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Center-focus and Hopf bifurcation for a class of quartic Kukles-like systems

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Abstract
In the present article, we solve the center-focus problem for a class of quartic Kukles-like systems with third-order nilpotent singularities and prove the existence of five limit cycles in the neighborhood of the origin.
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1 Introduction

One of the most classical problems in the qualitative theory of planar analytic differential systems is to characterize the local phase portrait at an isolated singular point. This problem has been solved except if the singular point is a center or a focus. The problem in distinguishing between a center and a focus is called the *center-focus problem*. Once we have made a distinction between a center and a focus, another problem is to find the number of limit cycles bifurcated from the focus.

If a real analytic system has a nilpotent center at the origin, then after a linear change of variables and a rescaling of time variable, it can be written in the following form:

$$\begin{aligned} \dot{x} &= y + X(x,y), \\ \dot{y} &= Y(x,y), \end{aligned} \tag{1.1}$$

where $X(x,y)$ and $Y(x,y)$ are real analytic functions without constant and linear terms, defined in a neighborhood of the origin. Takens [1] proved that the Lyapunov system can be formally transformed into a generalized Liénard system. Moussu [2] found the $\mathcal{C}^\infty$ normal form for analytic nilpotent centers. Stróżyna and Żołądek [3] studied orbital normal forms for analytic planar vector fields with nilpotent singularity. Álvarez and Gasull [4] proved that the generalized Liénard system could be simplified even more by a reparameterization of the time. At the same time, Giacomini *et al.* [5, 6] showed that the analytic nilpotent systems with a center can be expressed as limit of non-degenerate systems with a center. The conditions of center and isochronous center at the origin for a class of non-analytic quintic systems are studied in [7].

Remember that the so-called stability problem and the center problem can be solved for smooth non-degenerate critical points via the Lyapunov constants. On the other hand the same problem but for nilpotent singular point is far to be solved in general, due to the invalidation of classical methods. In 2011, Liu and Li [8] dealt with the integral factor method for solving the above mentioned problems of third-order nilpotent singularities. This method is based into a different way of computing the so-called *quasi-Lyapunov constants*. Using the integral factor method, [9] investigated center conditions and bifurcation of limit cycles at the nilpotent critical point in a class of septic polynomial differential systems.

In this work, employing the integral factor method, we study the center-focus discrimination and Hopf bifurcation defined in a neighborhood of a third-order nilpotent singular point in a class of quartic Kukles-like systems with the form

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{1.2}$$

The rest of paper is organized as follows. In Section 2, we give some preliminary knowledge presented in [8], which is helpful throughout the paper. In Section 3, we compute the first several quasi-Lyapunov constants at the third-order nilpotent singular point of system (1.2) and provide sufficient and necessary conditions in order that system (1.2) have a center in a neighborhood of the origin. We end this paper in Section 4 by applying Theorem 2.1 to generate limit cycles for system (1.2).

2 Some preliminary results

Before state our results we need to introduce some well-known definitions, lemmas, and theorems.

In canonical coordinates the Lyapunov system with the origin as a nilpotent critical point can be written in the form

$$\begin{aligned} \frac{dx}{dt} &= y + \sum_{i+j=2}^{\infty} a_{ij}x^iy^j = X(x,y), \\ \frac{dy}{dt} &= \sum_{i+j=2}^{\infty} b_{ij}x^iy^j = Y(x,y). \end{aligned} \tag{2.1}$$

Suppose that the function $y = y(x)$ satisfies $X(x, y) = 0$, $y(0) = 0$. Lyapunov proved (see for instance [10]) that the origin of system (2.1) is a monodromic critical point (*i.e.*, a center or a focus) if and only if

$$\begin{aligned} &Y\big(x, y(x)\big) = \alpha x^{2n+1} + o\big(x^{2n+1}\big), \quad \alpha < 0, \\ &\left[\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial x}\right]_{y=y(x)} = \beta x^n + o\big(x^n\big), \\ &\beta^2 + 4(n+1)\alpha < 0, \end{aligned} \tag{2.2}$$

where n is a positive integer.

Definition 2.1 Let $y = f(x) = -a_{20}x^2 + o(x^2)$ be the unique solution of the function equation $X(x, f(x)) = 0$, $f(0) = 0$ at a neighborhood of the origin. If there are an integer m and a non-zero real number α, such that

$$Y\bigl(x, f(x)\bigr) = \alpha x^m + o\bigl(x^m\bigr), \tag{2.3}$$

we say that the origin is a high-order singular point of system (2.1) with the multiplicity m.

By using the results in [10], we attain the following conclusion.

Lemma 2.1 *The origin of system* (2.1) *is a three-order singular point which is a saddle point or a center, if and only if* $b_{20} = 0$, $(2a_{20} - b_{11})^2 + 8b_{30} < 0$.

When the condition in Lemma 2.1 holds, we can assume that

$$a_{20} = \mu, \qquad b_{20} = 0, \qquad b_{11} = 2\mu, \qquad b_{30} = -2. \tag{2.4}$$

Otherwise, by letting $(2a_{20} - b_{11})^2 + 8b_{30} = -16\lambda^2$, $2a_{20} + b_{11} = 4\lambda\mu$ and making the transformation $\xi = \lambda x$, $\eta = \lambda y + \frac{1}{4}(2a_{20} - b_{11})\lambda x^2$, we obtain the mentioned result.

From (2.4), system (2.1) becomes the following real autonomous planar system:

$$\begin{aligned} \frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{\infty} a_{ij} x^i y^j = X(x, y), \\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{\infty} b_{ij} x^i y^j = Y(x, y). \end{aligned} \tag{2.5}$$

Write

$$X(x, y) = y + \sum_{k=2}^{\infty} X_k(x, y), \qquad Y(x, y) = \sum_{k=2}^{\infty} Y_k(x, y), \tag{2.6}$$

where for $k = 1, 2, \ldots,$

$$X_k(x, y) = \sum_{i+j=k} a_{ij} x^i y^j, \qquad Y_k(x, y) = \sum_{i+j=k} b_{ij} x^i y^j. \tag{2.7}$$

By using the transformation of generalized polar coordinates

$$x = r\cos\theta, \qquad y = r^2 \sin\theta, \tag{2.8}$$

system (2.5) becomes

$$\begin{aligned} \frac{dr}{dt} &= \frac{\cos\theta[\sin\theta(1 - 2\cos^2\theta) + \mu(\cos^2\theta + 2\sin^2\theta)]}{1 + \sin^2\theta} r^2 + o\bigl(r^2\bigr), \\ \frac{d\theta}{dt} &= \frac{-r}{2(1 + \sin^2\theta)(\cos^4\theta + \sin^2\theta)} + o(r). \end{aligned} \tag{2.9}$$

Thus, we have

$$\frac{dr}{d\theta} = \frac{-\cos\theta[\sin\theta(1 - 2\cos^2\theta) + \mu(\cos^2\theta + 2\sin^2\theta)]}{2(\cos^4\theta + \sin^2\theta)} r + o(r). \tag{2.10}$$

Let

$$r = \tilde{r}(\theta, h) = \sum_{k=1}^{\infty} \nu_k(\theta) h^k \tag{2.11}$$

be a solution of (2.10) satisfying the initial condition $r|_{\theta=0} = h$, where h is small and

$$\begin{aligned} &\nu_1(\theta) = \left(\cos^4\theta + \sin^2\theta\right)^{\frac{-1}{4}} \exp\left(\frac{-\mu}{2} \arctan \frac{\sin\theta}{\cos^2\theta}\right), \\ &\nu_1(k\pi) = 1, \quad k = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{2.12}$$

Because for all sufficiently small r, we have $d\theta/dt < 0$. In a small neighborhood, we can define the successor function of system (2.5) as follows:

$$\Delta(h) = \tilde{r}(-2\pi, h) - h = \sum_{k=2}^{\infty} \nu_k(-2\pi) h^k. \tag{2.13}$$

We have the following result.

Lemma 2.2 *For any positive integer m, $\nu_{2m+1}(-2\pi)$ has the form*

$$\nu_{2m+1}(-2\pi) = \sum_{k=1}^{m} \zeta_k^{(m)} \nu_{2k}(-2\pi), \tag{2.14}$$

where $\zeta_k^{(m)}$ is a polynomial of $\nu_j(\pi)$, $\nu_j(2\pi)$, $\nu_j(-2\pi)$ $(j = 2, 3, \ldots, 2m)$ with rational coefficients.

It is differential from the center-focus problem for the elementary critical points, we know from Lemma 2.2 that when $k > 1$ for the first non-zero $\nu_k(-2\pi)$, k is an even integer.

Definition 2.2

1. For any positive integer m, $\nu_{2m}(-2\pi)$ is called the mth focal value of system (2.5) in the origin.
2. If $\nu_2(-2\pi) \neq 0$, then the origin of system (2.5) is called one-order weakened focus. In addition, if there is an integer $m > 1$, such that $\nu_2(-2\pi) = \nu_4(-2\pi) = \cdots = \nu_{2m-2}(-2\pi) = 0$, but $\nu_{2m}(-2\pi) \neq 0$, then the origin is called a m-order weakened focus of system (2.5).
3. If for all positive integer m, we have $\nu_{2m}(-2\pi) = 0$, then the origin of system (2.5) is called a center.

Definition 2.3 Let f_k, g_k be two bounded functions with respect to μ and all a_{ij}, b_{ij}, $k = 1, 2, \ldots$. If for some integer m, there exist $\xi_1^{(m)}, \xi_2^{(m)}, \ldots, \xi_{m-1}^{(m)}$, which are continuous bounded functions with respect to μ and all a_{ij}, b_{ij}, $i = 1, 2, \ldots$, such that

$$f_m = g_m + \left(\xi_1^{(m)} f_1 + \xi_2^{(m)} f_2 + \cdots + \xi_{m-1}^{(m)} f_{m-1}\right). \tag{2.15}$$

We say that f_m is equivalent to g_m, denoted by $f_m \sim g_m$.

If $f_1 = g_1$ and for all positive integers $m, f_m \sim g_m$, we say that the function sequences $\{f_m\}$ and $\{g_m\}$ are equivalent, denoted by $\{f_m\} \sim \{g_m\}$.

We know from Lemma 2.2 and Definition 2.2 that for the sequence $\{\nu_k(-2\pi)\}$, $k \geq 2$, we have $\nu_{2k+1}(-2\pi) \sim 0$, $k = 1, 2, \ldots$.

We next state the results concerning with bifurcation of limit cycles of system (2.5). Consider the perturbed system of (2.5)

$$\frac{dx}{dt} = \delta x + X(x,y), \qquad \frac{dy}{dt} = 2\delta y + Y(x,y), \tag{2.16}$$

where $X(x,y)$, $Y(x,y)$ are given by (2.6). Clearly, when $0 < |\delta| \ll 1$, in a neighborhood of the origin, there exist one elementary node at the origin and two complex critical points of system (2.16) at (x_1, y_1) and (x_2, y_2), where

$$x_{1,2} = \frac{-\delta}{\mu \pm i} + o(\delta), \qquad y_{1,2} = \frac{\pm i\delta^2}{(\mu \pm i)^2} + o\big(\delta^2\big). \tag{2.17}$$

When $\delta \to 0$, one elementary node and two complex critical points coincide to become a three-order critical point. Let

$$r = \tilde{r}(\theta, h, \delta) = \nu_0(\theta, \delta) + \sum_{k=1}^{\infty} \nu_k(\theta, \delta) h^k \tag{2.18}$$

be a solution of system (2.16) satisfying the initial condition $r|_{\theta=0} = h$, where h is sufficiently small and

$$\nu_0(0, \delta) = 0, \qquad \nu_1(0, \delta) = 1, \qquad \ldots, \qquad \nu_k(0, \delta) = 0, \quad k = 2, 3, \ldots. \tag{2.19}$$

We have

$$\nu_0(\theta, \delta) = A(\theta)\delta + o(\delta), \tag{2.20}$$

where

$$A(\theta) = \frac{-\nu_1(\theta, 0)}{2} \int_0^{\theta} \frac{(1 + \sin^2\theta)\, d\theta}{\nu_1(\theta, 0)(\cos^4\theta + \sin^2\theta)}. \tag{2.21}$$

Hence, when $0 < h \ll 1$, $|\theta| < 4\pi$, $\delta = o(h)$, $\tilde{r}(\theta, h, \delta) = \nu_1(\theta, 0)h + o(h)$ and

$$\nu_0(-2\pi, \delta) = A(-2\pi)\delta + o(\delta), \tag{2.22}$$

where

$$A(-2\pi) = \frac{1}{2} \int_0^{2\pi} \frac{1 + \sin^2\theta}{(\cos^4\theta + \sin^2\theta)^{\frac{3}{4}}} \exp\left(\frac{\mu}{2} \arctan \frac{\sin\theta}{\cos^2\theta}\right) d\theta > 0. \tag{2.23}$$

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y + \sum_{k+j=2}^{\infty} a_{kj}(\gamma)x^k y^j, \\ \frac{dy}{dt} &= 2\delta y + \sum_{k+j=2}^{\infty} b_{kj}(\gamma)x^k y^j, \end{aligned} \tag{2.24}$$

where $\gamma = \{\gamma_1, \gamma_2, \dots, \gamma_{m-1}\}$ is $(m-1)$-dimensional parameter vector. Let $\gamma_0 = \{\gamma_1^{(0)}, \gamma_2^{(0)}, \dots, \gamma_{m-1}^{(0)}\}$ be a point at the parameter space. Suppose that for $\|\gamma - \gamma_0\| \ll 1$, the functions of the right hand of system (2.24) are power series of x, y with a non-zero convergence radius and have continuous partial derivatives with respect to γ. In addition,

$$a_{20}(\gamma) \equiv \mu, \qquad b_{20}(\gamma) \equiv 0, \qquad b_{11}(\gamma) \equiv 2\mu, \qquad b_{30}(\gamma) \equiv -2. \tag{2.25}$$

For an integer k, letting $\nu_{2k}(-2\pi, \gamma)$ be the k-order focal value of the origin of system $(2.24)_{\delta=0}$.

Theorem 2.1 *If for $\gamma = \gamma_0$, the origin of system $(2.24)_{\delta=0}$ is a m-order weak focus,and the Jacobin*

$$\frac{\partial(\nu_2, \nu_4, \dots, \nu_{2m-2})}{\partial(\gamma_1, \gamma_2, \dots, \gamma_{m-1})}\Big|_{\gamma=\gamma_0} \neq 0, \tag{2.26}$$

then there exist two positive numbers δ^* and γ^*, such that for $0 < |\delta| < \delta^*$, $0 < \|\gamma - \gamma_0\| < \gamma^*$, in a neighborhood of the origin, system (2.24) has at most m limit cycles which enclose the origin (an elementary node) $O(0,0)$. In addition, under the above conditions, there exist $\tilde{\gamma}$, $\tilde{\delta}$, such that when $\gamma = \tilde{\gamma}$, $\delta = \tilde{\delta}$, there exist exactly m limit cycles of (2.24) in a small neighborhood of the origin.

We give the following key results, which define the quasi-Lyapunov constants and provide a way of computing them.

Theorem 2.2 *For system (2.5), one can construct successively a formal series*

$$M(x,y) = y^2 + \sum_{k+j=3}^{\infty} c_{kj} x^k y^j, \tag{2.27}$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}} \sum_{m=1}^{\infty} (2m - 4s - 1)\lambda_m x^{2m+4}, \tag{2.28}$$

i.e.,

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) M - (s+1)\left(\frac{\partial M}{\partial x} X + \frac{\partial M}{\partial y} Y\right) = \sum_{m=1}^{\infty} \lambda_m (2m - 4s - 1) x^{2m+4}. \tag{2.29}$$

where s is a given positive integer,

$$c_{30} = 0, \qquad c_{40} = 1, \tag{2.30}$$

and

$$\{\nu_{2m}(-2\pi)\} \sim \{\sigma_m \lambda_m\}, \tag{2.31}$$

with

$$\sigma_m = \frac{1}{2}\int_0^{2\pi} \frac{(1+\sin^2\theta)\cos^{2m+4}\theta}{(\cos^4\theta+\sin^2\theta)^2}\nu_1^{2m-1}(\theta)\,d\theta > 0. \tag{2.32}$$

We see from (2.27) and (2.30) that when (2.8) holds, $M = y^2 + x^4 + o(r^4)$.

Definition 2.4 For system (2.5), λ_m is called the mth quasi-Lyapunov constant of the origin.

Theorem 2.3 *For any positive integer s and a given number sequence*

$$\{c_{0\beta}\}, \quad \beta \geq 3, \tag{2.33}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{2.34}$$

such that

$$\frac{\partial}{\partial x}\left(\frac{X}{M^{s+1}}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{M^{s+1}}\right) = \frac{1}{M^{s+2}}\sum_{m=5}^{\infty}\omega_m(s,\mu)x^m, \tag{2.35}$$

where for all k, $M_k(x,y)$ is a k-homogeneous polynomial in x, y and $s\mu = 0$.

Now, (2.35) can be written by

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty}\omega_m(s,\mu)x^m. \tag{2.36}$$

It is easy to see that (2.36) is linear with respect to the function M, so that we can easily find the following recursive formulas for the calculation of $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Theorem 2.4 *For $\alpha \geq 1$, $\alpha + \beta \geq 3$ in* (2.34) *and* (2.35), *$c_{\alpha\beta}$ can be uniquely determined by the recursive formula*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}). \tag{2.37}$$

For $m \geq 1$, $\omega_m(s,\mu)$ *can be uniquely determined by the recursive formula*

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.38}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \bigl[k-(s+1)(\alpha-k+1)\bigr]a_{kj}c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \bigl[j-(s+1)(\beta-j+1)\bigr]b_{kj}c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{2.39}$$

Notice that in (2.39), *we set*

$$\begin{aligned} &c_{00} = c_{10} = c_{01} = 0, \\ &c_{20} = c_{11} = 0, \qquad c_{02} = 1, \\ &c_{\alpha\beta} = 0, \quad \textit{if } \alpha < 0 \textit{ or } \beta < 0. \end{aligned} \tag{2.40}$$

We see from Theorem 2.4 that, by choosing $\{c_{\alpha\beta}\}$, such that

$$\omega_{2k+1}(s,\mu) = 0, \quad k = 1,2,\ldots, \tag{2.41}$$

we can obtain a solution group of $\{c_{\alpha\beta}\}$ of (2.41), thus, we have

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m-4s-1}. \tag{2.42}$$

Clearly, the recursive formulas presented by Theorem 2.4 is linear with respect to all $c_{\alpha\beta}$. Accordingly, it is convenient to realize the computations of quasi-Lyapunov constants by using a computer algebraic system like *Mathematica*.

3 Quasi-Lyapunov constants and center conditions

It is easy to see that the origin of system (1.2) is a third-order nilpotent singular point which is a center or a focus. Now we start the preparation of computing the quasi-Lyapunov constants at the origin of system (1.2).

Lemma 3.1 *Assume that s is a natural number. One can derive a power series* (2.34) *for system* (1.2) *under which* (2.36) *is satisfied, where*

$$c_{00} = 0, \qquad c_{10} = 0, \qquad c_{01} = 0, \qquad c_{20} = 0, \qquad c_{11} = 0, \qquad c_{02} = 1, \tag{3.1}$$

in addition, for any natural numbers α, β, $c_{\alpha\beta}$ *is given by the following recursive formula*:

$$\begin{aligned} c_{\alpha\beta} = {} & \bigl(-b_{40}(1+s)(2+\beta)c_{-5+\alpha,2+\beta} + 2(1+s)(2+\beta)c_{-4+\alpha,2+\beta} \\ & + b_{22}\bigl(2-(1+s)\beta\bigr)c_{-3+\alpha,\beta} \\ & + b_{21}\bigl(1-(1+s)(1+\beta)\bigr)c_{-3+\alpha,1+\beta} + b_{13}\bigl(3-(1+s)(-1+\beta)\bigr)c_{-2+\alpha,-1+\beta} \\ & + b_{12}\bigl(2-(1+s)\beta\bigr)c_{-2+\alpha,\beta} + b_{04}\bigl(4-(1+s)(-2+\beta)\bigr)c_{-1+\alpha,-2+\beta} \\ & + b_{03}\bigl(3-(1+s)(-1+\beta)\bigr)c_{-1+\alpha,-1+\beta}\bigr)/(s+1)/\alpha, \end{aligned} \tag{3.2}$$

and, for any natural number m, ω_m is given by the following recursive formula:

$$\begin{aligned}\omega_m &= -b_{40}(1+s)c_{-4+m,1} + 2(1+s)c_{-3+m,1} + b_{22}(3+s)c_{-2+m,-1} + b_{21}c_{-2+m,0} \\ &\quad + b_{13}\bigl(3+2(1+s)\bigr)c_{-1+m,-2} + b_{12}(3+s)c_{-1+m,-1} + b_{04}\bigl(4+3(1+s)\bigr)c_{m,-3} \\ &\quad + b_{03}\bigl(3+2(1+s)\bigr)c_{m,-2}.\end{aligned} \tag{3.3}$$

One of our main results is the following.

Theorem 3.1 *Consider system* (1.2); *the following are satisfied.*

(I) *By letting*

$$\begin{aligned} c_{03} &= 0, \\ c_{04} &= \frac{1}{75{,}600s(s+1)^2}\bigl(5{,}775b_{12}^2 - 3{,}415b_{22}b_{40} + 2{,}925b_{12}b_{40}^2 \\ &\quad - 3{,}402b_{40}^4 - 13{,}650b_{12}^2 s + 10{,}880b_{22}b_{40}s \\ &\quad + 17{,}325b_{12}b_{40}^2 s - 3{,}024b_{40}^4 s - 7{,}875b_{12}^2 s^2 - 6{,}095b_{22}b_{40}s^2 \\ &\quad - 9{,}405b_{12}b_{40}^2 s^2 + 4{,}158b_{40}^4 s^2 + 43{,}050b_{12}^2 s^3 - 20{,}390b_{22}b_{40}s^3 \\ &\quad - 23{,}805b_{12}b_{40}^2 s^3 + 3{,}780b_{40}^4 s^3\bigr), \\ c_{05} &= -\frac{b_{13}(175b_{12} + 216b_{40}^2 - 875b_{12}s - 24b_{40}^2 s + 1{,}190b_{12}s^2 - 240b_{40}^2 s^2)}{2{,}100(s+1)^2}, \end{aligned} \tag{3.4}$$

its first five quasi-Lyapunov constants are

$$\begin{aligned} \lambda_1 &= \frac{1}{3}b_{21}, \\ \lambda_2 &\sim \frac{6}{5}b_{03}, \\ \lambda_3 &\sim -\frac{65s-62}{35(4s-5)}b_{40}b_{13}, \\ \lambda_4 &\sim -\frac{2(217s^2-159s+44)}{315(s+1)(4s-7)}b_{22}b_{13}, \\ \lambda_5 &\sim -\frac{4(1{,}983s^2-2{,}183s+370)}{1{,}155(s+1)(4s-9)}b_{13}b_{04}, \end{aligned} \tag{3.5}$$

where in the expression of λ_k, we have already let $\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1} = 0$, $k = 2,3,4,5$.

(II) *It has a center at the origin if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. Furthermore this situation happens if and only if one of the following two conditions is satisfied*:

(i) $b_{21} = b_{03} = b_{13} = 0$;

(ii) $b_{21} = b_{03} = b_{40} = b_{22} = b_{04} = 0$.

Proof When condition (i) holds, system (1.2) could be written as

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -2x^3 + b_{40}x^4 + b_{12}xy^2 + b_{22}x^2y^2 + b_{04}y^4, \end{aligned} \tag{3.6}$$

the vector field of system (3.6) is symmetric with respect to axis x, so the origin is a center.

When condition (ii) holds, system (1.2) could be written as

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -x\left(2x^2 - b_{12}y^2 - b_{13}y^3\right), \end{aligned} \tag{3.7}$$

the vector field of system (3.7) is symmetric with respect to axis y, so the origin is also a center. □

4 Limit cycle bifurcation

Next we will prove that the third-order nilpotent singular $O(0,0)$ is at most a weak focus of order five, moreover, based on this conclusion, the perturbed system of (1.2) can produce five limit cycles enclosing an elementary node at the origin.

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 \neq 0$ easily yields.

Theorem 4.1 *For system* (1.2), *the origin is a five-order weak focus if and only if*

$$b_{21} = b_{03} = b_{40} = b_{22} = 0, \qquad b_{13}b_{04} \neq 0. \tag{4.1}$$

Consider the perturbed system of (1.2),

$$\begin{aligned} \frac{dx}{dt} &= \delta(\varepsilon)x + y, \\ \frac{dy}{dt} &= 2\delta(\varepsilon)y - 2x^3 + b_{21}(\varepsilon)x^2y + b_{12}(\varepsilon)xy^2 + b_{03}(\varepsilon)y^3 + b_{40}(\varepsilon)x^4 \\ &\quad + b_{22}(\varepsilon)x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{4.2}$$

In arriving at another main result, we only need to show that, when condition (4.1) holds, the Jacobian of the first four quasi-Lyapunov constants of system (1.2) with respect to b_{21}, b_{03}, b_{40}, b_{22} is not equal to zero. An easy computation shows that

$$\left.\frac{\partial(\lambda_1, \lambda_2, \lambda_3, \lambda_4)}{\partial(b_{21}, b_{03}, b_{40}, b_{22})}\right|_{(4.1)} = \frac{4(65s-62)(217s^2-159s+44)}{55{,}125(s+1)(4s-5)(4s-7)} b_{13}^2 \neq 0. \tag{4.3}$$

The above considerations imply the following main result.

Theorem 4.2 *If the origin of system* (1.2) *is a five-order weak focus, for* $0 < \delta \ll 1$, *making a small perturbation to the coefficients of system* (1.2), *then, for system* (4.2), *in a small neighborhood of the origin, there exist exactly five small amplitude limit cycles enclosing the origin* $O(0,0)$, *which is an elementary node*.

Example 4.1 Take

$$\begin{aligned} &\delta(\varepsilon) = \varepsilon^{30}, \qquad b_{21}(\varepsilon) = -\varepsilon^{20}, \qquad b_{12}(\varepsilon) = 0, \qquad b_{03}(\varepsilon) = \varepsilon^{12}, \\ &b_{40}(\varepsilon) = \varepsilon^{6}, \qquad b_{22}(\varepsilon) = -\varepsilon^{2}, \qquad b_{13} = c_1 \operatorname{sign}(c_1), \qquad b_{04} = c_2 \operatorname{sign}(c_2), \end{aligned} \tag{4.4}$$

where c_1, c_2 are arbitrary non-zero real constants, s (≥ 3) is a natural number.

Straightforward computations by using Theorem 3.1 give the first five quasi-Lyapunov constants at the origin of system (4.2):

$$\begin{aligned}
\lambda_1 &= -\frac{1}{3}\varepsilon^{20} + o(\varepsilon^{20}), \\
\lambda_2 &\sim \frac{6}{5}\varepsilon^{12} + o(\varepsilon^{12}), \\
\lambda_3 &\sim -\frac{65s-62}{35(4s-5)}c_1 \operatorname{sign}(c_1)\varepsilon^{6} + o(\varepsilon^{6}), \\
\lambda_4 &\sim \frac{2(217s^2-159s+44)}{315(s+1)(4s-7)}c_1 \operatorname{sign}(c_1)\varepsilon^{2} + o(\varepsilon^{2}), \\
\lambda_5 &\sim -\frac{4(1{,}983s^2-2{,}183s+370)}{1{,}155(s+1)(4s-9)}c_1 c_2 \operatorname{sign}(c_1)\operatorname{sign}(c_2) + o(1).
\end{aligned} \tag{4.5}$$

Then, for $0 < \varepsilon \ll 1$, system (4.2) has five limit cycles $\Gamma_k : r = \tilde{r}(\theta, h_k(\varepsilon))$ in a small neighborhood of the origin, where $h_k(\varepsilon) = O(\varepsilon^k)$, $k = 1, 2, 3, 4, 5$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YW completed the main part of this paper, CX corrected the main results. All authors read and approved the final manuscript.

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