Existence of periodic solutions for a prescribed mean curvature Liénard p-Laplacian equation with two delays

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Abstract

This paper is concerned with the prescribed mean curvature Liénard type p-Laplacian equation with two arguments. By employing Mawhin's coincidence degree theorem and the analysis techniques, some new existence results of periodic solutions are obtained. We also give an example to illustrate the application of our main results.

Keywords: periodic solution; Mawhin's continuation theorem; Liénard type p-Laplacian equation

1 Introduction

Recently, the study of the periodic solutions of prescribed mean curvature equations has become very active; see [1–5] and the references therein. Meanwhile, the Liénard equation, Liénard system, and p-Laplacian equations are also studied by many people; see for example [6–8]. These kinds of equations have wide applications in many fields, such as physics, mechanics, and engineering.

In [9] Wang studied the following prescribed mean curvature Rayleigh equation with one deviating argument:

$$\left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)' + f\big(t, x'(t)\big) + g\big(t, x\big(t - \tau(t)\big)\big) = e(t) \tag{1.1}$$

under the assumptions:

$$\begin{aligned} & f(t, u) \geq a|u|^r, \quad \forall (t, u) \in R^2 \quad \text{and} \\ & g(t, u) - e(t) \geq -m_1|u| - m_2, \quad \forall t \in R, x \geq d. \end{aligned} \tag{1.2}$$

By using the theory of coincidence degree, the author obtained some sufficient conditions for the existence and uniqueness of periodic solution in the case of $\tau(t) = 0$.

Stimulated by [9], we study the periodic solutions for a kind of prescribed mean curvature Liénard p-Laplacian equation with two deviating arguments in the form

$$\left(\varphi_p \left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}} \right) \right)' + f\big(x(t)\big)x'(t) + g\big(x\big(t - \tau(t)\big)\big) + h\big(x\big(t - \gamma(t)\big)\big) = e(t), \tag{1.3}$$

where $\varphi_p(s) = |s|^{p-2}s, p \geq 2, \tau, \gamma$ are continuous functions with period T, $\tau'(t) < 1$, $\gamma'(t) < 1$, and $\int_0^T e(t)\,dt = 0, f,g,h \in C^1(R,R)$.

To the best of our knowledge, there are few results on this topic. The purpose of this paper is to establish a criterion to guarantee the existence of T-periodic solution. Our methods and results are different from the corresponding ones of [9]. So our results are essentially new.

2 Preliminaries

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, that is, $X = \operatorname{Ker} L \oplus X_1$ and $Y = \operatorname{Im} L \oplus Y_1$. Furthermore, let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ be the continuous projectors. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P = L|_{D(L) \cap X_1}$ is invertible. Denote by K the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \Phi$. A map $N : \bar{\Omega} \to Y$ is called L-compact in $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and the operator $K(I - Q)N : \bar{\Omega} \to X$ is compact. The following, Mawhin's continuation theorem, is well known.

Lemma 2.1 (Gaines and Mawhin [10]) *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded subset and $N : \bar{\Omega} \to Y$ is L-compact in $\bar{\Omega}$. If all the following conditions hold*:

(1) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$;

(2) $Nx \notin \operatorname{Im} L, \forall x \in \partial\Omega \cap \operatorname{Ker} L$;

(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism.*

Then the equation $Lx = Nx$ has a solution in $\bar{\Omega} \cap D(L)$.

To use Mawhin's continuation theorem to study (1.3), we firstly transform (1.3) into the following form:

$$\begin{cases} x_1'(t) = \dfrac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}}, \\ x_2'(t) = -f(x_1(t))\dfrac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}} - g(x_1(t-\tau(t))) - h(x_1(t-\gamma(t))) + e(t). \end{cases} \tag{2.1}$$

Denote by $\phi(x) = \dfrac{\varphi_q(x)}{\sqrt{1-\varphi_q^2(x)}}$, $\varphi_q(s) = |s|^{q-2}s$, $1/p + 1/q = 1$, then prescribed mean curvature p-Laplacian operator $\varphi_p(\frac{x}{\sqrt{1+x^2}})$ is marked by $\phi^{-1}(x)$. Clearly, if $x(t) = (x_1(t),x_2(t))^\top$ is a T-periodic solution to (2.1), then $x_1(t)$ must be a T-periodic solution to (1.3).

Set $C_T^1 = \{x : x \in C(R,R^2), x(t+T) \equiv x(t)\}$ with the norm $|x|_0 = \max_{t\in[0,T]}|x(t)|$, $X = Y = \{x(t) = (x_1(t),x_2(t))^\top \in C(R,R^2) : x(t+T) \equiv x(t)\}$ with the norm $\|x\| = \max\{|x_1|_0,|x_2|_0\}$, $|x|_2 = (\int_0^T x^2(t)\,dt)^{1/2}$. Then X and Y are Banach spaces. Let

$$L : D(L) \subset X \to Y, \quad Lx = x' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}, \tag{2.2}$$

$$N : X \to Y,$$

$$\text{where } Nx = \begin{pmatrix} \phi(x_2(t)) \\ -f(x_1(t))\phi(x_2(t)) - g(x_1(t-\tau(t))) - h(x_1(t-\gamma(t))) + e(t) \end{pmatrix}. \tag{2.3}$$

It is easy to see that $\operatorname{Ker} L = R^2$, $\operatorname{Im} L = \{x : x \in Y, \int_0^T x(s)\,ds = 0\}$. So L is a Fredholm operator with index zero. Define projectors $P : X \to \operatorname{Ker} L$ and $Q : Y \to \operatorname{Im} Q$ by

$$Px = \frac{1}{T}\int_0^T x(s)\,ds, \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds,$$

and let K be the inverse of $L|_{\operatorname{Ker} P \cap D(L)}$. Obviously $\operatorname{Ker} L = \operatorname{Im} Q = R^2$,

$$[Ky]^{-1}(t) = \int_0^T k(t,s)y(s)\,ds, \tag{2.4}$$

where

$$k(t,s) = \begin{cases} \frac{s}{T}, & 0 \le s < t \le T, \\ \frac{s-T}{T}, & 0 \le t \le s \le T. \end{cases}$$

From (2.3) and (2.4), one can easily see that N is L-compact on Ω, where Ω is an open bounded subset of X. The following lemma is useful to estimate *a priori* bounds of periodic solutions of (2.1).

Lemma 2.2 (Liu *et al.* [11]) *Suppose that $x(t) \in C_T^1$, and that there is a point $t_* \in R$ such that $x(t_*) = 0$, then*

$$\int_0^T |x(t)|^2\,dt \le \frac{T^2}{\pi^2}\int_0^T |x'(t)|^2\,dt.$$

Lemma 2.3 ([12]) *Let $\tau \in C_T^1$, $|\tau'|_0 < 1$. If $x(t) \in C_T^1$ then*

$$\int_0^T |x(t - \tau(t))|^2\,dt \le \tau_0 \int_0^T |x(t)|^2\,dt,$$

where $\tau_0 = \frac{1}{1-|\tau'|_0}$, $\gamma(t)$ satisfies the above inequality and $\gamma_0 = \frac{1}{1-|\gamma'|_0}$.

At the end of this section, we list the basic assumptions which will be used in Section 3.

(H$_1$) $(g(x_1) - g(x_2))(x_1 - x_2) > 0$, $x_1, x_2 \in R$ and $x_1 \ne x_2$, $(h(x_1) - h(x_2))(x_1 - x_2) > 0$, $x_1, x_2 \in R$ and $x_1 \ne x_2$.

(H$_2$) There is a constant $d > 0$ such that $x(g(x) + h(x)) > 0$ if and only if $|x| > d$.

(H$_3$) There exist nonnegative constants k_1, k_2, c_1, c_2 such that

$$|g(x)| \le k_1|x| + c_1, \quad |x| > d;$$
$$|h(x)| \le k_2|x| + c_2, \quad |x| > d.$$

(H$_4$) There is a constant $\sigma > 0$ such that $|f(s)| \ge \sigma$ for all $s \in R$.

3 Main results and the proof

In this section we state the main results and give its proof.

Theorem 3.1 *Suppose the assumptions* (H$_1$)-(H$_4$) *hold. Then* (1.3) *has at least one T-periodic solution provided $\frac{T}{\pi}(k_1\sqrt{\tau_0} + k_2\sqrt{\gamma_0}) < \sigma$.*

Proof Let $\Omega_1 = \{x \in X : Lx = \lambda Nx, \lambda \in (0,1)\}$, suppose $x(t) \in \Omega_1$, then

$$
\begin{cases}
x_1'(t) = \lambda \dfrac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}} \triangleq \lambda \phi(x_2(t)), \\
x_2'(t) = -\lambda f(x_1(t)) \dfrac{\varphi_q(x_2(t))}{\sqrt{1-\varphi_q^2(x_2(t))}} - \lambda g(x_1(t - \tau(t))) - \lambda h(x_1(t - \gamma(t))) + \lambda e(t).
\end{cases}
\tag{3.1}
$$

Integrating both sides of the second equation of (3.1) from 0 to T, we have

$$
\int_0^T \big(g(x_1(t - \tau(t))) + h(x_1(t - \gamma(t))) \big) \, dt = 0,
$$

which implies that there exists a point $\xi \in [0, T]$ such that

$$
g(x_1(\xi - \tau(\xi))) + h(x_1(\xi - \gamma(\xi))) = 0.
\tag{3.2}
$$

Now we claim that there must exist a point $t_0 \in R$ such that

$$
|x_1(t_0)| \le d.
\tag{3.3}
$$

In fact, let $\xi - \tau(\xi) = t_1$, $\xi - \gamma(\xi) = t_2$, then from (3.2), $g(x_1(t_1)) + h(x_1(t_2)) = 0$. If $x_1(t_1) = x_1(t_2)$ holds, (3.3) is clearly true. Now assume $x_1(t_1) < x_1(t_2)$.

Set $F(x) \triangleq g(x) + h(x)$, according to (H$_1$), one has

$$
\begin{aligned}
F(x_1(t_1)) &= g(x_1(t_1)) + h(x_1(t_1)) \\
&< g(x_1(t_1)) + h(x_1(t_2)) \\
&< g(x_1(t_2)) + h(x_1(t_2)) = F(x_1(t_2)),
\end{aligned}
$$

namely $F(x_1(t_1)) < 0 < F(x_1(t_2))$, then there must exist a t_0 such that $F(x_1(t_0)) = 0$. That is, $g(x_1(t_0)) + h(x_1(t_0)) = 0$. According to (H$_2$), (3.3) holds. If $x_1(t_1) > x_1(t_2)$ holds, by a similar method, we can show that (3.3) is also true.

Let $\tilde{t} = nT + t_0$, where $\tilde{t} \in [0, T]$ and n is an integer. Then

$$
|x_1(t)| = \left| x_1(\tilde{t}) + \int_{\tilde{t}}^t x_1'(s) \, ds \right| \le d + \int_0^T |x_1'(s)| \, ds \quad \text{for all } s \in [0, T].
$$

Thus we get

$$
|x_1|_0 = \max_{t \in [0,T]} |x_1(t)| \le d + \sqrt{T} |x_1'|_2.
\tag{3.4}
$$

On the other hand, multiplying both sides of the second equation of (3.1) by $x_1'(t)$ and integrating over $[0, T]$, we have

$$
\begin{aligned}
\int_0^T \left(\phi^{-1}\left(\frac{x_1'(t)}{\lambda} \right) \right)' x_1'(t) \, dt &+ \int_0^T f(x_1(t))(x_1'(t))^2 \, dt \\
&+ \lambda \int_0^T g(x_1(t - \tau(t))) x_1'(t) \, dt + \lambda \int_0^T h(x_1(t - \gamma(t))) x_1'(t) \, dt \\
&= \lambda \int_0^T e(t) x_1'(t) \, dt,
\end{aligned}
\tag{3.5}
$$

where $\phi^{-1}(\frac{x_1'(t)}{\lambda}) = \varphi_p(\frac{\frac{x_1'(t)}{\lambda}}{\sqrt{1+(\frac{x_1'(t)}{\lambda})^2}})$. Furthermore, in view of (H$_4$), we have

$$\left| \int_0^T f(x_1(t))(x_1'(t))^2 \, dt \right|$$

$$= \int_0^T |f(x_1(t))| |x_1'(t)|^2 \, dt$$

$$\leq \left| \int_0^T \left(\phi^{-1}\left(\frac{x_1'(t)}{\lambda} \right) \right)' x_1'(t) \, dt \right| + \lambda \left| \int_0^T g(x_1(t-\tau(t))) x_1'(t) \, dt \right|$$

$$+ \lambda \left| \int_0^T h(x_1(t-\gamma(t))) x_1'(t) \, dt \right| + \lambda \left| \int_0^T e(t) x_1'(t) \, dt \right|. \qquad (3.6)$$

Denote $w(t) = \phi^{-1}(x_1'(t)/\lambda)$, then

$$\int_0^T \left(\phi^{-1}(x_1'(t)/\lambda) \right)' x_1'(t) \, dt = \lambda \int_0^T \phi(\omega(t)) \, d\omega(t) = 0.$$

Together with (3.6) and the fact $\lambda \in (0,1)$ yields

$$\sigma \int_0^T |x_1'(t)|^2 \, dt \leq \left| \int_0^T g(x_1(t-\tau(t))) x_1'(t) \, dt \right| + \left| \int_0^T h(x_1(t-\gamma(t))) x_1'(t) \, dt \right|$$

$$+ \left| \int_0^T e(t) x_1'(t) \, dt \right|.$$

Set

$$E_1 = \{t \in [0,1], |x_1(t-\tau(t))| \leq d\}, \qquad E_2 = \{t \in [0,1], |x_1(t-\tau(t))| > d\},$$

$$E_3 = \{t \in [0,1], |x_1(t-\gamma(t))| \leq d\}, \qquad E_4 = \{t \in [0,1], |x_1(t-\gamma(t))| > d\}.$$

In view of Lemma 2.3, we obtain

$$\sigma |x_1'|_2^2 \leq \int_{E_1} |g(x_1(t-\tau(t)))| |x_1'(t)| \, dt + \int_{E_2} |g(x_1(t-\tau(t)))| |x_1'(t)| \, dt$$

$$+ \int_{E_3} |h(x_1(t-\gamma(t)))| |x_1'(t)| \, dt + \int_{E_4} |h(x_1(t-\gamma(t)))| |x_1'(t)| \, dt$$

$$+ \int_0^T |e(t)| |x_1'(t)| \, dt$$

$$\leq g_d \int_0^T |x_1'(t)| \, dt + h_d \int_0^T |x_1'(t)| \, dt + k_1 \int_0^T |x_1(t-\tau(t))| |x_1'(t)| \, dt$$

$$+ k_2 \int_0^T |x_1(t-\gamma(t))| |x_1'(t)| \, dt + c_1 \int_0^T |x_1'(t)| \, dt + c_2 \int_0^T |x_1'(t)| \, dt$$

$$+ |e|_\infty \int_0^T |x_1'(t)| \, dt$$

$$\leq (g_d + h_d + c_1 + c_2 + |e|_\infty) \sqrt{T} |x_1'|_2 + k_1 \sqrt{\tau_0} |x_1'|_2 |x_1|_2$$

$$+ k_2 \sqrt{\gamma_0} |x_1'|_2 |x_1|_2, \qquad (3.7)$$

where τ_0, γ_0 are defined by Lemma 2.3, and

$$g_d = \max_{|x| \le d} |g(x)|, \qquad h_d = \max_{|x| \le d} |h(x)|, \qquad |e|_\infty = \max_{t \in [0,T]} |e(t)|.$$

From (3.3), there exists a point $t_0 \in R$ such that $|x_1(t_0)| \le d$. Let $\varpi(t) = x_1(t) - x_1(t_0)$, we have $\varpi(t + T) = \varpi(t)$, $\varpi'(t) = x_1'(t)$, and $\varpi(t_0) = 0$; then, by Lemma 2.2,

$$|\varpi|_2 \le \frac{T}{\pi} |\varpi'|_2 = \frac{T}{\pi} |x_1'|_2.$$

By the Minkowski inequality, we get

$$
\begin{aligned}
|x_1|_2 &= \left(\int_0^T |x_1(t)|^2 \, dt \right)^{\frac{1}{2}} = \left(\int_0^T |\varpi(t) + x_1(t_0)|^2 \, dt \right)^{\frac{1}{2}} \\
&\le \left(\int_0^T |x_1(t_0)|^2 \, dt \right)^{\frac{1}{2}} + \left(\int_0^T |\varpi(t)|^2 \, dt \right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.8}
$$

So

$$|x_1|_2 \le \sqrt{T} d + |\varpi|_2 \le \sqrt{T} d + \frac{T}{\pi} |x_1'|_2.$$

Combining (3.7) and (3.8), we obtain

$$
\begin{aligned}
\sigma |x_1'|_2^2 &\le (g_d + h_d + c_1 + c_2 + |e|_\infty) \sqrt{T} |x_1'|_2 + (k_1 \sqrt{\tau_0} + k_2 \sqrt{\gamma_0}) |x_1'|_2 \left(\sqrt{T} d + \frac{T}{\pi} |x_1'|_2 \right) \\
&= (g_d + h_d + c_1 + c_2 + |e|_\infty) \sqrt{T} |x_1'|_2 + (k_1 \sqrt{\tau_0} + k_2 \sqrt{\gamma_0}) \sqrt{T} d |x_1'|_2 \\
&\quad + \frac{T}{\pi} (k_1 \sqrt{\tau_0} + k_2 \sqrt{\gamma_0}) |x_1'|_2^2.
\end{aligned}
\tag{3.9}
$$

Since $\frac{T}{\pi}(k_1 \sqrt{\tau_0} + k_2 \sqrt{\gamma_0}) < \sigma$, there is a constant $M_0 > 0$ independent of λ such that $|x_1'|_2 \le M_0$, i.e., $|x_1|_0 \le d + \sqrt{T} |x_1'|_2 \le d + \sqrt{T} M_0 := M_1$.

By the first equation of (3.1), we have

$$\int_0^T \lambda \frac{\varphi_q(x_2(t))}{\sqrt{1 - \varphi_q^2(x_2(t))}} \, dt = 0,$$

together with $\varphi_q(0) = 0$, which implies that there is a constant $\eta \in [0, T]$ such that $x_2(\eta) = 0$. Hence

$$|x_2|_0 \le \int_0^\eta |x_2'(s)| \, ds \le \int_0^T |x_2'(s)| \, ds.
\tag{3.10}$$

By the two equations of (3.1) and Hölder's inequality, we obtain

$$
\begin{aligned}
\int_0^T |x_2'(s)| \, ds &\le \int_0^T |f(x_1(t)) x_1'(t)| \, dt + \lambda \int_0^T |g(x_1(t - \tau(t)))| \, dt \\
&\quad + \lambda \int_0^T |h(x_1(t - \gamma(t)))| \, dt + \lambda \int_0^T |e(t)| \, dt
\end{aligned}
$$

$$\leq \int_0^T \left| f\big(x_1(t)\big) \right| \left| x_1'(t) \right| dt + g_{M_1} T + h_{M_1} T + |e|_\infty T$$

$$< |f|_0 M_0 \sqrt{T} + \big(g_{M_1} + h_{M_1} + |e|_\infty\big) T, \tag{3.11}$$

where $|f|_0 = \max_{|x_1| \leq M_1} |f(x_1)|$, $g_{M_1} = \max_{|x_1| \leq M_1} |g(x_1)|$, $h_{M_1} = \max_{|x_1| \leq M_1} |h(x_1)|$. By (3.10), (3.11), $|x_2|_0 \leq (g_{M_1} + h_{M_1} + |e|_\infty)T + |f|_0 M_0 \sqrt{T} + d =: M_2$.

Let $\Omega_2 = \{x \in \operatorname{Ker} L : Nx \in \operatorname{Im} L\}$. If $x \in \Omega_2$, then $QNx = 0$. In view of $\int_0^T e(t)\,dt = 0$, we have

$$\begin{cases} \dfrac{\varphi_q(x_2)}{\sqrt{1-\varphi_q^2(x_2)}} = 0, \\ \int_0^T g(x_1)\,dt + \int_0^T h(x_1)\,dt = 0. \end{cases} \tag{3.12}$$

So $x_2 = 0 \leq M_2$. Together with (H$_2$) yields

$$|x_1| \leq d \leq M_2.$$

Let $\Omega = \{x : x = (x_1, x_2) \in X, |x_1|_0 < M_1 + 1, |x_2|_0 < M_2 + 1\}$. Then $\Omega \supset (\Omega_1 \cup \Omega_2)$ and Ω is a bounded open set of X. So (1) and (2) of Lemma 2.1 are satisfied.

In the next step we show that condition (3) of Lemma 2.1 holds. Define a linear isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ by $J(x_1, x_2) = (-x_2, x_1)$, and let

$$H(v, \mu) := \mu v + \frac{1-\mu}{T} JQNv, \quad (v, \mu) \in \Omega \times [0,1].$$

The direct computation and (H$_2$) show that for $(x, \mu) \in \partial(\Omega \cap \operatorname{Ker} L) \times [0,1]$,

$$x^\top H(x, \mu) = \mu\big(x_1^2 + x_2^2\big) + \frac{1-\mu}{T}\big(g(x_1) + h(x_1)\big)x_1 + \frac{1-\mu}{T} \frac{|x_2|^{q-2} x_2^2}{\sqrt{1-\varphi_q^2(x_2)}} > 0.$$

Thus, $x^\top H(x, \mu) \neq 0$ for $(x, \mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0,1]$, which implies

$$\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(x, 0), \Omega \cap \operatorname{Ker} L, 0\} = \deg\{H(x, 1), \Omega \cap \operatorname{Ker} L, 0\}$$

$$= \deg\{I, \Omega \cap \operatorname{Ker} L, 0\} \neq 0.$$

Condition (3) of Lemma 2.1 holds. By Lemma 2.1, the equation $Lx = Nx$ has a solution. This completes the proof of Theorem 3.1. □

We can use a similar method to conclude the following result, the details are omitted.

Theorem 3.2 *Suppose* (H$_3$), (H$_4$), *and the following assumptions hold*:

(H$_1'$) $(g(x_1) - g(x_2))(x_1 - x_2) < 0$, $x_1, x_2 \in R$, *and* $x_1 \neq x_2$, $(h(x_1) - h(x_2))(x_1 - x_2) < 0$, $x_1, x_2 \in R$, *and* $x_1 \neq x_2$.

(H$_2'$) *There is a constant* $d > 0$ *such that* $x(g(x) + h(x)) < 0$ *if and only if* $|x| > d$.

Then (1.3) *has at least one T-periodic solution if* $\frac{T}{\pi}(k_1\sqrt{\tau_0} + k_2\sqrt{\gamma_0}) < \sigma$.

4 Uniqueness results

It is difficult to establish the uniqueness of the T-periodic solution of (1.3), but in the special case we obtain the uniqueness of (1.3).

Theorem 4.1 *Assume* (H$_1'$) *holds, and* $\tau(t) = \varepsilon_1$, $\gamma(t) = \varepsilon_2$ (ε_1, ε_2 *are sufficiently small constants*). *Then* (1.3) *has at most one T-periodic solution.*

Proof Denote

$$I(x) = \int_0^x f(w)\,dw, \qquad y(t) = \varphi_p\left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}}\right) + I\big(x(t)\big).$$

Then (1.3) can be converted into the following form:

$$\begin{cases} x'(t) = \frac{\varphi_q(y(t) - I(x(t)))}{\sqrt{1 - \varphi_q^2(y(t) - I(x(t)))}}, \\ y'(t) = -g(x(t - \varepsilon_1)) - h(x(t - \varepsilon_2)) + e(t). \end{cases} \tag{4.1}$$

Suppose that $x_i(t)$ ($i = 1, 2$) are two T-periodic solutions of (1.3). Then from (4.1), we have

$$\begin{cases} x_i'(t) = \frac{\varphi_q(y_i(t) - I(x_i(t)))}{\sqrt{1 - \varphi_q^2(y_i(t) - I(x_i(t)))}}, \\ y_i'(t) = -g(x_i(t - \varepsilon_1)) - h(x_i(t - \varepsilon_2)) + e(t), \quad i = 1, 2. \end{cases} \tag{4.2}$$

Let $u(t) = x_1(t) - x_2(t)$, $v(t) = y_1(t) - y_2(t)$, which together with (4.2) yields

$$\begin{cases} u'(t) = \frac{\varphi_q(y_1(t) - I(x_1(t)))}{\sqrt{1 - \varphi_q^2(y_1(t) - I(x_1(t)))}} - \frac{\varphi_q(y_2(t) - I(x_2(t)))}{\sqrt{1 - \varphi_q^2(y_2(t) - I(x_2(t)))}}, \\ v'(t) = -(g(x_1(t - \varepsilon_1)) - g(x_2(t - \varepsilon_1))) \\ \qquad - (h(x_1(t - \varepsilon_2)) - h(x_2(t - \varepsilon_2))). \end{cases} \tag{4.3}$$

Now, we claim that

$$v(t) \leq 0, \quad \forall t \in R.$$

We argue by contradiction, in view of $v \in C^2[0, T]$ and $v(t + T) = v(t)$, $\forall t \in R$, we obtain $\max_{t \in R} v(t) > 0$.

Then there must exist $t^* \in R$ such that

$$\begin{cases} v(t^*) = \max_{t \in [0, T]} v(t) = \max_{t \in R} v(t) > 0, \\ v'(t^*) = -(g(x_1(t^* - \varepsilon_1)) - g(x_2(t^* - \varepsilon_1))) \\ \qquad - (h(x_1(t^* - \varepsilon_2)) - h(x_2(t^* - \varepsilon_2))) = 0, \\ v''(t^*) = -(g'(x_1(t^* - \varepsilon_1))x_1'(t^* - \varepsilon_1) - g'(x_2(t^* - \varepsilon_1))x_2'(t^* - \varepsilon_1)) \\ \qquad - (h'(x_1(t^* - \varepsilon_2))x_1'(t^* - \varepsilon_2) - h'(x_2(t^* - \varepsilon_2))x_2'(t^* - \varepsilon_2)) \leq 0. \end{cases} \tag{4.4}$$

Noticing (H$_1'$), that is, $g'(x) < 0$, $h'(x) < 0$, ε_1, ε_2 sufficiently small, it follows from the second equation of (4.4) that

$$x_1\big(t^* - \varepsilon_1\big) \equiv x_2\big(t^* - \varepsilon_1\big), \qquad x_1\big(t^* - \varepsilon_2\big) \equiv x_2\big(t^* - \varepsilon_2\big).$$

Then from the third equation of (4.4), we get

$$v''\big(t^*\big) = -g'\big(x_1\big(t^* - \varepsilon_1\big)\big)\big(x_1'\big(t^* - \varepsilon_1\big) - x_2'\big(t^* - \varepsilon_1\big)\big)$$
$$- h'\big(x_1\big(t^* - \varepsilon_2\big)\big)\big(x_1'\big(t^* - \varepsilon_2\big) - x_2'\big(t^* - \varepsilon_2\big)\big)$$

$$= -g'\left(x_1\left(t^* - \varepsilon_1\right)\right)\left(\frac{\varphi_q(y_1(t^* - \varepsilon_1) - I(x_1(t^* - \varepsilon_1)))}{\sqrt{1 - \varphi_q^2(y_1(t^* - \varepsilon_1) - I(x_1(t^* - \varepsilon_1)))}}\right.$$

$$\left. - \frac{\varphi_q(y_2(t^* - \varepsilon_1) - I(x_2(t^* - \varepsilon_1)))}{\sqrt{1 - \varphi_q^2(y_2(t^* - \varepsilon_1) - I(x_2(t^* - \varepsilon_1)))}}\right)$$

$$- h'\left(x_1\left(t^* - \varepsilon_2\right)\right)\left(\frac{\varphi_q(y_1(t^* - \varepsilon_2) - I(x_1(t^* - \varepsilon_1)))}{\sqrt{1 - \varphi_q^2(y_1(t^* - \varepsilon_2) - I(x_1(t^* - \varepsilon_2)))}}\right.$$

$$\left. - \frac{\varphi_q(y_2(t^* - \varepsilon_2) - I(x_2(t^* - \varepsilon_2)))}{\sqrt{1 - \varphi_q^2(y_2(t^* - \varepsilon_2) - I(x_2(t^* - \varepsilon_2)))}}\right).$$

In view of $g'(x_1(t^* - \varepsilon_1)) < 0$, $h'(x_1(t^* - \varepsilon_2)) < 0$, $v(t^*) = y_1(t^*) - y_2(t^*) > 0$, $\frac{x}{\sqrt{1-x^2}}$, $\varphi_q(x)$ are increasing on x, and ε_1, ε_2 are sufficiently small, thus we have

$$v''\left(t^*\right) > 0.$$

This contradicts the third equation of (4.4), which implies that

$$v(t) = y_1(t) - y_2(t) \le 0, \quad \forall t \in R.$$

By using a similar argument, we can also show that $y_2(t) - y_1(t) \le 0$.
Hence, we obtain

$$y_1(t) \equiv y_2(t), \quad \forall t \in R.$$

Then from (4.3) and $g'(x) < 0$, we have $x_1(t - \varepsilon_1) \equiv x_2(t - \varepsilon_1)$, $x_1(t - \varepsilon_2) \equiv x_2(t - \varepsilon_2)$, $\forall t \in R$. That is

$$x_1(t) \equiv x_2(t).$$

Therefore, (1.3) has at most one T-periodic solution. The proof of Theorem 4.1 is now complete. □

From Theorem 3.2 and Theorem 4.1, we see that (1.3) has an unique T-periodic solution when $\tau(t) = \varepsilon_1$, $\gamma(t) = \varepsilon_2$.

5 An example
In this section we give an example to illustrate the application of Theorem 3.1.

Example 5.1 Consider the prescribed mean curvature Liénard equation with two deviating arguments:

$$\left(\varphi_4\left(\frac{x'(t)}{\sqrt{1 + x'^2(t)}}\right)\right)' + f(x(t))x'(t) + g\left(x\left(t - \frac{\cos 3\pi t}{4\pi}\right)\right) + h\left(x\left(t - \frac{\sin 3\pi t}{4\pi}\right)\right)$$

$$= \sin 3\pi t, \tag{5.1}$$

where $f(x) = \sigma + x^2$, $\sigma > 5$, $\tau(t) = \frac{\cos 3\pi t}{4\pi}$, $\gamma(t) = \frac{\sin 3\pi t}{4\pi}$, $e(t) = \sin 3\pi t$, $g(x) = \frac{85}{16}x + 6$, $h(x) = 5x + 2$. So $\tau_0 = \gamma_0 = 4$. Choosing $T = \frac{2}{3}$, $d = 10$, $k_1 = \frac{85}{16}$, $k_2 = 5$, then $(k_1\sqrt{\tau_0} + k_2\sqrt{\gamma_0})\frac{T}{\pi} \approx 4.38 < 5 < \sigma$. It is obvious that (H$_1$), (H$_2$), (H$_3$), and (H$_4$) hold. Then it follows from Theorem 3.1 that (5.1) has at least one $\frac{2}{3}$-periodic solution.

Authors' contributions
All authors have equally contributed in obtaining new results in this article and also read and approved the final manuscript.

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