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Existence of analytic invariant curves for a complex planar mapping near resonance

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Abstract

In this paper a 2-dimensional mapping is investigated in the complex field $\mathbb{C}$ for the existence of analytic invariant curves. Employing the method of majorant series, we need to discuss the eigenvalue α of the mapping at a fixed point. Besides the hyperbolic case $|\alpha| \neq 1$, we focus on those α on the unit circle S^1, *i.e.*, $|\alpha| = 1$. We discuss not only those α at resonance, *i.e.*, at a root of the unity, but also those α near resonance under the Brjuno condition.

Keywords: invariant curves; geometric difference equation; majorant series; Brjuno condition; resonance

1 Introduction

It is well known that a common and useful method to understand behaviors of a dynamical system generated by iteration of a self-mapping is to find a simple invariant structure in its phase space and to describe the dynamics on it. Invariant manifold is one of such structures and, in particular, invariant curve is the main object for 2-dimensional systems and easier to be discussed deeply. The existence of real analytic closed invariant curves for 2-dimensional area-preserving mappings has been investigated by many authors [1–7]. In this paper, we deal with the existence of analytic invariant curves for a 2-dimensional complex mapping $T : \mathbb{C}^2 \to \mathbb{C}^2$, $(z, w) \mapsto (\zeta, \omega)$, defined by

$$\begin{cases} \zeta = az + bw + \phi(z, w), \\ \omega = cz + dw + \psi(z, w), \end{cases} \tag{1}$$

where a, b, c, d are complex constants, $b \neq 0$, $ad - bc \neq 0$, and the power series

$$\phi(z, w) = \sum_{i+j\geq 2} \phi_{ij} z^i w^j \quad \text{and} \quad \psi(z, w) = \sum_{i+j\geq 2} \psi_{ij} z^i w^j$$

converge in a neighborhood of the origin. Clearly, the mapping T has a fixed point $O = (0, 0)$ with the Jacobian matrix

$$A = DT(0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

at O. The characteristic polynomial is

$$P_A(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Observe that the function $w = f(z)$ is an invariant curve of T if and only if f satisfies the functional equation

$$cz + df(z) + \psi\big(z, f(z)\big) = f\big[az + bf(z) + \phi\big(z, f(z)\big)\big], \quad z \in \mathbb{C}. \tag{2}$$

Since $b \neq 0$ and the analytic equation

$$u = az + by + \phi(z, y) \tag{3}$$

can be uniquely solved for y in the way

$$y = -\frac{a}{b}z + \frac{1}{b}u + \Lambda(z, u), \tag{4}$$

where Λ is analytic in a neighborhood of the origin and $\operatorname{ord}\Lambda \geq 2$. If we define

$$h(z) := az + bf(z) + \phi\big(z, f(z)\big), \tag{5}$$

then by (3) and (4)

$$f(z) = -\frac{a}{b}z + \frac{1}{b}h(z) + \Lambda\big(z, h(z)\big), \tag{6}$$

and hence from (2)

$$h\big(h(z)\big) - (a + d)h(z) + (ad - bc)z = \Theta\big(z, h(z), h\big(h(z)\big)\big), \tag{7}$$

where the function

$$\begin{aligned} \Theta\big(z, h(z), h\big(h(z)\big)\big) &= bd\Lambda\big(z, h(z)\big) - b\Lambda\big(h(z), h\big(h(z)\big)\big) \\ &\quad + b\psi\left(z, -\frac{a}{b}z + \frac{1}{b}h(z) + \Lambda\big(z, h(z)\big)\right) \end{aligned}$$

and the power series

$$\Theta(z_0, z_1, z_2) = \sum_{i+j+k \geq 2} \Theta_{i,j,k} z_0^i z_1^j z_2^k$$

are analytic in a polydisc.

The transformation (it is called the Schröder transformation)

$$h(z) = g\big(\alpha g^{-1}(z)\big) \tag{8}$$

with $\alpha \in \mathbb{C}$ for h yields

$$g\big(\alpha^2 z\big) - (a + d)g(\alpha z) + (ad - bc)g(z) = \Theta\big(g(z), g(\alpha z), g\big(\alpha^2 z\big)\big), \tag{9}$$

a geometric difference equation.

In order to get an analytic solution of (7), we need to find an invertible analytic solution of equation (9) for possible choices of α. This implies that the desired solution satisfies $g(0) = 0$ and $g'(0) \neq 0$. Therefore, without loss of generality, we can assume that

$$g(z) = z + \sum_{n\geq 2} \gamma_n z^n. \tag{10}$$

Substituting (10) into (9) we get

$$P_A(\alpha) = 0 \tag{11}$$

and, for $n \geq 2$,

$$\begin{aligned} P_A(\alpha^n)\gamma_n = \sum_{i+j+k\geq 2} &\Theta_{i,j,k} P_{n,i,j,k}\big(\gamma_2, \dots, \gamma_{n-1}; \\ &\alpha\gamma_2, \dots, \alpha\gamma_{n-1}; \alpha^2\gamma_2, \dots, \alpha^2\gamma_{n-1}\big), \end{aligned} \tag{12}$$

where $P_{n,i,j,k}$ is a homogeneous polynomial with positive coefficients in the variables $\gamma_2, \dots, \gamma_{n-1}; \alpha\gamma_2, \dots, \alpha\gamma_{n-1}; \alpha^2\gamma_2, \dots, \alpha^2\gamma_{n-1}$.

Note equation (11), we have

$$P_A(\alpha^n) = (\alpha^n - \alpha)Q_n(\alpha), \tag{13}$$

where $Q_n(\alpha) = \alpha^n + \alpha - (a + d)$. Hence, for all $n \geq 2$, (12) can be rewritten as

$$\begin{aligned} (\alpha^n - \alpha)Q_n(\alpha)\gamma_n = \sum_{i+j+k\geq 2} &\Theta_{i,j,k} P_{n,i,j,k}\big(\gamma_2, \dots, \gamma_{n-1}; \\ &\alpha\gamma_2, \dots, \alpha\gamma_{n-1}; \alpha^2\gamma_2, \dots, \alpha^2\gamma_{n-1}\big). \end{aligned} \tag{14}$$

In this paper, the complex α in (9) is chosen in $\sigma(A) := \{\lambda \in \mathbb{C} \mid P_A(\lambda) = 0\}$ and satisfies the following hypotheses:

(H1) $0 < |\alpha| \neq 1$.

(H2) $\alpha = e^{2\pi i\theta}$, where $\theta \in \mathbb{R}\backslash\mathbb{Q}$ is a Brjuno number ([8] and [9]), *i.e.*, $B(\theta) = \sum_{k=0}^{\infty} \frac{\log q_{k+1}}{q_k} < \infty$, where $\{p_k/q_k\}$ denotes the sequence of partial fractions of the continued fraction expansion of θ which is said to satisfy the *Brjuno condition*.

(H3) $\alpha = e^{2\pi iq/p}$ for some integers $p \in \mathbb{N}$ with $p \geq 2$ and $q \in \mathbb{Z}\backslash\{0\}$, and $\alpha \neq e^{2\pi il/k}$ for all $1 \leq k \leq p - 1$ and $l \in \mathbb{Z}\backslash\{0\}$.

Observe that α is off the unit circle S^1 in the case of (H1) but on S^1 in the rest of the cases. More difficulties are encountered for α on S^1, as mentioned in the so-called small-divisor problem (seen in [10], p.22 and p.146 and [11]). In the case where α is a Diophantine number, *i.e.*, there exist constants $\zeta > 0$ and $\sigma > 0$ such that $|\alpha^n - 1| \geq \zeta^{-1}n^{-\sigma}$ for all $n \geq 1$, the number $\alpha \in S^1$ is 'far' from all roots of the unity and was considered in different settings [12–14]. In recent work [15] the case of (H3), where α is a root of the unity, was also discussed for a general class of iterative equations. Since then, one has been striving to give a result of analytic solutions for those α 'near' a root of the unity, *i.e.*, neither being roots of the unity nor satisfying the Diophantine condition. The Brjuno condition in (H2) provides such a chance for us. As stated in [16], for a real number θ, we denote by $[\theta]$ its integer part, and let $\{\theta\} = \theta - [\theta]$. Then every irrational number θ has a unique expression of Gauss's continued fraction

$$\theta = a_0 + \theta_0 = a_0 + \frac{1}{a_1 + \theta_1} = \cdots,$$

denoted simply by $\theta = [a_0, a_1, \ldots, a_n, \ldots]$, where a_j's and θ_j's are calculated by the algorithm: (a) $a_0 = [\theta]$, $\theta_0 = \{\theta\}$, and (b) $a_n = [\frac{1}{\theta_{n-1}}]$, $\theta_n = \{\frac{1}{\theta_{n-1}}\}$ for all $n \geq 1$. Define the sequences $(p_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as follows:

$$q_{-2} = 1, \qquad q_{-1} = 0, \qquad q_n = a_n q_{n-1} + q_{n-2},$$
$$p_{-2} = 0, \qquad p_{-1} = 1, \qquad p_n = a_n p_{n-1} + p_{n-2}.$$

It is easy to show that $p_n/q_n = [a_0, a_1, \ldots, a_n]$. Thus, to every $\theta \in \mathbb{R}\backslash\mathbb{Q}$ we associate, using its convergence, an arithmetical function $B(\theta) = \sum_{n\geq 0} \frac{\log q_{n+1}}{q_n}$. We say that θ is a Brjuno number or that it satisfies the Brjuno condition if $B(\theta) < +\infty$. The Brjuno condition is weaker than the Diophantine condition. For example, if $a_{n+1} \leq ce^{a_n}$ for all $n \geq 0$, where $c > 0$ is a constant, then $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ is a Brjuno number but is not a Diophantine number. So, the case (H2) contains both a Diophantine condition and a condition which expresses that α is near resonance.

In this paper, we consider the Brjuno condition instead of the Diophantine one. We discuss not only the cases (H1) and (H3) but also (H2) for analytic invariant curves of the mapping T defined in (1).

2 Geometric difference equation under (H1)

Theorem 1 *Assume that $\alpha \in \sigma(A)$ and* (H1) *holds. Then equation* (9) *has an analytic solution $g(z)$ of the form* (10) *in a neighborhood of the origin.*

Proof We first consider the case $0 < |\alpha| < 1$. Since $ad - bc \neq 0$ and $\lim_{n\to\infty} \alpha^n = 0$, there is $\Theta_0 > 0$ such that

$$\left|P_A\left(\alpha^n\right)\right| \geq \Theta_0 \quad \text{for all } n \geq 2.$$

Define a new sequence $\{\beta_n\}_{n=1}^{\infty}$ by $\beta_1 = 1$ and

$$\beta_n = \Theta_0^{-1} \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}(\beta_2, \ldots, \beta_{n-1}; \beta_2, \ldots, \beta_{n-1}; \beta_2, \ldots, \beta_{n-1}), \quad n \geq 2.$$

A simple inductive proof shows that $|\gamma_n| \leq \beta_n$ for all $n \geq 1$. If

$$\Theta_1(z_0, z_1, z_2) := \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| z_0^i z_1^j z_2^k,$$

then Θ_1 is convergent in a polydisc. Furthermore, if we set $\widehat{\Theta}(z_0, z_1, z_2) = \Theta_0^{-1}\Theta_1(z_0, z_1, z_2)$, the power series $\widehat{\Theta}(z_0, z_1, z_2)$ converge also in a polydisc. If $\beta(z) = \sum_{n\geq 1} \beta_n z^n$, then we have

$$\Theta_0 \beta(z) = \Theta_0 z + \widehat{\Theta}\left(\beta(z), \beta(z), \beta(z)\right).$$

Define the function

$$F(z,w) = \Theta_0(z-w) + \widehat{\Theta}(w,w,w)$$

for (z,w) from a neighborhood of $(0,0)$, then $\beta(z)$ satisfies

$$F\bigl(z,\beta(z)\bigr) = 0. \tag{15}$$

In view of $F(0,0)=0$, $F'_w(0,0) = -\Theta_0 \neq 0$, and the implicit function theorem, there exists a unique function $\Phi(z)$, analytic in a neighborhood of the origin, such that

$$\Phi(0) = 0, \qquad \Phi'(0) = -\frac{F'_z(0,0)}{F'_w(0,0)} = 1$$

and $F(z,\Phi(z)) = 0$. According to (15), we have $\beta(z) = \Phi(z)$. This proves that the series (10) is an analytic solution of (9) in a neighborhood of the origin.

Now we consider the case $|\alpha| > 1$. In this case, the formal power series (10) satisfies

$$g(z) - (a+d)g(\mu z) + (ad-bc)g\bigl(\mu^2 z\bigr) = \Theta^*\bigl(g(z), g(\mu z), g\bigl(\mu^2 z\bigr)\bigr) \tag{16}$$

with $\mu = 1/\alpha$ and $\Theta^*(z_0,z_1,z_2) = \Theta(z_2,z_1,z_0)$. With obvious notations we have

$$P_A^*(\lambda) = 1-(a+d)\lambda + (ad-bc)\lambda^2 = \lambda^2 P_A\left(\frac{1}{\lambda}\right).$$

Note that $|\mu| < 1$ and $ad - bc \neq 0$, it follows that there is a positive constant Θ_0 such that $|P_A^*(\mu^n)| \geq \Theta_0$ for all $n \geq 2$. Then the result in the case $|\alpha| > 1$ is obtained by applying the result in the case $|\alpha| < 1$. This completes the proof. □

3 Geometric difference equation under (H2)

In this section we discuss the existence of analytic solutions of the geometric difference equation (9) under (H2). In order to introduce Davie's lemma, we need to recall some facts in [17] briefly. Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$ and $(q_n)_{n\in\mathbb{N}}$ be the sequence of partial denominators of Gauss's continued fraction for θ as in the Introduction. As in [16], let

$$A_k = \left\{ n \geq 0 \;\middle|\; \|n\theta\| \leq \frac{1}{8q_k} \right\}, \qquad E_k = \max\left(q_k, \frac{q_{k+1}}{4}\right), \qquad \eta_k = \frac{q_k}{E_k}.$$

Let A_k^* be the set of integers $j \geq 0$ such that either $j \in A_k$ or for some j_1 and j_2 in A_k, with $j_2 - j_1 < E_k$, one has $j_1 < j < j_2$ and q_k divides $j - j_1$. For any integer $n \geq 0$, define

$$l_k(n) = \max\left((1+\eta_k)\frac{n}{q_k} - 2, (m_n\eta_k + n)\frac{1}{q_k} - 1 \right),$$

where $m_n = \max\{j \mid 0 \leq j \leq n, j \in A_k^*\}$. We then define the function $h_k : \mathbb{N} \to \mathbb{R}_+$ as follows:

$$h_k(n) = \begin{cases} \frac{m_n + \eta_k n}{q_k} - 1 & \text{if } m_n + q_k \in A_k^*, \\ l_k(n) & \text{if } m_n + q_k \notin A_k^*. \end{cases}$$

Let $g_k(n) := \max(h_k(n), [\frac{n}{q_k}])$, and define $k(n)$ by the condition $q_{k(n)} \leq n \leq q_{k(n)+1}$. Clearly, $k(n)$ is nondecreasing. Then we are able to state the following result.

Lemma 1 (Davie's lemma [17]) *Let* $K(n) = n \log 2 + \sum_{j=0}^{k(n)} g_j(n) \log(2q_{j+1})$. *Then*

(a) *there is a universal constant* $\gamma > 0$ (*independent of* n *and* θ) *such that*

$$K(n) \leq n\left(\sum_{j=0}^{k(n)} \frac{\log q_{j+1}}{q_j} + \gamma\right),$$

(b) $K(n_1) + K(n_2) \leq K(n_1 + n_2)$ *for all* n_1 *and* n_2,

(c) $-\log|\alpha^n - 1| \leq K(n) - K(n-1)$.

The main result of this section is the following theorem.

Theorem 2 *Assume that* $\alpha \in \sigma(A)$, $|a + d| > 2$ *and* (H2) *holds. Then equation* (9) *has an analytic solution* $g(z)$ *of the form* (10) *in a neighborhood of the origin.*

Proof Since $|a + d| > 2$ for all $n \geq 2$, it follows from (14) that

$$|\gamma_n| \leq L\left|\alpha^{n-1} - 1\right|^{-1} \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}\big(|\gamma_2|, \ldots, |\gamma_{n-1}|; |\gamma_2|, \ldots, |\gamma_{n-1}|; |\gamma_2|, \ldots, |\gamma_{n-1}|\big) \tag{17}$$

with $L = (|a + d| - 2)^{-1}$. Define a sequence $\{C_n\}_{n=1}^{\infty}$ by $C_1 = 1$ and

$$C_n = L \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}(C_2, \ldots, C_{n-1}; C_2, \ldots, C_{n-1}; C_2, \ldots, C_{n-1}), \quad n \geq 2. \tag{18}$$

Similar to the proof in Theorem 1, using the implicit function theorem, we can prove that the power series $R(z) = z + \sum_{n=2}^{\infty} C_n z^n$ is convergent in a neighborhood of the origin. Thus there is a positive constant ϱ such that

$$C_n \leq \varrho^n, \quad n = 2, 3, \ldots.$$

Now, we can deduce, by induction, that $|\gamma_n| \leq C_n e^{K(n-1)}$ for $n \geq 1$, where $K : \mathbb{N} \to \mathbb{R}$ is defined in Lemma 1. In fact $|\gamma_1| = 1 = C_1$. For a proof by induction, we assume that $|\gamma_j| \leq C_j e^{K(j-1)}$, $j \leq n - 1$. According to Lemma 1, it follows from (17) and (18) that

$$\begin{aligned} |\gamma_n| &\leq L\left|\alpha^{n-1} - 1\right|^{-1} \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}\big(C_2 e^{K(1)}, \ldots, C_{n-1} e^{K(n-2)}; \\ &\quad C_2 e^{K(1)}, \ldots, C_{n-1} e^{K(n-2)}; C_2 e^{K(1)}, \ldots, C_{n-1} e^{K(n-2)}\big) \\ &\leq \left|\alpha^{n-1} - 1\right|^{-1} e^{K(n-2)} L \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}(C_2, \ldots, C_{n-1}; \\ &\quad C_2, \ldots, C_{n-1}; C_2, \ldots, C_{n-1}) \\ &\leq C_n e^{K(n-1)} \end{aligned}$$

as required.

Note that $K(n) \leq n(B(\theta) + \gamma)$ for some universal constant $\gamma > 0$. Then

$$|\gamma_n| = C_n e^{K(n-1)} \leq \varrho^n e^{(n-1)(B(\theta)+\gamma)},$$

that is,

$$\lim_{n\to\infty} \sup\bigl(|\gamma_n|\bigr)^{\frac{1}{n}} \leq \lim_{n\to\infty} \sup\bigl(\varrho^n e^{(n-1)(B(\theta)+\gamma)}\bigr)^{\frac{1}{n}} = \varrho e^{B(\theta)+\gamma}.$$

This implies that the convergence radius of (10) is at least $(\varrho e^{B(\theta)+\gamma})^{-1}$. This completes the proof. □

4 Geometric difference equation under (H3)

The next theorem is devoted to the case of (H3), where α is not only on the unit circle in $\mathbb{C}$ but also a root of the unity. In this case neither the Diophantine condition nor the Brjuno condition is satisfied.

Define a sequence $\{B_n\}_{n=1}^{\infty}$ by $B_1 = 1$ and

$$\begin{aligned} B_n = L\Gamma \sum_{i+j+k\geq 2} |\Theta_{i,j,k}| P_{n,i,j,k}(B_2,\ldots,B_{n-1};\\ B_2,\ldots,B_{n-1};B_2,\ldots,B_{n-1}), \quad n \geq 2, \end{aligned} \tag{19}$$

where L is defined in (17) and

$$\Gamma = \max\bigl\{1, \bigl|\alpha^{j-1} - 1\bigr|^{-1} : j = 1,2,\ldots,p\bigr\}. \tag{20}$$

Moreover, let

$$\Omega(n,\alpha) := \sum_{i+j+k\geq 2} \Theta_{i,j,k} P_{n,i,j,k}\bigl(\gamma_2,\ldots,\gamma_{n-1};\alpha\gamma_2,\ldots,\alpha\gamma_{n-1};\alpha^2\gamma_2,\ldots,\alpha^2\gamma_{n-1}\bigr).$$

Theorem 3 *Assume that $\alpha \in \sigma(A)$, $|a + d| > 2$ and* (H3) *holds. If $\Omega(lp + 1,\alpha) = 0$ for all $l \in \mathbb{N} = \{1,2,\ldots\}$, then equation* (9) *has an analytic solution of the form*

$$g(z) = z + \sum_{n=lp+1,l\in\mathbb{N}} \zeta_n z^n + \sum_{n\neq lp+1,l\in\mathbb{N}} \gamma_n z^n$$

in a neighborhood of the origin, where ζ_{lp+1} is an arbitrary constant satisfying the inequality $|\gamma_{lp+1}| \leq B_{lp+1}$, and the sequence $\{B_n\}_{n=1}^{\infty}$ is defined in (19). *Otherwise, if $\Omega(lp + 1,\alpha) \neq 0$ for some $l = 1,2,\ldots$, then equation* (9) *has no analytic solution in any neighborhood of the origin.*

Proof As in the proof of Theorem 1, we seek for a power series solution of (9) of the form (10). Obviously, (11)-(14) hold again. If $\Omega(lp+1,\alpha) \neq 0$ for some natural number l, then (14) does not hold for $n = lp + 1$ since $\alpha^{lp} - 1 = 0$. In that case, (9) has no formal solutions.

If $\Omega(lp + 1,\alpha) = 0$, then there are infinitely many choices of corresponding γ_{lp+1} in (14) and the power series $\sum_{n=1}^{\infty} \gamma_n z^n$ forms a family of functions of infinitely many parameters. We can arbitrarily choose $\gamma_{lp+1} = \zeta_{lp+1}$ such that $|\zeta_{lp+1}| \leq B_{lp+1}$, $l = 1,2,\ldots$. In what follows, we prove that the series $\sum_{n=1}^{\infty} \gamma_n z^n$ has a nonzero radius of convergence. First of all, note that

$$\left|\alpha^{n-1} - 1\right|^{-1} \le \Gamma.$$

It follows from (14) that for all $n \neq lp + 1$, $l = 1, 2, \dots$,

$$|\gamma_n| \le L\Gamma \sum_{i+j+k \ge 2} |\Theta_{i,j,k}| P_{n,i,j,k}\big(|\gamma_2|, \dots, |\gamma_{n-1}|; |\gamma_2|, \dots, |\gamma_{n-1}|; |\gamma_2|, \dots, |\gamma_{n-1}|\big). \tag{21}$$

Further, we can show that

$$|\gamma_n| \le B_n, \quad n = 1, 2, \dots. \tag{22}$$

In fact, for an inductive proof, we assume that $|\gamma_\nu| \le B_\nu$ for all $1 \le \nu \le n$. When $n = lp$, we have $|\gamma_{n+1}| = |\zeta_{n+1}| \le B_{n+1}$. On the other hand, when $n \neq lp$, from (21) we get

$$|\gamma_{n+1}| \le L\Gamma \sum_{i+j+k \ge 2} |\Theta_{i,j,k}| P_{n,i,j,k}(B_2, \dots, B_{n-1}; B_2, \dots, B_{n-1}; B_2, \dots, B_{n-1}) = B_{n+1},$$

implying (22). Moreover, as in the proof of Theorem 1, we can prove that the series $\sum_{n=1}^{\infty} B_n z^n$ converges in a neighborhood of the origin. Thus the series $z + \sum_{n \ge 2} \gamma_n z^n$ has a nonzero radius of convergence. This completes the proof. □

5 Analyticity of invariant curves

In this section, we will state and prove our main results.

Theorem 4 *Suppose that one of the conditions in Theorems* 1-3 *is fulfilled. Then equation* (2) *has a solution of the form*

$$f(x) = -\frac{a}{b} z + \frac{1}{b} g\big(\alpha g^{-1}(z)\big) + \Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big),$$

where Λ *is defined in* (4) *and* g *is an invertible analytic solution of equation* (9).

Proof By Theorems 1-3, we can find an analytic solution g of the geometric difference equation (9) in the form of (10) such that $g(0) = 0$ and $g'(0) = \eta \neq 0$. Clearly, the inverse g^{-1} is analytic in a neighborhood of the point $g(0) = 0$. Let

$$f(x) = -\frac{a}{b} z + \frac{1}{b} g\big(\alpha g^{-1}(z)\big) + \Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big),$$

which is also analytic in a neighborhood of the origin. From (9), it is easy to see that

$$\begin{aligned} &f\big(az + bf(z) + \phi\big(z, f(z)\big)\big) \\ &\quad = f\big[g\big(\alpha g^{-1}(z)\big)\big] \\ &\quad = -\frac{a}{b} g\big(\alpha g^{-1}(z)\big) + \frac{1}{b} g\big(\alpha^2 g^{-1}(z)\big) + \Lambda\big(g\big(\alpha g^{-1}(z)\big), g\big(\alpha^2 g^{-1}(z)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{b}\big[dg\big(\alpha g^{-1}(z)\big) - adz + bcz + \Theta\big(z, g\big(\alpha g^{-1}(z)\big), g\big(\alpha^2 g^{-1}(z)\big)\big) \\
&\quad + b\Lambda\big(g\big(\alpha g^{-1}(z)\big), g\big(\alpha^2 g^{-1}(z)\big)\big)\big] \\
&= \frac{1}{b}\bigg[dg\big(\alpha g^{-1}(z)\big) - adz + bcz + bd\Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big) \\
&\quad + b\psi\left(z, -\frac{a}{b}z + \frac{1}{b}g\big(\alpha g^{-1}(z)\big) + \Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big)\right)\bigg] \\
&= cz + d\left(-\frac{a}{b}z + \frac{1}{b}g\big(\alpha g^{-1}(z)\big) + \Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big)\right) \\
&\quad + \psi\left(z, -\frac{a}{b}z + \frac{1}{b}g\big(\alpha g^{-1}(z)\big) + \Lambda\big(z, g\big(\alpha g^{-1}(z)\big)\big)\right) \\
&= cz + df(z) + \psi\big(z, f(z)\big).
\end{aligned}
$$

The proof is complete. □

Competing interests
The author declares that he has no competing interests.

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References
1. Birkhoff, GD: Dynamical systems with degrees of freedom. Trans. Am. Math. Soc. **18**, 199-300 (1917)
2. Birkhoff, GD: Surface transformations and their dynamical applications. Acta Math. **43**, 1-119 (1920)
3. Diamond, PM: Analytic invariants of mappings of two variables. J. Math. Anal. Appl. **27**, 601-608 (1969)
4. Fontich, E: Transversal homoclinic points of a class of conservative diffeomorphisms. J. Differ. Equ. **87**, 1-27 (1990)
5. Kuczma, M: Functional Equations in a Single Variable. Polish Scientific Publishers, Warsaw (1968)
6. Moser, J: The analytic invariants of an area-preserving mapping near a hyperbolic fix point. Commun. Pure Appl. Math. **9**, 673-692 (1956)
7. Moser, J: On invariant curves of area-preserving maps of an annulus. Nachr. Akad. Wiss. Gött. **1**, 1-20 (1962)
8. Bjuno, AD: Analytic form of differential equations. Trans. Mosc. Math. Soc. **25**, 131-288 (1971)
9. Marmi, S, Moussa, P, Yoccoz, JC: The Brjuno functions and their regularity properties. Commun. Math. Phys. **186**, 265-293 (1997)
10. Berger, MS: Nonlinearity and Functional Analysis. Academic Press, New York (1977)
11. Siegel, CL: Vorlesungen über Himmelsmechanik. Springer, Berlin (1956)
12. Si, J: On analytic solutions of the equation of invariant curves. C. R. Math. Acad. Sci. Soc. R. Can. **XVII**, 49-52 (1995)
13. Si, J, Wang, XP, Zhang, W: Analytic invariant curves for a planar map. Appl. Math. Lett. **15**, 561-573 (2002)
14. Si, J, Zhang, W: Analytic solutions of a functional equation for invariant curves. J. Math. Anal. Appl. **259**, 83-93 (2001)
15. Si, J, Zhang, W: Analytic solutions of a nonlinear iterative equation near neutral fixed points and poles. J. Math. Anal. Appl. **283**, 373-388 (2003)
16. Carletti, T, Marmi, S: Linearization of Analytic and Non-Analytic Germs of Diffeomorphisms of $(\mathbb{C}, 0)$. Bull. Soc. Math. Fr. **128**, 69-85 (2000)
17. Davie, AM: The critical function for the semistandard map. Nonlinearity **7**, 219-229 (1994)

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