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# Stabilization of company's income modeled by a system of discrete stochastic equations

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**Abstract**

The paper deals with a system of difference equations where the coefficients depend on Markov chains. The functional equations for a particular density and the moment equations for the system are derived and used in the investigation of mode stability of company's income. An application of the results is illustrated by two models.

## 1 Introduction

As early as the beginning of the 20th century, it was discovered that, even in sequences of equally distributed random variables, a marginal distribution may quite naturally occur other than normal. Most of the underlying laws behind such occurrences can only be understood on the basis of the theory of Markov chains. A random process is called a Markov chain if the way it passes from one state to another depends only on the current state regardless of the states preceding it. The set of states is finite or countable. Using stochastic approach, one can investigate a number of aspects relating to a variety of phenomena in finance and economics.

In terms of the modern theory of finance [1–6], efficient ways must be found for controlling the financial resources using such categories as 'time', 'profit', and 'risk'.

With a stochastic approach, models describing the profit of a company may be created using Markov chains. The first such financial models were developed in the second half of the 20th century [7, 8].

The paper constructs mathematical models of complex economic systems working in conditions of uncertainty. Incomplete or distorted information, too few observations, structure changing over time, stochastic nature of the impact of the external environment, these and other factors generate conditions of uncertainty for the system. The problems encountered in creating and solving mathematical models are caused by the fact that the input-output data type contains nonlinearities and perturbations. The paper focuses on the construction of moment equations to determine the mean value of the guaranteed profit of a company. The theoretical results are applied to two models of the profit of a company.

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## 2 Moment equations

Let $(\Omega, \mathcal{F}, F, \mathbb{P})$ be a filtered probability space (stochastic basis) consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration

$$F = \{\mathcal{F}_t, \forall t \geq 0\} \subset \mathcal{F}$$

(for definitions see, *e.g.*, [9]). The space $\Omega$ is called a sample space, $\mathcal{F}$ is the set of all possible events (a $\sigma$-algebra) that may occur to moment $t$, and $\mathbb{P}$ is some probability measure on $\Omega$. Such a random space plays a fundamental role in the construction of models in economics, finance *etc.*

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider an initial problem formulated for the stochastic dynamic system with random coefficients in the form

$$x_{n+1} = A(\xi_{n+1}, \xi_n)x_n + B(\xi_{n+1}, \xi_n), \quad n = 1, 2, \dots, \tag{1}$$

$$x_0 = \varphi(\omega), \tag{2}$$

where $A$ is an $m \times m$ matrix with random elements, $B$ is an $m$-dimensional column vector with random elements, $\xi_n$ is the Markov chain of a finite number of states $\theta_1, \theta_2, \dots, \theta_q$ with probabilities $p_k(n) = P\{\xi_n = \theta_k\}$, $k = 1, 2, \dots, q$, $n = 1, 2, \dots$, that satisfy the system of difference equations

$$p(n+1) = \Pi p(n), \tag{3}$$

$p(n) = (p_1(n), p_2(n), \dots, p_k(n))^T$, $\Pi = (\pi_{ks})_{k,s=1}^{q}$ is a $q \times q$ transition matrix and $\varphi \colon \Omega \to \mathbb{R}^m$.

If the random variable $\xi_{n+1}$ is in state $\theta_k$, $k = 1, 2, \dots, q$, and the random variable $\xi_n$ is in state $\theta_s$, $s = 1, 2, \dots, q$, we denote

$$A_{ks} = A(\theta_k, \theta_s), \qquad B_{ks} = B(\theta_k, \theta_s), \quad k, s = 1, 2, \dots, q,$$

and assume that there exist inverse matrices $A_{ks}^{-1}$.

The state $m$-dimensional column vector-function $x_n$, $n = 1, 2, \dots$, is called a solution of system (1) within the meaning of a strong solution if it satisfies (1) with initial condition (2) [10].

Our task is to derive the moment equations of system (1) to be used for determining the mode stability of the income of a company.

We define the moments of the first and second order of a solution $x_n$, $n = 1, 2, \dots$, of (1) before deriving the moment equations.

In the sequel, $\mathbb{E}_m$ denotes an $m$-dimensional Euclidean space, $m$-dimensional row vector-functions $f_k(n, z)$, $n = 1, 2, \dots$, $k = 1, 2, \dots, q$, $z \in \mathbb{E}_m$ are the particular probability density functions of $x_n$, $n = 1, 2, \dots$, determined by the formula (see in [11])

$$\int_{\mathbb{E}_m} f_k(n, z)\, dz = P\{x_n \in \mathbb{E}_m, \xi_n = \theta_k\}, \tag{4}$$

and they satisfy the following equations:

$$f_k(n+1, z) = \sum_{s=1}^{q} \pi_{ks} f_s\bigl(n, A_{ks}^{-1}(z - B_{ks})\bigr) \det A_{ks}^{-1}. \tag{5}$$

**Definition 1** The vector function

$$E^{(1)}\{x_n\} = \sum_{k=1}^{q} E_k^{(1)}\{x_n\},$$

where

$$E_k^{(1)}\{x_n\} = \int_{\mathbb{E}_m} z f_k(n,z)\,dz, \quad k = 1,2,\dots,q, \tag{6}$$

is called moment of the first order for a solution $x_n$, $n = 1,2,\dots,$ of (1). The values $E_k^{(1)}\{x_n\}$, $k = 1,2,\dots,q$, are called particular moments of the first order.

**Definition 2** The matrix function

$$E^{(2)}\{x_n\} = \sum_{k=1}^{q} E_k^{(2)}\{x_n\},$$

where

$$E_k^{(2)}\{x_n\} = \int_{\mathbb{E}_m} zz^* f_k(n,z)\,dz, \quad k = 1,2,\dots,q, \tag{7}$$

is called moment of the second order for a solution $x_n$, $n = 1,2,\dots,$ of (1). The values $E_k^{(2)}\{x_n\}$, $k = 1,2,\dots,q$, are called particular moments of the second order.

**Theorem 1** *Systems of moment equations of the first or second orders for a solution $x_n$, $n = 1,2,\dots,$ of (1) are of the form*

$$E_k^{(1)}\{x_{n+1}\} = \sum_{s=1}^{q} \pi_{ks}\bigl(A_{ks}E_k^{(1)}\{x_n\} + B_{ks}p_s(n)\bigr), \tag{8}$$

$$E_k^{(2)}\{x_{n+1}\} = \sum_{s=1}^{q} \pi_{ks}\bigl(A_{ks}E_k^{(2)}\{x_n\}A_{ks}^* + A_{ks}E_s^{(1)}B_{ks}^* + B_{ks}E_s^{(1)}A_{ks}^* + B_{ks}B_{ks}^* p_s(n)\bigr), \tag{9}$$

*respectively.*

*Proof* Multiplying equation (5) by $z$ and integrating them on the Euclidean space $\mathbb{E}_m$, we obtain the system

$$\int_{\mathbb{E}_m} z f_k(n+1,z)\,dz = \sum_{s=1}^{q} \pi_{ks} \int_{\mathbb{E}_m} z f_s\bigl(n, A_{ks}^{-1}(z - B_{ks})\bigr) \det A_{ks}^{-1}\,dz. \tag{10}$$

Using the substitution $Y_{ks} = A_{ks}^{-1}z$, integrating by parts, in regard to $\int_{\mathbb{E}_m} f_k(n+1,z)\,dz = p_k(n)$, we get, as in the proof of Theorem 2 in [2], systems of moment equations (8).

In the same way, the system of moment equations (9) can be derived. This means that equation (5) is multiplied by $zz^*$ and integrated by parts on the Euclidean space $\mathbb{E}_m$. □

## 3 Model problem 1

We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$ as a space of trade relations. The changes in the level of the income of a company can be modeled by using stochastic difference equations. Such a convenient mathematical model is the scalar case of initial problem (1), (2), that is,

$$x_{n+1} = x_n + b(\xi_{n+1}, \xi_n), \quad n = 1, 2, \dots, \tag{11}$$
$$x_0 = \varphi(\omega).$$

The stochastic equation (11) describes the graph of the income of a company with the initial value of income $x_0$. Here the inhomogeneity $b(\xi_{n+1}, \xi_n)$ represents the conditions in which a company works. For example, the value $b(\theta_k, \theta_s) = b_{ks}$, $k, s = 1, 2, \dots, q$, means the transition from one state $\theta_k$ of company activities, say, a crisis, to another state $\theta_s$, say, the post-crisis situation.

The transition probabilities satisfy system (3), and moment equations (8), (9) in the scalar case take the form

$$E^{(1)}\{x_{n+1}\} = \Pi E^{(1)}\{x_n\} + B_1 p(n), \tag{12}$$
$$E^{(2)}\{x_{n+1}\} = \Pi E^{(2)}\{x_n\} + 2B_2 E^{(1)}\{x_n\} + B_2 p(n), \tag{13}$$

where

$$B_1 = \begin{pmatrix} \pi_{11}b_{11} & \pi_{12}b_{12} & \cdots & \pi_{1q}b_{1q} \\ \pi_{21}b_{21} & \pi_{22}b_{22} & \cdots & \pi_{2q}b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{q1}b_{q1} & \pi_{q2}b_{q2} & \cdots & \pi_{qq}b_{qq} \end{pmatrix}, \qquad B_2 = \begin{pmatrix} \pi_{11}b_{11}^2 & \pi_{12}b_{12}^2 & \cdots & \pi_{1q}b_{1q}^2 \\ \pi_{21}b_{21}^2 & \pi_{22}b_{22}^2 & \cdots & \pi_{2q}b_{2q}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{q1}b_{q1}^2 & \pi_{q2}b_{q2}^2 & \cdots & \pi_{qq}b_{qq}^2 \end{pmatrix}.$$

The first or the second moments can be obtained by solving systems of difference equations (3) and (12) or (3) and (13) respectively, for example, by using a numerical method.

**Example 1** The real mean value of the income can be determined in a particular case. Assume that a company, trying to overcome a crisis, may get into three possible states corresponding to the three possible methods used:

1. salary adjustment,
2. number of employees,
3. credit.

Let the transition matrix have the form

$$\Pi = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix}.$$

In accordance with the annual report of the company, the income values under the transition from one state to another are determined by the following coefficients:

$$b_{12} = 2, \qquad b_{13} = 4, \qquad b_{21} = 4, \qquad b_{23} = 4, \qquad b_{31} = 6, \qquad b_{32} = 4.$$

Then, given the initial values $E^{(1)}\{x_0\} = (0,0,0)^T$ and $p(0) = (1,0,0)^T$ for the first moment, we get

$$E^{(1)}\{x_1\} = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix},$$

further,

$$E^{(1)}\{x_2\} = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0,5 \\ 0,5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2,5 \\ 2 \end{pmatrix},$$

and so on.

By using an iterative method, after a finite number of steps, the mean value of the income can be obtained. This is approximately 4.33 with a variance of 1.11.

**Remark 1** The processes described by system (11) can be controlled introducing a control function $U_n$.

Let us develop the idea mentioned in the above remark. On a bounded area $G \subset \mathbb{E}_m$, we consider the following stochastic dynamic system with random coefficients:

$$x_{n+1} = A(\xi_{n+1}, \xi_n)x_n + B(\xi_{n+1}, \xi_n)U_n, \quad n = 1, 2, \dots, \tag{14}$$

where the vectors $U_n$ belong to the set of control $U$ (see [12]). We define the functional

$$J = E\left\{\sum_{n=0}^{\infty} x_n^* Q(\xi_n)x_n + U_n^* L(\xi_n)U_n\right\}, \tag{15}$$

where the matrices $Q$, $L$ with Markov elements are symmetric and positive definite. The functional $J$ is called the quality criterion of control vectors $U_n$. The control function $U_n$ in the form

$$U_n = S(\xi_n)x_n, \quad n = 1, 2, \dots, \tag{16}$$

which minimizes the quality criterion (15) with respect to equation (11), is called the optimal control.

We denote $S(\theta_k) = S_k$, $Q(\theta_k) = Q_k$, $L(\theta_k) = L_k$, $k = 1, 2, \dots, q$. If we use the method developed in [13], we obtain the following.

**Theorem 2** *Let there exist the optimal control* (16) *that minimizes the quality criterion* (15) *with respect to equation* (11). *Then the matrices $S_k$ are determined by the system*

$$S_k = -\left(L_k + \sum_{s=1}^{q} \pi_{sk} B_{sk}^* C_s B_{sk}\right)^{-1} \cdot \sum_{s=1}^{q} \pi_{sk} B_{sk}^* C_s A_{sk}, \tag{17}$$

*where the matrices* $C_s$, $k = 1, 2, \dots, q$, *satisfy the system of matrix equations*

$$C_k = Q_k + \sum_{s=1}^{q} \pi_{sk} A_{sk}^* C_s A_{sk} + \sum_{s=1}^{q} \pi_{sk} A_{sk}^* C_s B_{sk} \left( L_k + \sum_{s=1}^{q} \pi_{sk} B_{sk}^* C_s B_{sk} \right)^{-1} \sum_{s=1}^{q} \pi_{sk} B_{sk}^* C_s A_{sk}$$

*with known matrices* $A_{sk}$, $B_{sk}$.

The theorem gives the necessary conditions for an optimal solution of system (14) making it possible to transform the problem of synthesis of the optimal control to the problem of determining the matrices $S_k$ in system (17).

## 4 Model problem 2

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider an initial problem formulated for a nonhomogenous difference equation of the form

$$x_{n+1} = x_n + b(\xi_n), \quad n = 1, 2, \dots, \qquad x_0 = o, \tag{18}$$

where $o$ is an $m$-dimensional zero vector describing the behavior of the random value $x_n$, which stands for the company income at a moment $n$.

Assume that a company is managed in accordance with the behavior of a Markov chain $\xi_n$ with two possible states $\theta_1$, $\theta_2$ and the transition matrix

$$\Pi = \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}, \quad 0 \le \lambda \le 1. \tag{19}$$

If the random variable is in state $\theta_1$, the company makes a profit, if the random variable is in state $\theta_2$, the company suffers a loss. Let the value of the company's profit or loss be expressed as

$$b(\theta_1) = \beta, \qquad b(\theta_2) = -\beta$$

and the initial state be $x_0 = o$. Then equations (12) take the form

$$\begin{aligned} E_1^{(1)}\{x_{n+1}\} &= (1-\lambda)\left( E_1^{(1)}\{x_n\} + \frac{\beta}{2} \right) + \lambda \left( E_2^{(1)}\{x_n\} - \frac{\beta}{2} \right), \\ E_2^{(1)}\{x_{n+1}\} &= \lambda \left( E_1^{(1)}\{x_n\} + \frac{\beta}{2} \right) + (1-\lambda)\left( E_2^{(1)}\{x_n\} - \frac{\beta}{2} \right). \end{aligned} \tag{20}$$

If we consider the initial values

$$E_1^{(1)}\{x_0\} = E_2^{(1)}\{x_0\} = o, \tag{21}$$

we obtain

$$E_1^{(1)}\{x_{n+1}\} + E_2^{(1)}\{x_{n+1}\} = E^{(1)}\{x_{n+1}\} = o,$$
$$E_1^{(1)}\{x_{n+1}\} - E_2^{(1)}\{x_{n+1}\} = (1-2\lambda)\bigl(E_1^{(1)}\{x_n\} - E_2^{(1)}\{x_n\} + \beta\bigr).$$

From the first equation, it is easy to see that $E^{(1)}\{x_{n+1}\} = o$, which means that the company will be left without the expected net profit in the above conditions.

Moment of the second order $E^{(2)}\{x_{n+1}\}$ for the above conditions can be obtained from equations (13) in the form

$$E^{(2)}\{x_n\} = n\beta^2 + \frac{\beta^2}{2\lambda^2}\bigl(n(1-2\lambda) - (n+1)(1-2\lambda)^2 + (1-2\lambda)^{n+2}\bigr). \tag{22}$$

Thence, if, say, $\lambda = 0.5$, the moment of the second order is $E^{(2)}\{x_n\} = n\beta^2$, which means that the company's net profit ranges according to the three-sigma rule in the interval

$$-3\beta\sqrt{n} \leq x_n \leq 3\beta\sqrt{n}.$$

## 5 Concluding remarks

While the theoretical results as such are original and valuable too, the present models confirm the practical importance of the methods devised to study discrete systems with random parameters. The construction of moment equations is a classical method and the results shown imply that its use in investigating different types of equations brings elegant results in modeling problems in the theory of finance.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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