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Weak generalized and numerical solution for a quasilinear pseudo-parabolic equation with nonlocal boundary condition

Irem Baglan[1] and Fatma Kanca[2*]

*Correspondence: fatma.kanca@khas.edu.tr
[2]Department of Management Information Systems, Kadir Has University, Istanbul, 34083, Turkey
Full list of author information is available at the end of the article

Abstract

This paper investigates the one dimensional mixed problem with nonlocal boundary conditions, for the quasilinear parabolic equation. Under some natural regularity and consistency conditions on the input data, the existence, uniqueness, convergence of the weak generalized solution, and also continuous dependence upon the data of the solution are shown by using the generalized Fourier method. We construct an iteration algorithm for the numerical solution of this problem.

1 Introduction

D denotes the domain

$$D := \{0 < x < 1, 0 < t < T\}.$$

Consider the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial x^2 \partial t} = f(x,t,u), \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in [0,1], \tag{2}$$

the nonlocal boundary condition

$$u(0,t) = u(1,t), \quad t \in [0,T], \tag{3}$$

$$u_x(1,t) = 0, \quad t \in [0,T], \tag{4}$$

for a quasilinear pseudo-parabolic equation with the nonlinear source term $f = f(x,t,u)$.

The functions $\varphi(x)$ and $f(x,t,u)$ are given functions on $[0,1]$ and $\bar{D} \times (-\infty,\infty)$, respectively. $\varepsilon \in [0,\varepsilon_0]$ is a small parameter, $\varepsilon_0 \geq 0$.

Denote the solution of problem (1)-(4) by $u = u(x,t,\varepsilon)$.

This problem was investigated with different boundary conditions by various researchers by using Fourier methods [1].

In this study, we consider the initial-boundary value problem (1)-(4) with nonlocal boundary conditions (2)-(3). The periodic nature of (2)-(3) type boundary conditions is demonstrated in [2]. In this study, we prove the existence, uniqueness convergence of the weak generalized solution, continuous dependent upon the data of the solution; and we construct an iteration algorithm for the numerical solution of this problem. We analyze computationally convergence of the iteration algorithm, as well as treating a test example.

We will use the weak solution approach from [3] for the considered problem (1)-(4).

According to [4, 5] we assume the following definitions.

Definition 1 The function $v(x,t) \in C^2(\bar{D})$ is called test function if it satisfies the following conditions:

$$v(x,T) = 0, \qquad v(0,t) = v(1,t), \qquad v_x(1,t) = 0, \quad \forall t \in [0,T] \text{ and } \forall x \in [0,1].$$

Definition 2 The function $u(x,t,\varepsilon) \in C(\bar{D})$ satisfying the integral identity

$$\begin{aligned} &\int_0^T \int_0^1 \left[\left(\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} - \varepsilon \frac{\partial^3 v}{\partial x^2 \, \partial t} \right) u + f(x,t,u) v \right] dx\, dt \\ &\quad + \int_0^1 \varphi(x) \left[v(x,0) - \varepsilon \frac{\partial^3 v(x,0)}{\partial x^2} \right] dx = 0, \end{aligned} \tag{5}$$

for arbitrary test function $v = v(x,t)$, is called a generalized (weak) solution of problem (1)-(4).

2 Reducing the problem to countable system of integral equations

Consider the following system of functions on the interval $[0,1]$:

$$\begin{aligned} &X_0(x) = 2, \qquad X_{2k-1}(x) = 4\cos(2\pi kx), \\ &X_{2k}(x) = 4(1-x)\sin(2\pi kx), \quad k = 1,2,\ldots, \end{aligned} \tag{6}$$

$$Y_0(x) = x, \qquad Y_{2k-1}(x) = x\cos(2\pi kx), \qquad Y_{2k}(x) = \sin(2\pi kx), \qquad k = 1,2,\ldots. \tag{7}$$

The system of functions (6) and (7) arise in [6] for the solution of a nonlocal boundary value problem in heat conduction.

It is easy to verify that the systems of function (6) and (7) are biorthonormal on $[0,1]$. They are also Riesz bases in $L_2[0,1]$ (see [7, 8]).

We will use the Fourier series representation of the weak solution to transform the initial-boundary value problem to the infinite set of nonlinear integral equations.

Any solution of (1)-(4) can be represented as

$$u(x,t,\varepsilon) = \sum_{k=1}^{\infty} u_k(t,\varepsilon) X_k(x), \tag{8}$$

where the functions $u_k(t,\varepsilon)$, $k=0,1,2,\ldots$, satisfy the following system of equations:

$$\begin{aligned}
u_0(t,\varepsilon) &= \varphi_0 + \int_0^t f_0(\tau)\,d\tau,\\
u_{2k}(t,\varepsilon) &= \varphi_{2k} e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t f_{2k}(\tau) e^{-\frac{(2\pi k)^2 (t-\tau)}{1+\varepsilon(2\pi k)^2}}\,d\tau,\\
u_{2k-1}(t,\varepsilon) &= \left(\varphi_{2k-1} - \varphi_{2k} + \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\varphi_{2k}\right) e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}\\
&\quad + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \left[f_{2k-1}(\tau) - \left(1 - \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\right)(t-\tau) f_{2k}(\tau)\right]\\
&\quad \cdot e^{-\frac{(2\pi k)^2 (t-\tau)}{1+\varepsilon(2\pi k)^2}}\,d\tau,
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
\varphi_k &= \int_0^1 \varphi(x) Y_k(x)\,dx,\\
f_k(x) &= \int_0^1 f(x,t,u) Y_k(x)\,dx.
\end{aligned}$$

Definition 3 Denote the set

$$\{u(t,\varepsilon)\} = \{u_0(t,\varepsilon), u_{2k}(t,\varepsilon), u_{2k-1}(t,\varepsilon), k=1,2,\ldots\},$$

of functions continuous on $[0,T]$ satisfying the condition

$$\max_{0\le t\le T}|u_0(t,\varepsilon)| + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t,\varepsilon)| + \max_{0\le t\le T}|u_{2k-1}(t,\varepsilon)|\right) < \infty,$$

by B. Let

$$\|u(t,\varepsilon)\| = \max_{0\le t\le T}|u_0(t,\varepsilon)| + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{2k}(t,\varepsilon)| + \max_{0\le t\le T}|u_{2k-1}(t,\varepsilon)|\right)$$

be the norm in B. It can be shown that B is a Banach space [9].

We denote the solution of the nonlinear system (9) by $\{u(t,\varepsilon)\}$.

Theorem 4

(1) *Assume the function* $f(x,t,u)$ *is continuous with respect to all arguments in* $\bar{D}\times(-\infty,\infty)$ *and satisfies the following condition*:

$$|f(t,x,u) - f(t,x,\tilde{u})| \le b(t,x)|u-\tilde{u}|,$$

where $b(x,t)\in L_2(D)$, $b(x,t)\ge 0$,

(2) $f(x,t,0)\in C^2[0,1]$, $t\in[0,1]$,

(3) $\varphi(x)\in C^2[0,1]$.

Then the system (9) *has a unique solution in* D.

Proof For $N = 0, 1, \dots$, let us define an iteration for the system (9) as follows:

$$\begin{aligned}
u_0^{(N+1)}(t,\varepsilon) &= u_0^{(0)}(t,\varepsilon) + \int_0^t \int_0^1 f\Biggl(\xi,\tau,\Biggl(2u_0^{(N)}(\tau,\varepsilon) \\
&\quad + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau,\varepsilon)\cos 2\pi k\xi\bigr)\Biggr)\Biggr)\xi\,d\xi\,d\tau, \\
u_{2k}^{(N+1)}(t,\varepsilon) &= u_{2k}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\Biggl(\xi,\tau,\Biggl(2u_0^{(N)}(\tau,\varepsilon) \\
&\quad + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau,\varepsilon)\cos 2\pi k\xi\bigr)\Biggr)\Biggr) \\
&\quad \cdot \sin 2\pi k\xi\,d\xi\,d\tau, \\
u_{2k-1}^{(N+1)}(t,\varepsilon) &= u_{2k-1}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\Biggl(\xi,\tau,\Biggl(2u_0^{(N)}(\tau,\varepsilon) \\
&\quad + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau,\varepsilon)\cos 2\pi k\xi\bigr)\Biggr)\Biggr) \\
&\quad \cdot \xi\cos 2\pi k\xi\,d\xi\,d\tau + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\Biggl(\xi,\tau,\Biggl(2u_0^{(N)}(\tau) \\
&\quad + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau)\cos 2\pi k\xi\bigr)\Biggr)\Biggr) \\
&\quad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau \\
&\quad - \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f\Biggl(\xi,\tau,\Biggl(2u_0^{(N)}(\tau,\varepsilon) \\
&\quad + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau,\varepsilon)\cos 2\pi k\xi\bigr)\Biggr)\Biggr) \\
&\quad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau,
\end{aligned} \tag{10}$$

where, for simplicity, we let

$$\begin{aligned}
&Au^{(N)}(\xi,\tau,\varepsilon) \\
&\quad = 2u_0^{(N)}(\tau,\varepsilon) + 4\sum_{k=1}^{\infty}\bigl(u_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + u_{2k-1}^{(N)}(\tau,\varepsilon)\cos 2\pi k\xi\bigr), \\
&u_0^{(N+1)}(t,\varepsilon) = u_0^{(0)}(t,\varepsilon) + \int_0^t \int_0^1 f\bigl(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr)\xi\,d\xi\,d\tau, \\
&u_{2k}^{(N+1)}(t,\varepsilon) = u_{2k}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\bigl(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr) \\
&\qquad \cdot \sin 2\pi k\xi\,d\xi\,d\tau,
\end{aligned} \tag{11}$$

$$
\begin{aligned}
u_{2k-1}^{(N+1)}(t,\varepsilon) &= u_{2k-1}^{(0)}(t,\varepsilon) \\
&\quad + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\bigl(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr)\xi\cos 2\pi k\xi\,d\xi\,d\tau \\
&\quad + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\bigl(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr) \\
&\quad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau \\
&\quad - \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f\bigl(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr) \\
&\quad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau,
\end{aligned}
$$

where $u_0^{(0)}(t,\varepsilon) = \varphi_0$, $u_{2k}^{(0)}(t,\varepsilon) = \varphi_{2k} e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$, $u_{2k-1}^{(0)}(t,\varepsilon) = (\varphi_{2k-1} - \varphi_{2k} + \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\varphi_{2k}) \times e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$.

From the condition of the theorem we have $u^{(0)}(t,\varepsilon) \in B$. We will prove that the other sequential approximations satisfy this condition.

Let us write $N = 0$ in (11).

$$
u_0^{(1)}(t,\varepsilon) = u_0^{(0)}(t,\varepsilon) + \int_0^t\int_0^1 f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr)\,d\xi\,d\tau.
$$

Adding and subtracting $\int_0^t\int_0^1 f(\xi,\tau,0)\,d\xi\,d\tau$ to both sides of the last equation, we obtain

$$
\begin{aligned}
u_0^{(1)}(t,\varepsilon) &= u_0^{(0)}(t,\varepsilon) + \int_0^t\int_0^1 \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\,d\xi\,d\tau \\
&\quad + \int_0^t\int_0^1 f(\xi,\tau,0)\,d\xi\,d\tau.
\end{aligned}
$$

Applying the Cauchy Inequality to the last equation, we have

$$
\begin{aligned}
\bigl|u_0^{(1)}(t,\varepsilon)\bigr| &\le |\varphi_0| + \left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\int_0^1\bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^t d\tau\right)^{\frac{1}{2}}\left(\int_0^t\left\{\int_0^1 f(\xi,\tau,0)\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Lipschitzs Condition to the last equation, we have

$$
\begin{aligned}
\bigl|u_0^{(1)}(t,\varepsilon)\bigr| &\le |\varphi_0| + \sqrt{t}\left(\int_0^t\left\{\int_0^1 b(\xi,\tau)\bigl|Au^{(0)}(\xi,\tau,\varepsilon)\bigr|\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\quad + \sqrt{t}\left(\int_0^t\left\{\int_0^1 f(\xi,\tau,0)\,d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.
\end{aligned}
$$

Let us use

$$
\bigl|Au^{(0)}(\xi,\tau)\bigr| \le \bigl|u^{(0)}(\tau,\varepsilon)\bigr|.
$$

Taking the maximum of both sides of the last inequality yields the following:

$$\max_{0\le t\le T}\left|u_0^{(1)}(t,\varepsilon)\right| \le |\varphi_0| + \sqrt{T}\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\| + \sqrt{T}\left\|f(x,t,0)\right\|_{L_2(D)},$$

$$u_{2k}^{(1)}(t,\varepsilon) = \varphi_{2k}e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) \cdot \sin 2\pi k\xi\, d\xi\, d\tau.$$

Adding and subtracting $\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)\sin 2\pi k\xi\, d\xi\, d\tau$ to both sides of the last equation, we obtain

$$\begin{aligned} u_{2k}^{(1)}(t,\varepsilon) &= \varphi_{2k}e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} \\ &\quad + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr] \\ &\quad \cdot \sin 2\pi k\xi\, d\xi\, d\tau \\ &\quad + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)\sin 2\pi k\xi\, d\xi\, d\tau. \end{aligned}$$

Applying the Cauchy Inequality to the last equation, we have

$$\begin{aligned} &\left|u_{2k}^{(1)}(t,\varepsilon)\right| \\ &\quad \le |\varphi_{2k}| + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\, d\tau\right)^{\frac{1}{2}} \\ &\qquad \cdot \left(\int_0^t \left\{\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^1 \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\ &\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^1 f(\xi,\tau,0)\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}. \end{aligned}$$

Taking the summation of both sides with respect to k and using the Hölder Inequality yield the following:

$$\begin{aligned} \sum_{k=1}^{\infty}\left|u_{2k}^{(1)}(t,\varepsilon)\right| &\le \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{1}{2\sqrt{2}\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2(\sqrt{1+\varepsilon(2\pi k)^2})^2}\right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_0^t \sum_{k=1}^{\infty}\left\{\int_0^1 \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\ &\quad + \frac{1}{2\sqrt{2}\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2(\sqrt{1+\varepsilon(2\pi k)^2})^2}\right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_0^t \sum_{k=1}^{\infty}\left\{\int_0^1 f(\xi,\tau,0)\sin 2\pi k\xi\, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}. \end{aligned}$$

Using the Bessel Inequality in the last inequality, we obtain

$$\sum_{k=1}^{\infty}\left|u_{2k}^{(1)}(t,\varepsilon)\right| \leq \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{1}{4\sqrt{3}}\left(\int_0^t \sum_{k=1}^{\infty}\left\{\int_0^1 \left[f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right) - f(\xi,\tau,0)\right]\right.\right.$$
$$\left.\left. \cdot \sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}$$
$$+ \frac{1}{4\sqrt{3}}\left(\int_0^t \sum_{k=1}^{\infty}\left\{\int_0^1 f(\xi,\tau,0)\sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}}.$$

Applying the Lipschitzs Condition to the last equation and taking the maximum of both sides of the last inequality yield the following:

$$\sum_{k=1}^{\infty}\max_{0\leq t\leq T}\left|u_{2k}^{(1)}(t,\varepsilon)\right| \leq \sum_{k=1}^{\infty}|\varphi_{2k}| + \frac{1}{4\sqrt{3}}\left\|b(x,t)\right\|_{L_2(D)}\left\|u^{(0)}(t,\varepsilon)\right\|_B$$
$$+ \frac{1}{4\sqrt{3}}\left\|f(x,t,0)\right\|_{L_2(D)},$$

$$u_{2k-1}^{(1)}(t,\varepsilon) = \left(\varphi_{2k-1} - \varphi_{2k} + \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\varphi_{2k}\right)e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$$
$$+ \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)\xi\cos 2\pi k\xi \, d\xi \, d\tau$$
$$+ \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)$$
$$\cdot (t-\tau)\sin 2\pi k\xi \, d\xi \, d\tau$$
$$- \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)$$
$$\cdot (t-\tau)\sin 2\pi k\xi \, d\xi \, d\tau.$$

Adding and subtracting

$$\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)\xi\cos 2\pi k\xi \, d\xi \, d\tau,$$
$$\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)(t-\tau)\sin 2\pi k\xi \, d\xi \, d\tau,$$
$$\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f(\xi,\tau,0)(t-\tau)\sin 2\pi k\xi \, d\xi \, d\tau$$

to both sides of the last equation and applying the derivative to φ_{2k}, we obtain

$$u_{2k-1}^{(1)}(t,\varepsilon) = \left(\varphi_{2k-1} + \frac{1}{4\pi^2 k^2}\varphi_{2k}'' + \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\varphi_{2k}''\right)e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$$
$$+ \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \left[f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right) - f(\xi,\tau,0)\right]$$

$$\begin{aligned}
&\cdot \xi \cos 2\pi k\xi \, d\xi \, d\tau \\
&+ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)\xi \cos 2\pi k\xi \, d\xi \, d\tau \\
&+ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr] \\
&\cdot (t-\tau) \sin 2\pi k\xi \, d\xi \, d\tau \\
&+ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f(\xi,\tau,0)(t-\tau) \sin 2\pi k\xi \, d\xi \, d\tau \\
&- \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} \\
&\cdot \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr](t-\tau) \sin 2\pi k\xi \, d\xi \, d\tau \\
&+ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} \\
&\cdot f(\xi,\tau,0)(t-\tau) \sin 2\pi k\xi \, d\xi \, d\tau.
\end{aligned}$$

Applying the Cauchy Inequality to the last equation, we have the following:

$$\begin{aligned}
&\bigl|u_{2k-1}^{(1)}(t,\varepsilon)\bigr| \\
&\quad \le |\varphi_{2k-1}| + \frac{1}{4\pi^2k^2}\bigl|\varphi_{2k}''\bigr| + \frac{1}{1+\varepsilon(2\pi k)^2}\bigl|\varphi_{2k}''\bigr| \\
&\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^t \left\{\frac{1}{1+\varepsilon(2\pi k)^2} \int_0^1 \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\right.\right. \\
&\qquad \left.\left.\cdot \xi \cos 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \left\{\int_0^1 f(\xi,\tau,0)\xi \cos 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^t \left\{\frac{1}{1+\varepsilon(2\pi k)^2} \int_0^1 \bigl[f\bigl(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\bigr) - f(\xi,\tau,0)\bigr]\right.\right. \\
&\qquad \left.\left.\cdot (t-\tau) \sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau\right)^{\frac{1}{2}} \\
&\qquad \cdot \left(\int_0^t \left\{\frac{1}{1+\varepsilon(2\pi k)^2} \int_0^1 f(\xi,\tau,0)(t-\tau) \sin 2\pi k\xi \, d\xi\right\}^2 d\tau\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_0^t \left\{ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^1 \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} \left[f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right) - f(\xi,\tau,0) \right] \right. \\
&\left. \cdot (t-\tau)\sin 2\pi k\xi \, d\xi \right\}^2 d\tau^{\frac{1}{2}} \\
&+ \left(\int_0^t e^{-\frac{2(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \, d\tau \right)^{\frac{1}{2}} \\
&\cdot \left(\int_0^t \left\{ \frac{1}{1+\varepsilon(2\pi k)^2} \int_0^1 \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f(\xi,\tau,0)(t-\tau)\sin 2\pi k\xi \, d\xi \right\}^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned}
$$

Taking the summation of both sides with respect to k and using the Hölder, Bessel, and Lipschitzs Inequalities yields the following:

$$
\begin{aligned}
\sum_{k=1}^{\infty} \left| u_{2k-1}^{(1)}(t,\varepsilon) \right| \le{}& \sum_{k=1}^{\infty} |\varphi_{2k-1}| + \left(\frac{1}{4\sqrt{6}\pi} + \frac{\pi}{\sqrt{6}} \right) \sum_{k=1}^{\infty} \left| \varphi_{2k}'' \right| \\
&+ \frac{1}{4\sqrt{3}} \left(\int_0^t \int_0^1 b^2(\xi,\tau) \left| u^{(0)}(\tau,\varepsilon) \right|^2 d\xi \, d\tau \right)^{\frac{1}{2}} \\
&+ \frac{1}{4\sqrt{3}} \left(\int_0^t \int_0^1 f^2(\xi,\tau,0) \, d\xi \, d\tau \right)^{\frac{1}{2}} \\
&+ \frac{|t|}{4\sqrt{3}} \left(\int_0^t \int_0^1 b^2(\xi,\tau) \left| u^{(0)}(\tau,\varepsilon) \right|^2 d\xi \, d\tau \right)^{\frac{1}{2}} \\
&+ \frac{|t|}{4\sqrt{3}} \left(\int_0^t \int_0^1 f^2(\xi,\tau,0) \, d\xi \, d\tau \right)^{\frac{1}{2}} \\
&+ \frac{|t|\pi}{\sqrt{3}} \left(\int_0^t \int_0^1 b^2(\xi,\tau) \left| u^{(0)}(\tau,\varepsilon) \right|^2 d\xi \, d\tau \right)^{\frac{1}{2}} \\
&+ \frac{|t|\pi}{2\sqrt{3}} \left(\int_0^t \int_0^1 f^2(\xi,\tau,0) \, d\xi \, d\tau \right)^{\frac{1}{2}}.
\end{aligned}
$$

Taking the maximum of both sides of the last inequality yields the following:

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \max_{0\le t\le T} \left| u_{2k-1}^{(1)}(t,\varepsilon) \right| \\
&\quad \le \sum_{k=1}^{\infty} |\varphi_{2k-1}| + \left(\frac{1+4\pi^2}{4\sqrt{6}} \right) \sum_{k=1}^{\infty} \left| \varphi_{2k}'' \right| \\
&\qquad + \frac{1}{4\sqrt{3}} \left\| b(x,t) \right\|_{L_2(D)} \left\| u^{(0)}(t,\varepsilon) \right\|_B + \frac{1}{4\sqrt{3}} \left\| f(x,t,0) \right\|_{L_2(D)} \\
&\qquad + \frac{|T|}{4\sqrt{3}} \left\| b(x,t) \right\|_{L_2(D)} \left\| u^{(0)}(t,\varepsilon) \right\|_B + \frac{|T|}{4\sqrt{3}} \left\| f(x,t,0) \right\|_{L_2(D)} \\
&\qquad + \frac{|T|\pi}{2\sqrt{3}} \left\| b(x,t) \right\|_{L_2(D)} \left\| u^{(0)}(t,\varepsilon) \right\|_B + \frac{|T|\pi}{2\sqrt{3}} \left\| f(x,t,0) \right\|_{L_2(D)}.
\end{aligned}
$$

Finally we have the following inequality:

$$\begin{aligned}
\|u^{(1)}(t,\varepsilon)\|_B &= 2\max_{0\le t\le T}\left|u_0^{(1)}(t,\varepsilon)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(1)}(t,\varepsilon)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(1)}(t,\varepsilon)\right|\right) \\
&\le 2|\varphi_0| + 4\sum_{k=1}^{\infty}\left(|\varphi_{2k}| + |\varphi_{2k-1}|\right) + \left(\frac{1+4\pi^2}{\sqrt{6}}\right)\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right| \\
&\quad + \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\|b(x,t)\|_{L_2(D)}\|u^{(0)}(t,\varepsilon)\|_B\right) \\
&\quad + \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\|f(x,t,0)\|_{L_2(D)}.
\end{aligned}$$

Hence $u^{(1)}(t,\varepsilon)\in B$. In the same way, for a general value of N we have

$$\begin{aligned}
\|u^{(N)}(t,\varepsilon)\|_B &= 2\max_{0\le t\le T}\left|u_0^{(N)}(t,\varepsilon)\right| + 4\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{2k}^{(N)}(t,\varepsilon)\right| + \max_{0\le t\le T}\left|u_{2k-1}^{(N)}(t,\varepsilon)\right|\right) \\
&\le 2|\varphi_0| + 4\sum_{k=1}^{\infty}\left(|\varphi_{2k}| + |\varphi_{2k-1}|\right) + \left(\frac{1+4\pi^2}{\sqrt{6}}\right)\sum_{k=1}^{\infty}\left|\varphi_{2k}''\right| \\
&\quad + \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\|b(x,t)\|_{L_2(D)}\|u^{(N-1)}(t,\varepsilon)\|_B\right) \\
&\quad + \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\|f(x,t,0)\|_{L_2(D)},
\end{aligned}$$

considering the induction hypothesis that $u^{(N-1)}(t,\varepsilon)\in B$, we deduce that $u^{(N)}(t,\varepsilon)\in B$, and by the principle of mathematical induction we obtain

$$\left\{u(t,\varepsilon)\right\} = \left\{u_0(t,\varepsilon), u_{2k}(t,\varepsilon), u_{2k-1}(t,\varepsilon), k=1,2,\ldots\right\}\in B.$$

Now we prove that the iterations $u^{(N+1)}(t,\varepsilon)$ converge in B, as $N\to\infty$. We have

$$\begin{aligned}
&u^{(1)}(t,\varepsilon) - u^{(0)}(t,\varepsilon) \\
&\quad = 2\left(u_0^{(1)}(t,\varepsilon) - u_0^{(0)}(t,\varepsilon)\right) + 4\sum_{k=1}^{\infty}\left[\left(u_{2k}^{(1)}(t,\varepsilon) - u_{2k}^{(0)}(t,\varepsilon)\right) + \left(u_{2k-1}^{(1)}(t,\varepsilon) - u_{2k-1}^{(0)}(t,\varepsilon)\right)\right] \\
&\quad = 2\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)\xi\,d\xi\,d\tau \\
&\qquad + \frac{4}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)\xi\cos 2\pi k\xi\,d\xi\,d\tau \\
&\qquad + \frac{4}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right)(t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau \\
&\qquad - \frac{4}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right) \\
&\qquad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau.
\end{aligned}$$

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to the right side of (11), respectively, we obtain

$$\begin{aligned}
&\left|u^{(1)}(t,\varepsilon)-u^{(0)}(t,\varepsilon)\right| \\
&\quad\leq 2\left|u_0^{(1)}(t,\varepsilon)-u_0^{(0)}(t,\varepsilon)\right|+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(1)}(t,\varepsilon)-u_{2k}^{(0)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(1)}(t,\varepsilon)-u_{2k-1}^{(0)}(t,\varepsilon)\right|\right) \\
&\quad\leq \left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t,\varepsilon)\right| \\
&\qquad+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)\,d\xi\,d\tau\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\begin{aligned}
A_T&=\left[\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\left|u^{(0)}(t,\varepsilon)\right|\right. \\
&\qquad\left.+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)\,d\xi\,d\tau\right)^{\frac{1}{2}}\right], \\
u^{(2)}(t,\varepsilon)-u^{(1)}(t,\varepsilon)&=2\left(u_0^{(2)}(t,\varepsilon)-u_0^{(1)}(t,\varepsilon)\right) \\
&\qquad+4\sum_{k=1}^{\infty}\left[\left(u_{2k}^{(2)}(t,\varepsilon)-u_{2k}^{(1)}(t,\varepsilon)\right)+\left(u_{2k-1}^{(2)}(t,\varepsilon)-u_{2k-1}^{(1)}(t,\varepsilon)\right)\right].
\end{aligned}$$

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to the right hand side of (9), respectively, we obtain

$$\begin{aligned}
&\left|u^{(2)}(t,\varepsilon)-u^{(1)}(t,\varepsilon)\right| \\
&\quad\leq 2\left|u_0^{(2)}(t,\varepsilon)-u_0^{(1)}(t,\varepsilon)\right| \\
&\qquad+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t,\varepsilon)-u_{2k}^{(1)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(2)}(t,\varepsilon)-u_{2k-1}^{(1)}(t,\varepsilon)\right|\right) \\
&\quad\leq \left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}A_T, \\
&u^{(3)}(t,\varepsilon)-u^{(2)}(t,\varepsilon)=2\left(u_0^{(3)}(t,\varepsilon)-u_0^{(2)}(t,\varepsilon)\right) \\
&\qquad+4\sum_{k=1}^{\infty}\left[\left(u_{2k}^{(3)}(t,\varepsilon)-u_{2k}^{(2)}(t,\varepsilon)\right)+\left(u_{2k-1}^{(3)}(t,\varepsilon)-u_{2k-1}^{(2)}(t,\varepsilon)\right)\right].
\end{aligned}$$

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to the right hand side of (9), respectively, we obtain

$$\begin{aligned}
&\left|u^{(3)}(t,\varepsilon)-u^{(2)}(t,\varepsilon)\right| \\
&\quad\leq 2\left|u_0^{(3)}(t,\varepsilon)-u_0^{(2)}(t,\varepsilon)\right|+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(3)}(t,\varepsilon)-u_{2k}^{(2)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(3)}(t,\varepsilon)-u_{2k-1}^{(2)}(t,\varepsilon)\right|\right)
\end{aligned}$$

$$\leq \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}}{3}\right)\left(\int_0^t \int_0^1 b^2(\xi,\tau)\left|u^{(2)}(t,\varepsilon) - u^{(1)}(t,\varepsilon)\right|^2 d\xi\, d\tau\right)^{\frac{1}{2}}$$

$$\leq \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}}{3}\right)^2 A_T$$

$$\cdot \left[\int_0^t \int_0^1 b^2(\xi,\tau)\left(\int_0^\tau \int_0^1 b^2(\xi_1,\tau_1)\, d\xi_1\, d\tau_1\right) d\xi\, d\tau\right]^{\frac{1}{2}}$$

$$\leq \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}}{3}\right)^2 A_T \frac{1}{\sqrt{2}}\left[\left(\int_0^t \int_0^1 b^2(\xi,\tau)\, d\xi\, d\tau\right)^2\right]^{\frac{1}{2}}.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t,\varepsilon) - u^{(N)}(t,\varepsilon)\right|$$

$$\leq 2\left|u_0^{(N+1)}(t,\varepsilon) - u_0^{(N)}(t,\varepsilon)\right|$$

$$+ 4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t,\varepsilon) - u_{2k}^{(N)}(t,\varepsilon)\right| + \left|u_{2k-1}^{(N+1)}(t,\varepsilon) - u_{2k-1}^{(N)}(t,\varepsilon)\right|\right)$$

$$\leq \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t \int_0^1 b^2(\xi,\tau)\, d\xi\, d\tau\right)^2\right]^{\frac{N}{2}}$$

$$\leq \left(2\sqrt{T} + \frac{\sqrt{3}}{3} + \frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N A_T \frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|_{L_2(D)}^N. \tag{12}$$

Then the last inequality shows that the $u^{(N+1)}(t,\varepsilon)$ converge in B.

Now let us show $\lim_{N\to\infty} u^{(N+1)}(t,\varepsilon) = u(t,\varepsilon)$. Noting that

$$2\left|\int_0^t \int_0^1 \left\{f\left[\xi,\tau,Au(\xi,\tau,\varepsilon)\right] - f\left[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\right]\right\}\xi\, d\xi\, d\tau\right|$$

$$+ 4\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\right.$$

$$\left.\cdot \left\{f\left[\xi,\tau,Au(\xi,\tau,\varepsilon)\right] - f\left[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\right]\right\}\sin 2\pi k\xi\, d\xi\, d\tau\right|$$

$$+ 4\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\right.$$

$$\left.\cdot \left\{f\left[\xi,\tau,Au(\xi,\tau,\varepsilon)\right] - f\left[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\right]\right\}\xi\cos 2\pi k\xi\, d\xi\, d\tau\right|$$

$$+ 4\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\right.$$

$$\left.\cdot (t-\tau)\left\{f\left[\xi,\tau,Au(\xi,\tau,\varepsilon)\right] - f\left[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\right]\right\}\sin 2\pi k\xi\, d\xi\, d\tau\right|$$

$$+ 4\left|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\right.$$

$$\cdot (t-\tau)\bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f\bigl[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr]\bigr\}\sin 2\pi k\xi\, d\xi\, d\tau\Bigg|$$

$$\leq \left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\bigl\|b(x,t)\bigr\|_{L_2(D)}\bigl\|u(\tau,\varepsilon)-u^{(N)}(\tau,\varepsilon)\bigr\|_B,$$

it follows that if we prove $\lim_{N\to\infty}\|u(\tau,\varepsilon)-u^{(N)}(\tau,\varepsilon)\|_B=0$, then we may deduce that $u(t,\varepsilon)$ satisfies (9).

To this aim we estimate the difference $\|u(t,\varepsilon)-u^{(N+1)}(t,\varepsilon)\|_B$; after some transformation we obtain

$$\begin{aligned}
&\bigl|u(t,\varepsilon)-u^{(N+1)}(t,\varepsilon)\bigr| \\
&\quad = 2\bigl|u_0(t,\varepsilon)-u_0^{(N+1)}(t,\varepsilon)\bigr| \\
&\qquad + 4\sum_{k=1}^{\infty}\bigl(\bigl|u_{2k}(t,\varepsilon)-u_{2k}^{(N+1)}(t,\varepsilon)\bigr|+\bigl|u_{2k-1}(t,\varepsilon)-u_{2k-1}^{(N+1)}(t,\varepsilon)\bigr|\bigr) \\
&\quad \leq 2\left|\int_0^t\int_0^1\bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f\bigl[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr]\bigr\}\xi\, d\xi\, d\tau\right| \\
&\qquad + 4\Bigg|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \\
&\qquad \cdot \bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f(\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon))\bigr]\bigr\}\sin 2\pi k\xi\, d\xi\, d\tau\Bigg| \\
&\qquad + 4\Bigg|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \\
&\qquad \cdot \bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f\bigl[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr]\bigr\}\xi\cos 2\pi k\xi\, d\xi\, d\tau\Bigg| \\
&\qquad + 4\Bigg|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}} \\
&\qquad \cdot (t-\tau)\bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f\bigl[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr]\bigr\}\sin 2\pi k\xi\, d\xi\, d\tau\Bigg| \\
&\qquad + 4\Bigg|\sum_{k=1}^{\infty}\frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2} \\
&\qquad \cdot (t-\tau)\bigl\{f\bigl[\xi,\tau,Au(\xi,\tau,\varepsilon)\bigr]-f\bigl[\xi,\tau,Au^{(N)}(\xi,\tau,\varepsilon)\bigr]\bigr\}\sin 2\pi k\xi\, d\xi\, d\tau\Bigg|.
\end{aligned}$$

Adding and subtracting $f(\xi,\tau,Au^{(N+1)}(\xi,\tau,\varepsilon))$ under the appropriate integrals to the right hand side of the inequality we obtain

$$\begin{aligned}
\bigl|u(t,\varepsilon)-u^{(N+1)}(t,\varepsilon)\bigr| &\leq \left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right) \\
&\quad \cdot\left\{\int_0^t\int_0^1 b^2(\xi,\tau)\bigl|u(\tau,\varepsilon)-u^{(N+1)}(\tau,\varepsilon)\bigr|^2\, d\xi\, d\tau\right\}^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right) \\
&\cdot\left\{\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right\}^{\frac{1}{2}}\left\|u^{(N+1)}(t,\varepsilon)-u^{(N)}(t,\varepsilon)\right\|_B.
\end{aligned}
$$

Applying the Gronwall Inequality to the last inequality and using inequality (11), we have

$$
\begin{aligned}
&\left\|u(t,\varepsilon)-u^{(N+1)}(t,\varepsilon)\right\|_B \\
&\quad\leq\sqrt{\frac{2}{N!}}A_T\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^{(N+1)}\left\|b(x,t)\right\|_{L_2(D)}^{(N+1)} \\
&\qquad\cdot\exp\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2\left\|b(x,t))\right\|_{L_2(D)}^2.
\end{aligned}
\tag{13}
$$

For the uniqueness, we assume that problem (1)-(4) has two solutions u, v. Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to the right hand side of $|u(t,\varepsilon)-v(t,\varepsilon)|$, respectively, we obtain

$$
\begin{aligned}
&\left|u(t,\varepsilon)-v(t,\varepsilon)\right|^2 \\
&\quad\leq\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2\int_0^t\int_0^1 b^2(\xi,\tau)\left|u(\tau,\varepsilon)-v(\tau,\varepsilon)\right|^2 d\xi\,d\tau,
\end{aligned}
$$

applying the Gronwall Inequality to the last inequality we have

$$
u(t,\varepsilon)=v(t,\varepsilon).
$$

The theorem is proved. □

3 Solution of problem (1)-(4)

Using the solution of the system (9) we compose the series

$$
2u_0(t,\varepsilon)+4\sum_{k=1}^{\infty}\left[u_{2k}(t,\varepsilon)(1-x)\sin 2\pi kx+u_{2k-1}(t,\varepsilon)\cos 2\pi kx\right].
$$

It is evident that the series converges uniformly on D. Therefore the sum

$$
u(\xi,\tau,\varepsilon)=2u_0(\tau,\varepsilon)+4\sum_{k=1}^{\infty}\left[u_{2k}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi+u_{2k-1}(\tau,\varepsilon)\cos 2\pi k\xi\right],
$$

is continuous on D. We have

$$
u_l(\xi,\tau,\varepsilon)=2u_0(\tau,\varepsilon)+4\sum_{k=1}^{l}\left[u_{2k}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi+u_{2k-1}(\tau,\varepsilon)\cos 2\pi k\xi\right].
\tag{14}
$$

From the conditions of Theorem 4 and from

$$
\lim_{l\to\infty}u_l(\xi,\tau,\varepsilon)=u(\xi,\tau,\varepsilon),
$$

it follows that

$$\lim_{l\to\infty} f\bigl(\xi,\tau,u_l(\tau,\xi,\varepsilon)\bigr) = f\bigl(\xi,\tau,u(\xi,\tau,\varepsilon)\bigr).$$

Using

$$u_l(\xi,\tau,\varepsilon)$$

and

$$\varphi_l(x) = 2\varphi_0 + 4\sum_{k=1}^{l}\bigl[\varphi_{2k}(1-x)\sin 2\pi kx + \varphi_{2k-1}\cos 2\pi kx\bigr], \quad x\in[0,1],$$

on the left hand side of (5) we denote the obtained expression by J_l:

$$J_l = \int_0^T\int_0^1\biggl[\biggl(\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2}\varepsilon\frac{\partial^3 v}{\partial x^2\,\partial t}\biggr)u_{(l)}(x,t,\varepsilon) + f\bigl(x,t,u_{(l)}(x,t,\varepsilon)\bigr)v(x,t)\biggr]\,dx\,dt + \int_0^1 \varphi_{(l)}(x)\biggl[v(x,0) - \varepsilon\frac{\partial^3 v(x,0)}{\partial x^2}\biggr]\,dx. \tag{15}$$

Applying the integration by parts the formula on the right hand side, the last equation, and using the conditions of Theorem 4, we can show that

$$\lim_{l\to\infty} J_l = 0.$$

This shows that the function $u(x,t,\varepsilon)$ is a generalized (weak) solution of problem (1)-(4).

The following theorem shows the existence and uniqueness results for the generalized solutions of problem (1)-(4).

Theorem 5 *Under the assumptions of Theorem* 4, *problem* (1)-(4) *possesses a unique generalized solution* $u = u(x,t) \in C(\bar{D})$ *in the following form*:

$$u(x,t,\varepsilon) = 2u_0(t,\varepsilon) + 4\sum_{k=1}^{\infty}\bigl[u_{2k}(t,\varepsilon)(1-x)\sin 2\pi kx + u_{2k-1}(t,\varepsilon)\cos 2\pi kx\bigr].$$

4 Continuous dependence upon the data

In this section, we shall prove the continuous dependence of the solution $u = u(x,t,\varepsilon)$ using an iteration method.

Theorem 6 *Under the conditions of Theorem* 4, *the solution* $u = u(x,t,\varepsilon)$ *depends continuously upon the data*.

Proof Let $\phi = \{\varphi, f\}$ and $\overline{\phi} = \{\overline{\varphi}, \overline{f}\}$ be two sets of data which satisfy the conditions of Theorem 4. Let $u = u(x,t,\varepsilon)$ and $v = v(x,t,\varepsilon)$ be the solutions of problem (1)-(4) corresponding to the data ϕ and $\overline{\phi}$, respectively, and

$$\bigl|f(t,x,0) - \overline{f}(t,x,0)\bigr| \le \varepsilon_1, \quad \text{for } \varepsilon_1 \ge 0.$$

The solution $v = v(x,t,\varepsilon)$ is defined by the following forms, respectively:

$$\begin{aligned}
v_0(t,\varepsilon) &= \overline{\varphi}_0 + \int_0^t \overline{f}_0(\tau)\,d\tau,\\
v_{2k}(t,\varepsilon) &= \overline{\varphi}_{2k} e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t \overline{f}_{2k}(\tau) e^{-\frac{(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\,d\tau,\\
v_{2k-1}(t,\varepsilon) &= \left(\overline{\varphi}_{2k-1} - \overline{\varphi}_{2k} + \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\overline{\varphi}_{2k}\right) e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}} + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\\
&\quad \cdot \left[\overline{f}_{2k-1}(\tau) - \left(1 - \frac{(2\pi k)^2}{1+\varepsilon(2\pi k)^2}\right)(t-\tau)\overline{f}_{2k}(\tau)\right] d\tau,
\end{aligned}$$

where

$$\overline{\varphi}_k = \int_0^1 \overline{\varphi}(x) Y_k(x)\,dx, \qquad \overline{f}_k(x) = \int_0^1 \overline{f}(x,t,u) Y_k(x)\,dx.$$

For simplicity, let us write

$$\begin{aligned}
Av^{(N)}(\xi,\tau) &= 2v_0^{(N)}(\tau,\varepsilon) + 4\sum_{k=1}^{\infty}\left(v_{2k}^{(N)}(\tau,\varepsilon)(1-\xi)\sin 2\pi k\xi + v_{2k-1}^{(N)}(\tau)\cos 2\pi k\xi\right),\\
v_0^{(N+1)}(t,\varepsilon) &= v_0^{(0)}(t,\varepsilon) + \int_0^t\int_0^1 \overline{f}\left(\xi,\tau,Av^{(N)}(\xi,\tau,\varepsilon)\right)\xi\,d\xi\,d\tau,\\
v_{2k}^{(N+1)}(t,\varepsilon) &= v_{2k}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\\
&\quad \cdot \overline{f}\left(\xi,\tau,Av^{(N)}(\xi,\tau,\varepsilon)\right)\sin 2\pi k\xi\,d\xi\,d\tau,\\
v_{2k-1}^{(N+1)}(t,\varepsilon) &= v_{2k-1}^{(0)}(t,\varepsilon) + \frac{1}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\\
&\quad \cdot \overline{f}\left(\xi,\tau,Av^{(N)}(\xi,\tau,\varepsilon)\right)\xi\cos 2\pi k\xi\,d\xi\,d\tau\\
&\quad - \frac{4\pi k}{1+\varepsilon(2\pi k)^2}\int_0^t\int_0^1 e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\overline{f}\left(\xi,\tau,Av^{(N)}(\xi,\tau,\varepsilon)\right)\\
&\quad \cdot (t-\tau)\sin 2\pi k\xi\,d\xi\,d\tau,
\end{aligned}$$

where $v_0^{(0)}(t) = \overline{\varphi}_0$, $v_{2k}^{(0)}(t) = \overline{\varphi}_{2k} e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$, $v_{2k-1}^{(0)}(t) = (\overline{\varphi}_{2k-1} - 4\pi k\overline{\varphi}_{2k}) e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}$. From the condition of the theorem we have $v^{(0)}(t,\varepsilon) \in B$. We will prove that the other sequential approximations satisfy this condition. First of all, we consider $u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon)$:

$$\begin{aligned}
&u^{(1)}(t,\varepsilon) - v^{(1)}(t,\varepsilon)\\
&\quad = 2\left(u_0^{(1)}(t,\varepsilon) - v_0^{(1)}(t,\varepsilon)\right) + 4\sum_{k=1}^{\infty}\left[\left(u_{2k}^{(1)}(t,\varepsilon) - v_{2k}^{(1)}(t,\varepsilon)\right) + \left(u_{2k-1}^{(1)}(t,\varepsilon) - v_{2k-1}^{(1)}(t,\varepsilon)\right)\right]\\
&\quad = (\varphi_0 - \overline{\varphi}_0) + 2\int_0^t\int_0^1\left[f\left(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\right) - \overline{f}\left(\xi,\tau,Av^{(0)}(\xi,\tau,\varepsilon)\right)\right]\xi\,d\xi\,d\tau\\
&\qquad + 4(\varphi_{2k} - \overline{\varphi}_{2k}) e^{-(2\pi k)^2 t}
\end{aligned}$$

$$
\begin{aligned}
&+4\int_0^t\int_0^1\big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\big)-\overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau,\varepsilon)\big)\big]e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau\\
&+\big(4(\varphi_{2k-1}-\overline{\varphi}_{2k-1})-16\pi k(\varphi_{2k}-\overline{\varphi}_{2k})\big)e^{-\frac{(2\pi k)^2 t}{1+\varepsilon(2\pi k)^2}}\\
&+4\int_0^t\int_0^1\big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\big)-\overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau,\varepsilon)\big)\big]\\
&\cdot e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\xi\cos 2\pi k\xi\,d\xi\,d\tau\\
&-16\pi k\int_0^t\int_0^1(t-\tau)\big[f\big(\xi,\tau,Au^{(0)}(\xi,\tau,\varepsilon)\big)-\overline{f}\big(\xi,\tau,Av^{(0)}(\xi,\tau,\varepsilon)\big)\big]\\
&\cdot e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau.
\end{aligned}
$$

Adding and subtracting

$$
\begin{aligned}
&2\int_0^t\int_0^1 f(\xi,\tau,0)\xi\,d\xi\,d\tau,\qquad 2\int_0^t\int_0^1 \overline{f}(\xi,\tau,0)\xi\,d\xi\,d\tau,\\
&4\int_0^t\int_0^1 f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau,\\
&4\int_0^t\int_0^1 \overline{f}(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau,\\
&4\int_0^t\int_0^1 f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\xi\cos 2\pi k\xi\,d\xi\,d\tau,\\
&4\int_0^t\int_0^1 \overline{f}(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\xi\cos 2\pi k\xi\,d\xi\,d\tau,\\
&-16\pi k\int_0^t\int_0^1 f(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau,\\
&-16\pi k\int_0^t\int_0^1 \overline{f}(\xi,\tau,0)e^{\frac{-(2\pi k)^2(t-\tau)}{1+\varepsilon(2\pi k)^2}}\sin 2\pi k\xi\,d\xi\,d\tau
\end{aligned}
$$

to both sides of the last equation.

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to the right side of (11), respectively, we obtain

$$
\begin{aligned}
&\big|u^{(1)}(t,\varepsilon)-v^{(1)}(t,\varepsilon)\big|\\
&\quad\le 2\big|u_0^{(1)}(t,\varepsilon)-v_0^{(1)}(t,\varepsilon)\big|+4\sum_{k=1}^{\infty}\big(\big|u_{2k}^{(1)}(t,\varepsilon)-v_{2k}^{(1)}(t,\varepsilon)\big|+\big|u_{2k-1}^{(1)}(t,\varepsilon)-v_{2k-1}^{(1)}(t,\varepsilon)\big|\big)\\
&\quad\le 2\max|\varphi_0-\overline{\varphi}_0|+4\sum_{k=1}^{\infty}\max|\varphi_{2k}-\overline{\varphi}_{2k}|+\max|\varphi_{2k-1}-\overline{\varphi}_{2k-1}|\\
&\qquad+\frac{2\sqrt{6}|T|}{3}\sum_{k=1}^{\infty}\max\big|\varphi_{2k}''-\overline{\varphi}_{2k}''\big|\\
&\qquad\cdot\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\big|u^{(0)}(t,\varepsilon)\big|
\end{aligned}
$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}\left|v^{(0)}(t,\varepsilon)\right|$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 f^2(\xi,\tau,0)-\overline{f}^2(\xi,\tau,0)\,d\xi\,d\tau\right)^{\frac{1}{2}},$$

$$\|\varphi-\overline{\varphi}\|=2\max|\varphi_0-\overline{\varphi}_0|+4\sum_{k=1}^{\infty}\left[\max|\varphi_{2k}-\overline{\varphi}_{2k}|+\max|\varphi_{2k-1}-\overline{\varphi}_{2k-1}|\right]$$

$$+\left(\frac{1+4\pi^2}{\sqrt{6}}\right)\sum_{k=1}^{\infty}\max\left|\varphi''_{2k}-\overline{\varphi}''_{2k}\right|.$$

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to $u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)$, respectively, we obtain

$$\begin{aligned}
&\left|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)\right| \\
&\quad\le 2\left|u_0^{(2)}(t,\varepsilon)-v_0^{(2)}(t,\varepsilon)\right|+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(2)}(t,\varepsilon)-v_{2k}^{(2)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(2)}(t,\varepsilon)-v_{2k-1}^{(2)}(t,\varepsilon)\right|\right) \\
&\quad\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}A_T \\
&\qquad+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^{\frac{1}{2}}A_T.
\end{aligned}$$

Applying the Cauchy Inequality, the Hölder Inequality, the Lipschitzs Condition, and the Bessel Inequality to $u^{(3)}(t,\varepsilon)-v^{(3)}(t,\varepsilon)$, respectively, we obtain

$$\begin{aligned}
&\left|u^{(3)}(t,\varepsilon)-v^{(3)}(t,\varepsilon)\right| \\
&\quad\le 2\left|u_0^{(3)}(t,\varepsilon)-v_0^{(3)}(t,\varepsilon)\right|+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(3)}(t,\varepsilon)-v_{2k}^{(3)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(3)}(t,\varepsilon)-v_{2k-1}^{(3)}(t,\varepsilon)\right|\right) \\
&\quad\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right) \\
&\qquad\cdot\left(\int_0^t\int_0^1 b^2(\xi,\tau)\left|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)\right|^2 d\xi\,d\tau\right)^{\frac{1}{2}} \\
&\qquad+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right) \\
&\qquad\cdot\left(\int_0^t\int_0^1\overline{b}^2(\xi,\tau)\left|u^{(2)}(t,\varepsilon)-v^{(2)}(t,\varepsilon)\right|^2 d\xi\,d\tau\right)^{\frac{1}{2}} \\
&\quad\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2 A_T \\
&\qquad\cdot\left[\int_0^t\int_0^1 b^2(\xi,\tau)\left(\int_0^\tau\int_0^1 b^2(\xi_1,\tau_1)\,d\xi_1\,d\tau_1\right)d\xi\,d\tau\right]^{\frac{1}{2}} \\
&\qquad+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2 A_T
\end{aligned}$$

$$\cdot\left[\int_0^t\int_0^1 b^2(\xi,\tau)\left(\int_0^\tau\int_0^1 \overline{b}^2(\xi_1,\tau_1)\,d\xi_1\,d\tau_1\right)d\xi\,d\tau\right]^{\frac{1}{2}}$$

$$\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{1}{2}}$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^2 A_T\frac{1}{\sqrt{2}}\left[\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{1}{2}}.$$

In the same way, for a general value of N we have

$$\left|u^{(N+1)}(t,\varepsilon)-v^{(N+1)}(t,\varepsilon)\right|$$

$$\le 2\left|u_0^{(N+1)}(t,\varepsilon)-v_0^{(N+1)}(t,\varepsilon)\right|$$

$$+4\sum_{k=1}^{\infty}\left(\left|u_{2k}^{(N+1)}(t,\varepsilon)-v_{2k}^{(N+1)}(t,\varepsilon)\right|+\left|u_{2k-1}^{(N+1)}(t,\varepsilon)-v_{2k-1}^{(N+1)}(t,\varepsilon)\right|\right)$$

$$\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N \frac{A_T}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}}$$

$$\le\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N A_T\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|_{L_2(D)}^N$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N A_T\frac{1}{\sqrt{N!}}\left\|\overline{b}(x,t)\right\|_{L_2(D)}^N$$

$$\le A_T\cdot a_N=a_N\left(\|\varphi-\overline{\varphi}\|+C(t)+M_1\|f-\overline{f}\|\right)$$

where

$$a_N=\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N \frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 b^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}}$$

$$+\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N \frac{1}{\sqrt{N!}}\left[\left(\int_0^t\int_0^1 \overline{b}^2(\xi,\tau)\,d\xi\,d\tau\right)^2\right]^{\frac{N}{2}}$$

and

$$M_1=\left(2\sqrt{T}+\frac{\sqrt{3}}{3}+\frac{(1+2\sqrt{2}\pi)\sqrt{3}|T|}{3}\right)^N.$$

(The sequence a_N is convergent; then we can write $a_N\le M$, $\forall N$.)

It follows from the estimation [1] that $\lim_{N\to\infty}u^{(N+1)}(t)=u(t)$; then let $N\to\infty$ for the last equation,

$$\left|u(t)-v(t)\right|\le M\|\varphi-\overline{\varphi}\|+M_2\|f-\overline{f}\|,$$

where $M_2=M\cdot M_1$. If $\|f-\overline{f}\|\le\varepsilon$ and $\|\varphi-\overline{\varphi}\|\le\varepsilon$, then $|u(t,\varepsilon)-v(t,\varepsilon)|\le\varepsilon$. □

5 Numerical procedure for the nonlinear problem (1)-(4)

We construct an iteration algorithm for the linearization of problem (1)-(4) for $\varepsilon = 0$:

$$\frac{\partial u^{(n)}}{\partial t} - \frac{\partial^2 u^{(n)}}{\partial x^2} = f\left(x,t,u^{(n-1)}\right), \quad (x,t) \in D, \tag{16}$$

$$u^{(n)}(0,t) = u^{(n)}(1,t), \quad t \in [0,T], \tag{17}$$

$$u_x^{(n)}(1,t) = 0, \quad t \in [0,T], \tag{18}$$

$$u^{(n)}(x,0) = \varphi(x), \quad x \in [0,1]. \tag{19}$$

Let $u^{(n)}(x,t) = v(x,t)$ and $f(x,t,u^{(n-1)}) = \widetilde{f}(x,t)$. Then problem (16)-(19) can be written as a linear problem:

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \widetilde{f}(x,t), \quad (x,t) \in D, \tag{20}$$

$$v(0,t) = v(1,t), \quad t \in [0,T], \tag{21}$$

$$v_x(1,t) = 0, \quad t \in [0,T], \tag{22}$$

$$v(x,0) = \varphi(x), \quad x \in [0,1]. \tag{23}$$

We use the finite difference method to solve (20)-(23).

We subdivide the intervals $[0,1]$ and $[0,T]$ into M and N subintervals of equal lengths, $h = \frac{1}{M}$ and $\tau = \frac{T}{N}$, respectively.

Then we add the line $x = (M+1)h$ to generate the fictitious point needed for the second boundary condition.

We choose the implicit scheme, which is absolutely stable and has a second order accuracy in h and a first order accuracy in τ.

The implicit monotone difference scheme for (20)-(23) is as follows:

$$\frac{v_{i,j+1} - v_{i,j}}{\tau} = \frac{a^2}{h^2}(v_{i-1,j+1} - 2v_{i,j+1} + v_{i+1,j+1}) + \widetilde{f}_{i,j+1},$$

$$v_{i,0} = \varphi_i, \qquad v_{0,j} = v_{M,j}, \qquad v_{x,M_j} = 0,$$

where $0 \le i \le M$ and $1 \le j \le N$ are the indices for the spatial and time steps, respectively, $v_{i,j}$ is the approximation to $v(x_i,t_j)$, $f_{i,j} = f(x_i,t_j)$, $\varphi_i = \varphi(x_i)$, $x_i = ih$, $t_j = j\tau$ [10].

At the $t = 0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

6 Numerical examples

In this section, we will consider an example of numerical solution of the nonlinear problem (1)-(4).

These problems were solved by applying the iteration scheme and the finite difference scheme which were explained in Section 4. The condition

$$\operatorname{error}(i,j) := \left\| u_{i,j}^{(n+1)} - u_{i,j}^{(n)} \right\|_\infty$$

with $\operatorname{error}(i,j) := 10^{-3}$ was used as a stopping criterion for the iteration process.

Figure 1 The exact and numerical solutions of *u*(*x*, 1). The exact and numerical solutions of $u(x,1)$ for $\varepsilon = 0$, the exact solution is shown with a dashed line.

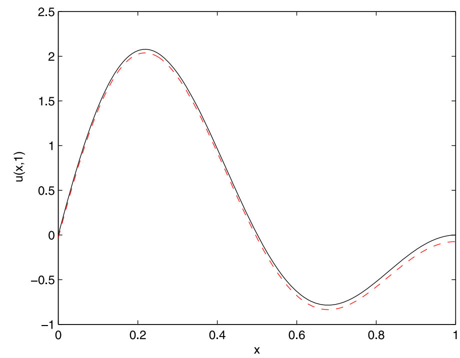


Figure 2 The exact and numerical solutions of *u*(*x*, 1). The exact and numerical solutions of $u(x,1)$ for $\varepsilon = 0.05$, the exact solution is shown with a dashed line.

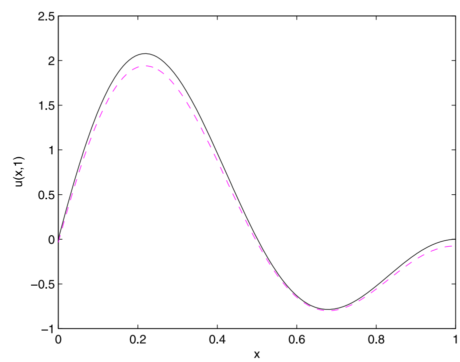
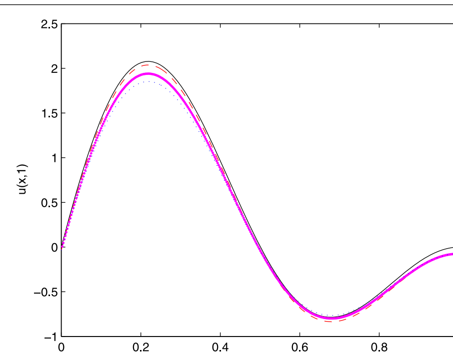


Figure 3 The exact and numerical solutions of *u*(*x*, 1). The exact and numerical solutions of $u(x,1)$, (−−) for $\varepsilon = 0$, (··) for $\varepsilon = 0.05$, (..) for $\varepsilon = 0.1$, the exact solution is shown with a dashed line.



Example 1 Consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \varepsilon \frac{\partial^3 u}{\partial x^2 \, \partial t} = \bigl(1 + (2\pi)^2(1+\varepsilon)\bigr)u + (1+\varepsilon)\bigl(4\pi \cos(2\pi x)\exp(t)\bigr),$$
$$u(x,0) = (1-x)\sin 2\pi x, \quad x \in [0,\pi],$$
$$u(0,t) = u(1,t), \quad t \in [0,T], \qquad u_x(1,t) = 0, \quad t \in [0,T].$$

It is easy to check that the analytical solution of this problem is

$$u(x,t) = e^t(1-x)\sin 2\pi x.$$

The comparisons between the analytical solution and the numerical finite difference solution for different ε values when $T = 1$ are shown in Figures 1 and 2.

We show in Figure 3 the analytical solution for $\varepsilon = 0$ and the numerical solution for $\varepsilon = 0$, $\varepsilon = 0,1$, $\varepsilon = 0,05$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Kocaeli University, Kocaeli, 41380, Turkey. [2]Department of Management Information Systems, Kadir Has University, Istanbul, 34083, Turkey.

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