot,v))$ satisfies

$$\begin{cases} D^\alpha\tilde{v}(x) = 0, \quad x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}\tilde{v}(x) = -a, \qquad \tilde{v}(1) = b. \end{cases}$$

From the uniqueness in problem (1.8), we deduce that

$$v = \omega - V\big(v\varphi(\cdot,v)\big). \tag{3.19}$$

Now let h be the function defined on $(0,1)$ by

$$h(x) = \begin{cases} \frac{v(x)\varphi(x,v(x))-u(x)\varphi(x,u(x))}{v(x)-u(x)} & \text{if } v(x) \ne u(x), \\ 0 & \text{if } v(x) = u(x). \end{cases}$$

Then by (H_3), $h \in \mathcal{B}^+((0,1))$ and by (3.17) and (3.19), we have

$$\big(I + V(h\cdot)\big)(v-u) = 0.$$

On the other hand, by (H_2), we remark that $h \le q$ and by (2.9) we deduce that

$$V\big(h|v-u|\big) \le 2V(qh_2) \le 2\alpha_q h_2 < \infty.$$

Hence by (3.12), we conclude that $u = v$. This completes the proof. □

Proof of Corollary 1.3 Let $\varphi(x,t) = \lambda p(x)f(t)$ and $\theta(t) = tf(t)$. It is clear that hypotheses (H_1) and (H_3) are satisfied. Since the function $q(x) := \lambda p(x)\max_{0\le\xi\le\omega(x)}\theta'(\xi)$ belongs to the class $\mathcal{K}_\alpha$, we have $\alpha_q \le \frac{1}{2}$ for λ sufficiently small. Moreover, by a simple computation, we obtain $\frac{d}{dt}[t(q(x)-\varphi(x,t\omega(x)))] = q(x) - \lambda p(x)\theta'(t\omega(x)) \ge 0$ for $t \in [0,1]$ and $x \in (0,1)$. This implies that the function φ satisfies hypothesis (H_2). So the result follows by Theorem 1.2. □

Example 3.7 Let $1 < \alpha \le 2$ and $a \ge 0$, $b \ge 0$ with $a + b > 0$. Let $\sigma \ge 0$, and p be a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-2)(1+\sigma)}(1-r)^{\alpha-1}p(r)\,dr < \infty.$$

Then for sufficiently small positive constant λ, the following problem:

$$\begin{cases} D^\alpha u(x) = \lambda p(x) u^{\sigma+1}(x), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = -a, & u(1) = b, \end{cases}$$

has a unique positive solution u in $C_{2-\alpha}([0,1])$ satisfying

$$c_0\omega(x) \le u(x) \le \omega(x).$$

Example 3.8 Let $1 < \alpha \le 2$ and $a \ge 0$, $b \ge 0$ with $a + b > 0$. Let $\sigma \ge 0$, $\gamma > 0$ and p be a positive continuous function on $(0,1)$ such that

$$\int_0^1 r^{(\alpha-2)(1+\sigma+\gamma)}(1-r)^{\alpha-1}p(r)\,dr < \infty.$$

Then for sufficiently small positive constant λ, the following problem:

$$\begin{cases} D^\alpha u(x) = \lambda p(x) u^{\sigma+1}(x)\log(1+u^\gamma(x)), & x \in (0,1), \\ \lim_{x\to 0^+} D^{\alpha-1}u(x) = -a, & u(1) = b, \end{cases}$$

has a unique positive solution u in $C_{2-\alpha}([0,1])$ satisfying

$$c_0\omega(x) \le u(x) \le \omega(x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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