Ahmad et al. *Advances in Difference Equations* 2014, **2014**:257
http://www.advancesindifferenceequations.com/content/2014/1/257

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions

Bashir Ahmad[1], Sotiris K Ntouyas[1,2] and Ahmed Alsaedi[1*]

*Correspondence: aalsaedi@hotmail.com
[1]Department of Mathematics, Faculty of Science, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate the existence of solutions for a nonlocal boundary value problem of fractional q-integro-difference inclusions of two fractional orders with Riemann-Liouville fractional q-integral boundary conditions. A new existence result is obtained by making use of a nonlinear alternative for contractive maps and is well illustrated with the aid of an example.

MSC: 34A60; 34A08

Keywords: fractional differential inclusions; nonlocal boundary conditions; fixed point theorems

1 Introduction

Nonlocal nonlinear boundary value problems of fractional order have been extensively investigated in recent years. Several results of interest ranging from theoretical analysis to asymptotic behavior and numerical methods for fractional differential equations are available in the literature on the topic. The introduction of fractional derivative in the mathematical modeling of many real world phenomena has played a key role in improving the integer-order mathematical models. One of the important factors accounting for the popularity of the subject is that differential operators of fractional-order help to understand the hereditary phenomena in many materials and processes in a better way than the corresponding integer-order differential operators. For examples and details in physics, chemistry, biology, biophysics, blood flow phenomena, control theory, wave propagation, signal and image processing, viscoelasticity, identification, fitting of experimental data, economics, *etc.*, we refer the reader to the texts [1–4]. Some recent work on fractional differential equations can be found in a series of papers [5–17] and the references cited therein.

Fractional q-difference equations, known as fractional analogue of q-difference equations, have recently been discussed by several researchers. For some recent work on the topic, see [18–32]. In a recent paper [33], the authors obtained some existence results for the Langevin type q-difference (integral) equation with two fractional orders and four-point nonlocal integral boundary conditions.

Initial and boundary value problems involving multivalued maps have been studied by many researchers. In fact, the multivalued (inclusions) problems appear in the mathematical modeling of a variety of problems in economics, optimal control, *etc.* and are widely studied by many authors, see [34–36] and the references therein. Recent works on fractional-order multivalued problems [37–43] clearly indicate the interest in the subject. In [44] the authors studied the existence of solutions for a problem of nonlinear fractional q-difference inclusions with nonlocal Robin (separated) conditions given by

$$\begin{aligned} &{}^cD_q^\alpha x(t) \in F\bigl(t,x(t)\bigr), \quad 0 \le t \le 1, 1 < q \le 2, \\ &\alpha_1 x(0) - \beta_1 D_q x(0) = \gamma_1 x(\eta_1), \qquad \alpha_2 x(1) + \beta_2 D_q x(1) = \gamma_2 x(\eta_2), \end{aligned}$$

where ${}^cD_q^\alpha$ is the fractional q-derivative of the Caputo type, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$ and $\alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{R}$ ($i = 1, 2$). However, the study of multivalued problems in the setting of fractional q-difference equations is still at an initial stage and needs to be explored further.

In this paper, motivated by [44], we consider the multivalued analogue of the problem addressed in [33]. Precisely we discuss the existence of solutions for a boundary value problem of fractional q-integro-difference inclusions with fractional q-integral boundary conditions given by

$$\begin{aligned} &{}^cD_q^\beta\bigl({}^cD_q^\gamma + \lambda\bigr)x(t) \in AF\bigl(t,x(t)\bigr) + BI_q^\xi G\bigl(t,x(t)\bigr), \quad t \in [0,1], \\ &x(0) = aI_q^{\alpha-1}x(\eta), \qquad x(1) = bI_q^{\alpha-1}x(\sigma), \quad \alpha > 2, 0 < \eta, \sigma < 1, \end{aligned} \tag{1.1}$$

where ${}^cD_q^\beta$ denotes the Caputo fractional q-difference operator of order β, $0 < q < 1$, $0 < \beta \le 1$, $0 < \gamma \le 1$, $0 < \xi < 1$, $F, G : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ are multivalued maps, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, a, b, A, B, α_1, β_1, σ_1, σ_2 are real constants and

$$I_q^\alpha x(\varrho) = \int_0^\varrho \frac{(\varrho - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} x(s)\, d_q s \quad (\varrho = \eta, \sigma).$$

Here we emphasize that the multivalued (inclusion) problem at hand is new in the sense that it involves fractional q-integro-difference inclusions of two fractional orders with four-point nonlocal Riemann-Liouville fractional q-integral boundary conditions (in contrast to the problem considered in [44]). Also our method of proof is different from the one employed in [44]. The paper is organized as follows. An existence result for problem (1.1), based on a nonlinear alternative for contractive maps, is established in Section 3. The background material for the problem at hand can be found in the related literature. However, for quick reference, we outline it in Section 2.

2 Preliminaries on fractional q-calculus

Here we recall some definitions and fundamental results on fractional q-calculus [45–47].

Let a q-real number denoted by $[a]_q$ be defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}, q \in \mathbb{R}^+ \setminus \{1\}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \qquad (a;q)_k = \prod_{i=0}^{k-1}\bigl(1 - aq^i\bigr), \quad k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x-y)^k$ is

$$(x-y)^{(0)}=1, \qquad (x-y)^{(k)}=\prod_{j=0}^{k-1}\left(x-yq^j\right), \quad k\in\mathbb{N}, x,y\in\mathbb{R}.$$

The q-gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y)=\frac{(1-q)^{(y-1)}}{(1-q)^{y-1}},$$

where $y\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$. Observe that $\Gamma_q(y+1)=[y]_q\Gamma_q(y)$. For any $x,y>0$, the q-beta function $B_q(x,y)$ is given by

$$B_q(x,y)=\int_0^1 t^{(x-1)}(1-qt)^{(y-1)}\,d_qt,$$

which, in terms of q-gamma function, can be expressed as

$$B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{2.1}$$

Definition 2.1 ([45]) Let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\beta\geq 0$ is $(I_q^0 f)(t)=f(t)$ and

$$I_q^\beta f(t):=\int_0^t \frac{(t-qs)^{(\beta-1)}}{\Gamma_q(\beta)}f(s)\,d_qs=t^\beta(1-q)^\beta\sum_{k=0}^{\infty}q^k\frac{(q^\beta;q)_k}{(q;q)_k}f\left(tq^k\right), \quad \beta>0, t\in[0,1].$$

Observe that $\beta=1$ in Definition 2.1 yields q-integral

$$I_qf(t):=\int_0^t f(s)\,d_qs=t(1-q)\sum_{k=0}^{\infty}q^kf\left(tq^k\right).$$

For more details on q-integral and fractional q-integral, see Section 1.3 and Section 4.2 respectively in [47].

Remark 2.2 The q-fractional integration possesses the semigroup property (Proposition 4.3 [47])

$$I_q^\gamma I_q^\beta f(t)=I_q^{\beta+\gamma}f(t); \quad \gamma,\beta\in\mathbb{R}^+. \tag{2.2}$$

Further, it has been shown in Lemma 6 of [46] that

$$I_q^\beta(x)^{(\sigma)}=\frac{\Gamma_q(\sigma+1)}{\Gamma_q(\beta+\sigma+1)}(x)^{(\beta+\sigma)}, \quad 0<x<a, \beta\in\mathbb{R}^+, \sigma\in(-1,\infty).$$

Before giving the definition of fractional q-derivative, we recall the concept of q-derivative.

We know that the q-derivative of a function $f(t)$ is defined as

$$(D_q f)(t) = \frac{f(t) - f(qt)}{t - qt}, \quad t \neq 0, \qquad (D_q f)(0) = \lim_{t\to 0}(D_q f)(t).$$

Furthermore,

$$D_q^0 f = f, \qquad D_q^n f = D_q\big(D_q^{n-1} f\big), \quad n = 1, 2, 3, \ldots. \tag{2.3}$$

Definition 2.3 ([47]) The Caputo fractional q-derivative of order $\beta > 0$ is defined by

$${}^cD_q^\beta f(t) = I_q^{\lceil\beta\rceil-\beta} D_q^{\lceil\beta\rceil} f(t),$$

where $\lceil\beta\rceil$ is the smallest integer greater than or equal to β.

Next we recall some properties involving Riemann-Liouville q-fractional integral and Caputo fractional q-derivative (Theorem 5.2 [47]).

$$I_q^\beta {}^cD_q^\beta f(t) = f(t) - \sum_{k=0}^{\lceil\beta\rceil-1} \frac{t^k}{\Gamma_q(k+1)}\big(D_q^k f\big)\big(0^+\big), \quad \forall t \in (0, a], \beta > 0; \tag{2.4}$$

$${}^cD_q^\beta I_q^\beta f(t) = f(t), \quad \forall t \in (0, a], \beta > 0. \tag{2.5}$$

To define the solution for problem (1.1), we need the following lemma.

Lemma 2.4 ([33]) *For a given $h \in C([0,1], \mathbb{R})$, the integral solution of the boundary value problem*

$$\begin{aligned} &{}^cD_q^\beta\big({}^cD_q^\gamma + \lambda\big)x(t) = h(t), \quad t \in [0,1], \\ &x(0) = aI_q^{\alpha-1}x(\eta), \qquad x(1) = bI_q^{\alpha-1}x(\sigma), \quad \alpha > 2, 0 < \eta, \sigma < 1, \end{aligned} \tag{2.6}$$

is given by

$$\begin{aligned} x(t) = {}& \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u) - \lambda x(u)\big)\,d_q u \\ & - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\left\{a \int_0^\eta \frac{(\eta - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u) - \lambda x(u)\big)\,d_q u\right) d_q s\right\} \\ & + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{b \int_0^\sigma \frac{(\sigma - qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u) - \lambda x(u)\big)\,d_q u\right) d_q s\right\} \\ & - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\big(I_q^\beta h(u) - \lambda x(u)\big)\,d_q u\right\}, \end{aligned} \tag{2.7}$$

where

$$\delta_1 = \frac{a\eta^{(\alpha-1)}}{\Gamma_q(\alpha)} - 1, \qquad \delta_2 = \frac{a\eta^{(\alpha+\gamma-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\alpha+\gamma)},$$

$$\delta_3 = \frac{b\sigma^{(\alpha-1)}}{\Gamma_q(\alpha)} - 1, \qquad \delta_4 = \frac{b\sigma^{(\alpha+\gamma-1)}\Gamma_q(\gamma+1)}{\Gamma_q(\alpha+\gamma)} - 1$$

and

$$\Delta = \delta_3\delta_2 - \delta_4\delta_1 \neq 0.$$

Let $\mathcal{C} = C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1] \to \mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)|, t \in [0,1]\}$. Also by $L^1([0,1],\mathbb{R})$ we denote the Banach space of measurable functions $x : [0,1] \to \mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^1 |x(t)| \, d_q t$.

In order to prove our main existence result, we make use of the following form of the nonlinear alternative for contractive maps [48, Corollary 3.8].

Theorem 2.5 *Let X be a Banach space, and D be a bounded neighborhood of $0 \in X$. Let $\mathcal{H}_1 : X \to \mathcal{P}_{cp,c}(X)$ (here $\mathcal{P}_{cp,c}(X)$ denotes the family of all nonempty, compact and convex subsets of X) and $\mathcal{H}_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ be two multi-valued operators such that*

(a) *$\mathcal{H}_1$ is a contraction, and*
(b) *$\mathcal{H}_2$ is upper semi-continuous (u.s.c.) and compact.*

Then, if $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, either

(i) *$\mathcal{H}$ has a fixed point in $\bar{D}$, or*
(ii) *there is a point $u \in \partial D$ and $\theta \in (0,1)$ with $u \in \theta\mathcal{H}(u)$.*

Definition 2.6 A multivalued map $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \to F(t,x)$ is measurable for each $x \in \mathbb{R}$,
(ii) $x \to F(t,x)$ is upper semi-continuous for almost all $t \in [0,1]$, and
(iii) for each real number $\rho > 0$, there exists a function $h_\rho \in L^1([0,1],\mathbb{R}^+)$ such that

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq h_\rho(t), \quad \text{a.e. } t \in [0,1]$$

for all $u \in \mathbb{R}$ with $\|u\| \leq \rho$.

Denote

$$S_{F,x} = \{v \in L^1([0,1],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ a.e. } t \in [0,1]\}.$$

Lemma 2.7 (Lasota and Opial [49]) *Let X be a Banach space. Let $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ be an L^1-Carathéodory multivalued map, and let Θ be a linear continuous mapping from $L^1([0,1],\mathbb{R})$ to $C([0,1],X)$. Then the operator*

$$\Theta \circ S_F : C([0,1],\mathbb{R}) \to \mathcal{P}_{cp,c}(C([0,1],\mathbb{R})), \quad x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x})$$

is a closed graph operator in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$.

3 Existence result

Before presenting the main results, we define the solutions of the boundary value problem (1.1).

Definition 3.1 A function $x \in AC^2([0,1],\mathbb{R})$ is said to be a solution of problem (1.1) if $x(0) = aI_q^{\alpha-1}x(\eta)$, $x(1) = bI_q^{\alpha-1}x(\sigma)$, and there exist functions $f \in S_{F,x}$, $g \in S_{G,x}$ such that

$$\begin{aligned} x(t) = {}& \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \biggl(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m)\,d_qm \\ & + B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm - \lambda x(u) \biggr)\,d_qu \\ & - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \biggl\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \biggl(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\ & \times f(m)\,d_qm + B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm - \lambda x(u) \biggr)\,d_qu \biggr)\,d_qs \biggr\} \\ & + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \biggl\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \biggl(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \\ & \times f(m)\,d_qm + B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm - \lambda x(u) \biggr)\,d_qu \biggr)\,d_qs \biggr\} \\ & - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \biggl\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \biggl(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} f(m)\,d_qm \\ & + B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm - \lambda x(u) \biggr)\,d_qu \biggr\}. \end{aligned}$$

In the sequel, we set

$$\begin{aligned} \chi_1 = {}& |A| \biggl\{ \frac{1+\alpha_2}{\Gamma_q(\beta+\gamma+1)} \\ & + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)} \bigl(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)} \bigr) \biggr\}, \end{aligned} \tag{3.1}$$

$$\chi_2 := \frac{1}{\Gamma_q(\gamma+1)}(1+\alpha_2) + \frac{1}{\Gamma_q(\gamma+\alpha)} \bigl(\alpha_1|a|\eta^{(\gamma+\alpha-1)} + \alpha_2|b|\sigma^{(\gamma+\alpha-1)} \bigr), \tag{3.2}$$

$$\begin{aligned} \omega_i := {}& |B| \bigl\{ (1+\alpha_2)\bigl(I_q^{(\beta+\gamma)} b_i\bigr)(1) + \alpha_1|a|\bigl(I_q^{(\beta+\gamma+\alpha-1)} b_i\bigr)(\eta) \\ & + \alpha_2|b|\bigl(I_q^{(\beta+\gamma+\alpha-1)} b_i\bigr)(\sigma) \bigr\}, \quad i = 1,2, \end{aligned} \tag{3.3}$$

and

$$\alpha_1 = \frac{|\delta_3 - \delta_4|}{|\Delta|}, \qquad \alpha_2 = \frac{|\delta_1 - \delta_2|}{|\Delta|}.$$

Theorem 3.2 *Assume that*

(H_1) $F : [0,1] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ *is an* L^1*-Carathéodory multivalued map*;
(H_2) *there exists a function* $k \in C([0,1],\mathbb{R}^+)$ *such that*

$$H\bigl(F(t,x),F(t,y)\bigr) \le k(t)\|x-y\| \quad \text{a.e. } t \in [0,1]$$

for all $x, y \in C([0,1],\mathbb{R})$ *and* $\chi_1\|k\| < 1$, *where* χ_1 *is given by* (3.1) *and* H *denotes the Hausdorff metric*;

(H_3) $G:[0,1]\times\mathbb{R}\to\mathcal{P}_{cp,c}(\mathbb{R})$ *is an* L^1*-Carathéodory multivalued map*;

(H_4) *there exist functions* $b_1,b_2\in L^1([0,1],\mathbb{R})$ *and a nondecreasing function* $\psi:\mathbb{R}^+\to(0,\infty)$ *such that*

$$\|G(t,x)\| := \sup\{|v| : v\in G(t,x)\}\le b_1(t)\psi(\|x\|)+b_2(t)\quad a.e.\ t\in[0,1]$$

for all $x\in\mathbb{R}$;

(H_5) *there exists a number* $M>0$ *such that*

$$\frac{(1-\chi_1\|k\|-|\lambda|\chi_2)M}{\chi_1F_0+\psi(M)\omega_1+\omega_2}>1, \tag{3.4}$$

where χ_1, χ_2, ω_i, $i=1,2$ *are given by* (3.1), (3.2) *and* (3.3) *respectively*, $F_0=\int_0^1\|F(t,0)\|\,dt$ *and* $\chi_1\|k\|+|\lambda|\chi_2<1$.

Then problem (1.1) *has a solution on* $[0,1]$.

Proof To transform problem (1.1) to a fixed point problem, let us introduce an operator $\mathfrak{L}:C([0,1],\mathbb{R})\longrightarrow\mathcal{P}(C([0,1],\mathbb{R}))$ as

$$\mathfrak{L}=\mathfrak{L}_1+\mathfrak{L}_2,$$

with

$$\mathfrak{L}_1(x)=\{h\in C([0,1],\mathbb{R}):h(t)=(\mathcal{F}_1x)(t)\},$$

where

$$\begin{aligned}
&(\mathcal{F}_1x)(t)\\
&\quad=\int_0^t\frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m)\,d_qm-\lambda x(u)\right)d_qu\\
&\qquad-\frac{[\delta_3t^\gamma-\delta_4]}{\Delta}\left\{a\int_0^\eta\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\right.\right.\right.\\
&\qquad\left.\left.\left.\times f(m)\,d_qm-\lambda x(u)\right)d_qu\right)d_qs\right\}\\
&\qquad+\frac{[\delta_1t^\gamma-\delta_2]}{\Delta}\left\{b\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\right.\right.\right.\\
&\qquad\left.\left.\left.\times f(m)\,d_qm-\lambda x(u)\right)d_qu\right)d_qs\right\}\\
&\qquad-\frac{[\delta_1t^\gamma-\delta_2]}{\Delta}\left\{\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u\frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m)\,d_qm-\lambda x(u)\right)d_qu\right\}
\end{aligned}$$

for $f\in S_{F,x}$ and

$$\mathfrak{L}_2(x)=\{h\in C([0,1],\mathbb{R}):h(t)=(\mathcal{F}_2x)(t)\},$$

where

$$\begin{aligned}
&(\mathcal{F}_2x)(t) \\
&\quad = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm \right) d_qu \\
&\qquad - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \left\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \right.\right.\right. \\
&\qquad \left.\left.\left. \times g(m)\,d_qm \right) d_qu \right) d_qs \right\} \\
&\qquad + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \right.\right.\right. \\
&\qquad \left.\left.\left. \times g(m)\,d_qm \right) d_qu \right) d_qs \right\} \\
&\qquad - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(B \int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} g(m)\,d_qm \right) d_qu \right\}
\end{aligned}$$

for $g \in S_{G,x}$.

We shall show that the operators $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfy all the conditions of Theorem 2.5 on $[0,1]$. For the sake of clarity, we split the proof into a sequence of steps and claims.

Step 1. We show that $\mathfrak{L}_1$ *is a multivalued contraction on* $C([0,1],\mathbb{R})$.

Let $x, y \in C([0,1],\mathbb{R})$ and $u_1 \in \mathfrak{L}_1(x)$. Then $u_1 \in \mathcal{P}(C([0,1],\mathbb{R}))$ and

$$\begin{aligned}
u_1(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} v_1(m)\,d_qm - \lambda x(u) \right) d_qu \\
&\quad - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \left\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\quad \left.\left.\left. \times v_1(m)\,d_qm - \lambda x(u) \right) d_qu \right) d_qs \right\} \\
&\quad + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\quad \left.\left.\left. \times v_1(m)\,d_qm - \lambda x(u) \right) d_qu \right) d_qs \right\} \\
&\quad - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} v_1(m)\,d_qm - \lambda x(u) \right) d_qu \right\}
\end{aligned}$$

for some $v_1 \in S_{F,x}$. Since $H(F(t,x),F(t,y)) \leq k(t)\|x-y\|$, there exists $w \in F(t,y)$ such that $|v_1(t) - w(t)| \leq k(t)\|x-y\|$. Thus the multivalued operator U is defined by $U(t) = S_{F,y} \cap K(t)$, where

$$K(t) = \left\{ w \in \mathbb{R} \,\middle|\, \left| v_1(t) - w(t) \right| \leq k(t)\|x-y\| \right\}$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U (which exists by Kuratowski-Ryll-Nardzewski's selection theorem [50, 51]). Then $v_2 \in F(t,y)$ and $|v_1(t) - v_2(t)| \leq k(t)\|x-y\|$ a.e. on $[0,1]$.

Define

$$\begin{aligned}
u_2(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} v_2(m)\, d_q m - \lambda x(u) \right) d_q u \\
&\quad - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \left\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\quad \left.\left.\left. \times v_2(m)\, d_q m - \lambda x(u) \right) d_q u \right) d_q s \right\} \\
&\quad + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\quad \left.\left.\left. \times v_2(m)\, d_q m - \lambda x(u) \right) d_q u \right) d_q s \right\} \\
&\quad - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} v_2(m)\, d_q m - \lambda x(u) \right) d_q u \right\}.
\end{aligned}$$

It follows that $u_2 \in \mathfrak{L}_1(y)$ and

$$\begin{aligned}
&\left| u_1(t) - u_2(t) \right| \\
&\quad = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \left| v_1(m) - v_2(m) \right| d_q m \right) d_q u \\
&\qquad - \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta} \left\{ a \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \left| v_1(m) - v_2(m) \right| d_q m \right) d_q u \right) d_q s \right\} \\
&\qquad + \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ b \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \left| v_1(m) - v_2(m) \right| d_q m \right) d_q u \right) d_q s \right\} \\
&\qquad - \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta} \left\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(A \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \left| v_1(m) - v_2(m) \right| d_q m \right) d_q u \right\} \\
&\quad \le |A| \left\{ \frac{1+\alpha_2}{\Gamma_q(\beta+\gamma+1)} \right. \\
&\qquad \left. + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)} \left(\alpha_1 |a| \eta^{(\beta+\gamma+\alpha-1)} + \alpha_2 |b| \sigma^{(\beta+\gamma+\alpha-1)} \right) \right\} \|k\| \|x-y\|.
\end{aligned}$$

Taking the supremum over the interval $[0,1]$, we obtain

$$\begin{aligned}
\|u_1 - u_2\| &\le |A| \left\{ \frac{1+\alpha_2}{\Gamma_q(\beta+\gamma+1)} \right. \\
&\qquad \left. + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)} \left(\alpha_1 |a| \eta^{(\beta+\gamma+\alpha-1)} + \alpha_2 |b| \sigma^{(\beta+\gamma+\alpha-1)} \right) \right\} \|k\| \|x-y\|.
\end{aligned}$$

Combining the previous inequality with the corresponding one obtained by interchanging the roles of x and y, we find that

$$H\big(\mathfrak{L}_1(x),\mathfrak{L}_1(y)\big) \le |A|\left\{\frac{1+\alpha_2}{\Gamma_q(\beta+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\big(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)}+\alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)}\big)\right\}\|k\|\|x-y\|$$

for all $x, y \in C([0,1],\mathbb{R})$. This shows that $\mathfrak{L}_1$ is a multivalued contraction as

$$\chi_1\|k\| = |A|\left\{\frac{1+\alpha_2}{\Gamma_q(\beta+\gamma+1)} + \frac{1}{\Gamma_q(\beta+\gamma+\alpha)}\big(\alpha_1|a|\eta^{(\beta+\gamma+\alpha-1)}+\alpha_2|b|\sigma^{(\beta+\gamma+\alpha-1)}\big)\right\}\|k\| < 1.$$

Step 2. We shall show that the operator $\mathfrak{L}_2$ *is u.s.c. and compact.* It is well known [52, Proposition 1.2] that a completely continuous operator having a closed graph is u.s.c. Therefore we will prove that $\mathfrak{L}_2$ is completely continuous and has a closed graph. This step involves several claims.

Claim I: $\mathfrak{L}_2$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$.

Let $B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$ be a bounded set in $C([0,1],\mathbb{R})$ and $u \in \mathfrak{L}_2(x)$ for some $x \in B_r$. Then we have

$$\begin{aligned}
&\big|\mathcal{F}_2(x)(t)\big| \\
&\quad\le \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big|g(m)\big|\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu \\
&\qquad + \alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right. \\
&\qquad \times \left.\left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big|g(m)\big|\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu\right)d_qs \\
&\qquad + \alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right. \\
&\qquad \times \left.\left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big|g(m)\big|\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu\right)d_qs \\
&\qquad + \alpha_2\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big|g(m)\big|\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu \\
&\quad\le \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad \times \left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big[b_1(m)\psi\big(\|x\|\big)+b_2(m)\big]\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu \\
&\qquad + \alpha_1|a|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right. \\
&\qquad \times \left.\left(|B|\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}\big[b_1(m)\psi\big(\|x\|\big)+b_2(m)\big]\,d_qm + |\lambda|\big|x(u)\big|\right)d_qu\right)d_qs \\
&\qquad + \alpha_2|b|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\right.
\end{aligned}$$

$$
\begin{aligned}
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \bigl[b_1(m)\psi\bigl(\|x\|\bigr) + b_2(m)\bigr]\, d_qm + |\lambda|\bigl|x(u)\bigr| \right) d_qu \biggr) d_qs \\
&+ \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} \bigl[b_1(m)\psi\bigl(\|x\|\bigr) + b_2(m)\bigr]\, d_qm + |\lambda|\bigl|x(u)\bigr| \right) d_qu \\
\le{}& \psi(r) \biggl\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m)\, d_qm \right) d_qu \\
&+ \alpha_1 |a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m)\, d_qm \right) d_qu \biggr) d_qs \\
&+ \alpha_2 |b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m)\, d_qm \right) d_qu \biggr) d_qs \\
&+ \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_1(m)\, d_qm \right) d_qu \biggr\} \\
&+ \biggl\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m)\, d_qm \right) d_qu \\
&+ \alpha_1 |a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m)\, d_qm \right) d_qu \biggr) d_qs \\
&+ \alpha_2 |b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\times \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m)\, d_qm \right) d_qu \biggr) d_qs \\
&+ \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \left(|B| \int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)} b_2(m)\, d_qm \right) d_qu \biggr\} \\
&+ r|\lambda| \biggl\{ \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\, d_qu + \alpha_1|a| \int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\, d_qu \right) d_qs \\
&+ \alpha_2|b| \int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)} \left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\, d_qu \right) d_qs + \alpha_2 \int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\, d_qu \biggr\} \\
\le{}& \psi(r)\omega_1 + \omega_2 + r|\lambda|\chi_2.
\end{aligned}
$$

Consequently,

$$
\|\mathcal{F}_2 x\| \le \psi(r)\omega_1 + \omega_2 + r|\lambda|\chi_2,
$$

and hence $\mathfrak{L}_2$ is bounded.

Claim II: $\mathfrak{L}_2$ *maps bounded sets into equicontinuous sets.*

As in the proof of Claim I, let B_r be a bounded set and $u \in \mathfrak{L}_2(x)$ for some $x \in B_r$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&\bigl|(\mathcal{F}_2x)(t_2) - (\mathcal{F}_2x)(t_1)\bigr| \\
&\quad\le \biggl|\int_0^{t_1} \frac{|(t_2-qu)^{\gamma-1} - (t_1-qu)^{\gamma-1}|}{\Gamma_q(\gamma)} \\
&\qquad\times \biggl(|B|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\bigl[b_1(m)\psi(r)+b_2(m)\bigr]\,d_qm + |\lambda|r\biggr)\,d_qu \\
&\qquad+ \int_{t_1}^{t_2} \frac{(t_2-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad\times \biggl(|B|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\bigl[b_1(m)\psi(r)+b_2(m)\bigr]\,d_qm + |\lambda|r\biggr)\,d_qu\biggr| \\
&\qquad+ \frac{1}{|\Delta|}\biggl\{|a|\bigl|\delta_3\bigl(t_2^\gamma - t_1^\gamma\bigr)\bigr|\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad\times \biggl(|B|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\bigl[b_1(m)\psi(r)+b_2(m)\bigr]\,d_qm + |\lambda|r\biggr)\,d_qu\biggr)\,d_qs \\
&\qquad+ |b|\bigl|\delta_1\bigl(t_2^\gamma - t_1^\gamma\bigr)\bigr|\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad\times \biggl(|B|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\bigl[b_1(m)\psi(r)+b_2(m)\bigr]\,d_qm + |\lambda|r\biggr)\,d_qu\biggr)\,d_qs \\
&\qquad+ \bigl|\delta_1\bigl(t_2^\gamma - t_1^\gamma\bigr)\bigr|\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)} \\
&\qquad\times \biggl(|B|\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\bigl[b_1(m)\psi(r)+b_2(m)\bigr]\,d_qm + |\lambda|r\biggr)\,d_qu\biggr\}.
\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $t_1 - t_2 \to 0$. Therefore it follows by the Arzelá-Ascoli theorem that $\mathfrak{L}_2 : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ is completely continuous.

Claim III: *Next we prove that $\mathfrak{L}_2$ has a closed graph.*

Let $x_n \to x_*$, $h_n \in \mathfrak{L}_2(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{L}_2(x_*)$. Associated with $h_n \in \mathcal{L}_2(x_n)$, there exists $v_n \in S_{G,x_n}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_n(t) &= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}v_n(m)\,d_qm\biggr)\,d_qu \\
&\quad- \frac{[\delta_3t^\gamma - \delta_4]}{\Delta}\biggl\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \\
&\quad\times v_n(m)\,d_qm\biggr)\,d_qu\biggr)\,d_qs\biggr\} \\
&\quad+ \frac{[\delta_1t^\gamma - \delta_2]}{\Delta}\biggl\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)} \\
&\quad\times v_n(m)\,d_qm\biggr)\,d_qu\biggr)\,d_qs\biggr\} \\
&\quad- \frac{[\delta_1t^\gamma - \delta_2]}{\Delta}\biggl\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}v_n(m)\,d_qm\biggr)\,d_qu\biggr\}.
\end{aligned}$$

Thus it suffices to show that there exists $\nu_* \in S_{G,x_*}$ such that for each $t \in [0,1]$,

$$\begin{aligned}
h_*(t) ={}& \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu_*(m)\,d_qm\right)d_qu \\
&- \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\left\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right. \\
&\left.\left.\left.\times \nu_*(m)\,d_qm\right)d_qu\right)d_qs\right\} \\
&+ \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right. \\
&\left.\left.\left.\times \nu_*(m)\,d_qm\right)d_qu\right)d_qs\right\} \\
&- \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu_*(m)\,d_qm\right)d_qu\right\}.
\end{aligned}$$

Let us consider the linear operator $\Theta : L^1([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ given by

$$\begin{aligned}
&\nu \mapsto \Theta(\nu)(t) \\
&= \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu(m)\,d_qm\right)d_qu \\
&- \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\left\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right. \\
&\left.\left.\left.\times \nu(m)\,d_qm\right)d_qu\right)d_qs\right\} \\
&+ \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right. \\
&\left.\left.\left.\times \nu(m)\,d_qm\right)d_qu\right)d_qs\right\} \\
&- \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu(m)\,d_qm\right)d_qu\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|h_n(t) - h_*(t)\| \\
&= \left\| \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\big(\nu_n(m)-\nu_*(m)\big)\,d_qm\right)d_qu\right. \\
&- \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\left\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right. \\
&\left.\left.\left.\times \big(\nu_n(m)-\nu_*(m)\big)\,d_qm\right)d_qu\right)d_qs\right\} \\
&+ \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\left\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\left(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\right.\right.\right.
\end{aligned}$$

$$\times \left(\nu_n(m) - \nu_*(m)\right) d_q m\Big) d_q u\Big) d_q s\Big\}$$
$$- \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}$$
$$\times \left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\left(\nu_n(m) - \nu_*(m)\right) d_q m\right) d_q u\Big\}\Big\| \to 0,$$

as $n \to \infty$. Thus, it follows by Lemma 2.7 that $\Theta \circ S_G$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{G,x_n})$. Since $x_n \to x_*$, therefore, we have

$$h_*(t) = \int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu_*(m)\, d_q m\right) d_q u$$
$$- \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\Big\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}$$
$$\times \nu_*(m)\, d_q m\Big) d_q u\Big) d_q s\Big\}$$
$$+ \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}$$
$$\times \nu_*(m)\, d_q m\Big) d_q u\Big) d_q s\Big\}$$
$$- \frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\nu_*(m)\, d_q m\right) d_q u\Big\}$$

for some $\nu_* \in S_{G,x_*}$.

Hence $\mathfrak{L}_2$ has a closed graph (and therefore has closed values). In consequence, $\mathfrak{L}_2$ is compact valued.

Therefore the operators $\mathfrak{L}_1$ and $\mathfrak{L}_2$ satisfy all the conditions of Theorem 2.5. So the conclusion of Theorem 2.5 applies and either condition (i) or condition (ii) holds. We show that conclusion (ii) is not possible. If $x \in \theta\mathfrak{L}_1(x) + \theta\mathfrak{L}_2(x)$ for $\theta \in (0,1)$, then there exist $f \in S_{F,x}$ and $g \in S_{G,x}$ such that

$$x(t) = \theta\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m)\, d_q m - \lambda x(u)\right) d_q u$$
$$- \frac{[\delta_3 t^\gamma - \delta_4]}{\Delta}\Big\{a\int_0^\eta \frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(A\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$
$$\times f(m)\, d_q m - \lambda x(u)\Big) d_q u\Big) d_q s\Big\}$$
$$+ \theta\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{b\int_0^\sigma \frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\Big(\int_0^s \frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\Big(A\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}$$
$$\times f(m)\, d_q m - \lambda x(u)\Big) d_q u\Big) d_q s\Big\}$$
$$- \theta\frac{[\delta_1 t^\gamma - \delta_2]}{\Delta}\Big\{\int_0^1 \frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(A\int_0^u \frac{(u-qm)^{(\beta-1)}}{\Gamma_q(\beta)}f(m)\, d_q m - \lambda x(u)\right) d_q u\Big\}$$
$$+ \theta\int_0^t \frac{(t-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\left(B\int_0^u \frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m)\, d_q m\right) d_q u$$

$$
\begin{aligned}
&-\lambda\frac{[\delta_3 t^\gamma-\delta_4]}{\Delta}\biggl\{a\int_0^\eta\frac{(\eta-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\\
&\times g(m)\,d_qm\biggr)\,d_qu\biggr)\,d_qs\biggr\}\\
&+\theta\frac{[\delta_1 t^\gamma-\delta_2]}{\Delta}\biggl\{b\int_0^\sigma\frac{(\sigma-qs)^{(\alpha-2)}}{\Gamma_q(\alpha-1)}\biggl(\int_0^s\frac{(s-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}\\
&\times g(m)\,d_qm\biggr)\,d_qu\biggr)\,d_qs\biggr\}\\
&-\theta\frac{[\delta_1 t^\gamma-\delta_2]}{\Delta}\biggl\{\int_0^1\frac{(1-qu)^{(\gamma-1)}}{\Gamma_q(\gamma)}\biggl(B\int_0^u\frac{(u-qm)^{(\beta+\xi-1)}}{\Gamma_q(\beta+\xi)}g(m)\,d_qm\biggr)\,d_qu\biggr\}.
\end{aligned}
$$

By hypothesis (H_2), for all $t\in[0,1]$, we have

$$
\begin{aligned}
\|F(t,x)\| &= H\bigl(F(t,x),0\bigr)\le H\bigl(F(t,x),F(t,0)\bigr)+H\bigl(F(t,0),0\bigr)\\
&\le H\bigl(F(t,x),F(t,0)\bigr)+\|F(t,0)\|.
\end{aligned}
$$

Hence, for any $a\in F(t,x)$,

$$
\begin{aligned}
|a| &\le \|F(t,x)\|\le H\bigl(F(t,x),F(t,0)\bigr)+\|F(t,0)\|\\
&\le k(t)\|x\|+\|F(t,0)\|
\end{aligned}
$$

for all $t\in[0,1]$. Then, by using the computations from Step 1 and Step 2, Claim I, we have

$$
|x(t)|\le\chi_1\bigl(\|k\|\|x\|+F_0\bigr)+\psi\bigl(\|x\|\bigr)\omega_1+\omega_2+|\lambda|\chi_2\|x\|,
$$

or

$$
\|x\|\le\chi_1\bigl(\|k\|\|x\|+F_0\bigr)+\psi\bigl(\|x\|\bigr)\omega_1+\omega_2+|\lambda|\chi_2\|x\|. \tag{3.5}
$$

Now, if condition (ii) of Theorem 2.5 holds, then there exist $\theta\in(0,1)$ and $x\in\partial B_M$ such that $x=\theta\mathfrak{L}(x)$. This implies that x is a solution with $\|x\|=M$ and consequently, inequality (3.5) yields

$$
\frac{(1-\chi_1\|k\|-|\lambda|\chi_2)M}{\chi_1F_0+\psi(M)\omega_1+\omega_2}\le 1,
$$

which contradicts (3.4). Hence, $\mathfrak{L}$ has a fixed point in $[0,1]$ by Theorem 2.5, which in fact is a solution of problem (1.1). This completes the proof. □

Example 3.3 Consider a nonlocal integral boundary value problem of fractional integro-differential equations given by

$$
\begin{cases}
{}^cD_q^{1/2}({}^cD_q^{1/2}+\frac{1}{8})x(t)\in F(t,x(t))+I^{3/4}G(t,x(t)), & t\in[0,1],\\
x(0)=I_q^2x(1/3), \qquad x(1)=\frac{1}{2}I_q^2x(2/3),
\end{cases} \tag{3.6}
$$

where $A=1$, $B=1$, $\beta=\gamma=q=b=1/2$, $a=1$, $\alpha=3$, $\eta=1/3$, $\sigma=2/3$, $\lambda=1/8$.

We found $\delta_1 = -0.925926$, $\delta_2 = 0.030136$, $\delta_3 = -0.851852$, $\delta_4 = -0.914762$, $\Delta = -0.872674$, $\alpha_1 = 0.072089$, $\alpha_2 = 1.095555$, $\chi_1 = 2.158402$, $\chi_2 = 2.37938$.

Let

$$F(t,x) = \left[\frac{1}{2} - \frac{\cos x}{(5+t)^2}, -\frac{1}{6}\right],$$
$$G(t,x) = \left[\frac{1}{4}\cos t^2 \sin\left(\frac{|x|}{2}\right) + \frac{e^{-x^2}(t^2+1)}{1+(t^2+1)} + \frac{1}{3}, \frac{1}{15}\sin\bigl(|x|\bigr) + \frac{(t^3+1)}{1+(t^3+1)}\right].$$

Then we have

$$\sup\bigl\{|u| : u \in F(t,x)\bigr\} \le \frac{1}{2} + \frac{1}{(5+t)^2}, \qquad H\bigl(F(t,x), F(t,\bar{x})\bigr) \le k(t)|x - \bar{x}|$$

with $k(t) = \frac{1}{(5+t)^2}$. Clearly, $\|k\| = 1/25$, $b_1(t) = \frac{1}{8}$, $b_2(t) = 1$, $\psi(M) = M$, $F_0 = 0.55$, $w_1 = 0.2698$, $w_2 = 2.1584$ and $\chi_1\|k\| + \chi_2|\lambda| < 1$. By the condition

$$\frac{(1 - \chi_1\|k\| - |\lambda|\chi_2)M}{\chi_1 F_0 + \psi(M)\omega_1 + \omega_2} > 1,$$

it is found that $M > M_1$, where $M_1 \simeq 9.65681$. Thus, all the assumptions of Theorem 3.2 are satisfied. Hence, the conclusion of Theorem 3.2 applies to problem (3.6).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each of the authors, BA, SKN and AA contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details

[1]Department of Mathematics, Faculty of Science, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece.

Acknowledgements

This paper was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of KAU.

Received: 15 July 2014 Accepted: 30 September 2014 Published: 13 October 2014

References

1. Samko, SG, Kilbas, AA, Marichev, OI: Fractional Integrals and Derivatives, Theory and Applications. Gordon & Breach, Yverdon (1993)
2. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
3. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
4. Baleanu, D, Diethelm, K, Scalas, E, Trujillo, JJ: Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos. World Scientific, Boston (2012)
5. Agarwal, RP, Zhou, Y, He, Y: Existence of fractional neutral functional differential equations. Comput. Math. Appl. **59**, 1095-1100 (2010)
6. Baleanu, D, Mustafa, OG, Agarwal, RP: On L^p-solutions for a class of sequential fractional differential equations. Appl. Math. Comput. **218**, 2074-2081 (2011)
7. Ahmad, B, Nieto, JJ: Riemann-Liouville fractional integro-differential equations with fractional nonlocal integral boundary conditions. Bound. Value Probl. **2011**, Article ID 36 (2011)
8. Ahmad, B, Ntouyas, SK, Alsaedi, A: New existence results for nonlinear fractional differential equations with three-point integral boundary conditions. Adv. Differ. Equ. **2011**, Article ID 107384 (2011)
9. Sudsutad, W, Tariboon, J: Existence results of fractional integro-differential equations with *m*-point multi-term fractional order integral boundary conditions. Bound. Value Probl. **2012**, Article ID 94 (2012)
10. O'Regan, D, Stanek, S: Fractional boundary value problems with singularities in space variables. Nonlinear Dyn. **71**, 641-652 (2013)

11. Ahmad, B, Ntouyas, SK, Alsaedi, A: A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. Math. Probl. Eng. **2013**, Article ID 320415 (2013)
12. Ahmad, B, Nieto, JJ: Boundary value problems for a class of sequential integrodifferential equations of fractional order. J. Funct. Spaces Appl. **2013**, Article ID 149659 (2013)
13. Zhang, L, Ahmad, B, Wang, G, Agarwal, RP: Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. **249**, 51-56 (2013)
14. Liu, X, Jia, M, Ge, W: Multiple solutions of a p-Laplacian model involving a fractional derivative. Adv. Differ. Equ. **2013**, Article ID 126 (2013)
15. Ahmad, B, Ntouyas, SK: Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2013**, 20 (2013)
16. Zhai, C, Xu, L: Properties of positive solutions to a class of four-point boundary value problem of Caputo fractional differential equations with a parameter. Commun. Nonlinear Sci. Numer. Simul. **19**, 2820-2827 (2014)
17. Punzo, F, Terrone, G: On the Cauchy problem for a general fractional porous medium equation with variable density. Nonlinear Anal. **98**, 27-47 (2014)
18. Ferreira, R: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ. **2010**, 70 (2010)
19. Goodrich, CS: Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. **61**, 191-202 (2011)
20. Ma, J, Yang, J: Existence of solutions for multi-point boundary value problem of fractional q-difference equation. Electron. J. Qual. Theory Differ. Equ. **2011**, 92 (2011)
21. Abdeljawad, T, Baleanu, D: Caputo q-fractional initial value problems and a q-analogue Mittag-Leffler function. Commun. Nonlinear Sci. Numer. Simul. **16**, 4682-4688 (2011)
22. Jarad, F, Abdeljawad, T, Gundogdu, E, Baleanu, D: On the Mittag-Leffler stability of q-fractional nonlinear dynamical systems. Proc. Rom. Acad., Ser. A: Math. Phys. Tech. Sci. Inf. Sci. **12**, 309-314 (2011)
23. Graef, JR, Kong, L: Positive solutions for a class of higher order boundary value problems with fractional q-derivatives. Appl. Math. Comput. **218**, 9682-9689 (2012)
24. Li, X, Han, Z, Sun, S: Existence of positive solutions of nonlinear fractional q-difference equation with parameter. Adv. Differ. Equ. **2013**, Artice ID 260 (2013)
25. Alsaedi, A, Ahmad, B, Al-Hutami, H: A study of nonlinear fractional q-difference equations with nonlocal integral boundary conditions. Abstr. Appl. Anal. **2013**, Article ID 410505 (2013)
26. Jarad, F, Abdeljawad, T, Baleanu, D: Stability of q-fractional non-autonomous systems. Nonlinear Anal., Real World Appl. **14**, 780-784 (2013)
27. Abdeljawad, T, Baleanu, D, Jarad, F, Agarwal, RP: Fractional sums and differences with binomial coefficients. Discrete Dyn. Nat. Soc. **2013**, Article ID 104173 (2013)
28. Ferreira, R: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**, 367-373 (2011)
29. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Existence of solutions for nonlinear fractional q-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. J. Franklin Inst. **351**, 2890-2909 (2014)
30. Agarwal, RP, Ahmad, B, Alsaedi, A, Al-Hutami, H: Existence theory for q-antiperiodic boundary value problems of sequential q-fractional integrodifferential equations. Abstr. Appl. Anal. **2014**, Article ID 207547 (2014)
31. Zhang, L, Baleanu, D, Wang, G: Nonlocal boundary value problem for nonlinear impulsive q_k-integrodifference equation. Abstr. Appl. Anal. **2014**, Article ID 478185 (2014)
32. Ahmad, B, Ntouyas, SK, Alsaedi, A, Al-Hutami, H: Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2014**, 26 (2014)
33. Ahmad, B, Nieto, JJ, Alsaedi, A, Al-Hutami, H: Boundary value problems of nonlinear fractional q-difference (integral) equations with two fractional orders and four-point nonlocal integral boundary conditions. Filomat (in press)
34. Abbasbandy, S, Nieto, JJ, Alavi, M: Tuning of reachable set in one dimensional fuzzy differential inclusions. Chaos Solitons Fractals **26**, 1337-1341 (2005)
35. Frigon, M: Systems of first order differential inclusions with maximal monotone terms. Nonlinear Anal. **66**, 2064-2077 (2007)
36. Nieto, JJ, Rodriguez-Lopez, R: Euler polygonal method for metric dynamical systems. Inf. Sci. **177**, 4256-4270 (2007)
37. Henderson, J, Ouahab, A: Fractional functional differential inclusions with finite delay. Nonlinear Anal. **70**, 2091-2105 (2009)
38. Chang, Y-K, Nieto, JJ: Some new existence results for fractional differential inclusions with boundary conditions. Math. Comput. Model. **49**, 605-609 (2009)
39. Cernea, A: On the existence of solutions for nonconvex fractional hyperbolic differential inclusions. Commun. Math. Anal. **9**(1), 109-120 (2010)
40. Agarwal, RP, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973-1033 (2010)
41. Ahmad, B, Ntouyas, SK: Some existence results for boundary value problems of fractional differential inclusions with non-separated boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2010**, 71 (2010)
42. Ntouyas, SK: Boundary value problems for nonlinear fractional differential equations and inclusions with nonlocal and fractional integral boundary conditions. Opusc. Math. **33**, 117-138 (2013)
43. Ahmad, B, Ntouyas, SK: An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 705809 (2014)
44. Ahmad, B, Ntouyas, SK: Existence of solutions for nonlinear fractional q-difference inclusions with nonlocal Robin (separated) conditions. Mediterr. J. Math. **10**, 1333-1351 (2013)
45. Agarwal, RP: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
46. Rajkovic, PM, Marinkovic, SD, Stankovic, MS: On q-analogues of Caputo derivative and Mittag-Leffler function. Fract. Calc. Appl. Anal. **10**, 359-373 (2007)
47. Annaby, MH, Mansour, ZS: q-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012)

48. Petryshyn, WV, Fitzpatric, PM: A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps. Trans. Am. Math. Soc. **194**, 1-25 (1974)
49. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)
50. Kuratowski, K, Ryll-Nardzewski, C: A general theorem on selectors. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 397-403 (1965)
51. Gorniewicz, L: Topological Fixed Point Theory of Multivalued Mappings. Springer, Dordrecht (2006)
52. Deimling, K: Multivalued Differential Equations. de Gruyter, Berlin (1992)

doi:10.1186/1687-1847-2014-257
Cite this article as: Ahmad et al.: **Existence of solutions for fractional q-integro-difference inclusions with fractional q-integral boundary conditions.** *Advances in Difference Equations* 2014 **2014**:257.