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Nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions

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Abstract
In this paper, we study the existence and uniqueness of solution for a problem consisting of a sequential nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions. A variety of fixed point theorems, such as Banach's fixed point theorem, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder degree theory, are used. Examples illustrating the obtained results are also presented.
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1 Introduction

In this paper, we concentrate on the study of existence and uniqueness of solution for the following nonlinear fractional Caputo-Langevin equation with nonlocal Riemann-Liouville fractional integral conditions:

$$\begin{aligned} &D^p\big(D^q+\lambda\big)x(t)=f\big(t,x(t)\big), \quad t\in[0,T],\\ &\sum_{i=1}^{m}\mu_i I^{\alpha_i}x(\eta_i)=\sigma_1,\\ &\sum_{j=1}^{n}\nu_j I^{\beta_j}x(\xi_j)=\sigma_2, \end{aligned} \tag{1.1}$$

where $0<p,q\leq 1$, $1<p+q\leq 2$, D^q and D^p are the Caputo fractional derivatives of order q and p, respectively, I^ϕ is the Riemann-Liouville fractional integral of order ϕ, where $\phi=\alpha_i,\beta_j>0$, $\eta_i,\xi_j\in(0,T)$ are given points, $\mu_i,\nu_j,\lambda,\sigma_1,\sigma_2\in\mathbb{R}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

The significance of studying problem (1.1) is that the nonlocal conditions are very general and include many conditions as special cases. In particular, if $\alpha_i=\beta_j=1$, for all $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, then the nonlocal condition of (1.1) reduces to

$$\begin{cases} \mu_1\int_0^{\eta_1}x(s)\,ds+\mu_2\int_0^{\eta_2}x(s)\,ds+\cdots+\mu_m\int_0^{\eta_m}x(s)\,ds=\sigma_1,\\ \nu_1\int_0^{\xi_1}x(s)\,ds+\nu_2\int_0^{\xi_2}x(s)\,ds+\cdots+\nu_n\int_0^{\xi_n}x(s)\,ds=\sigma_2, \end{cases} \tag{1.2}$$

and if $\sigma_1 = \sigma_2 = 0$, $m = n = 2$, $\mu_2, \nu_2 \neq 0$, then (1.2) is reduced to

$$\varepsilon_1 \int_0^{\eta_1} x(s)\,ds = \int_0^{\eta_2} x(s)\,ds, \qquad \varepsilon_2 \int_0^{\xi_1} x(s)\,ds = \int_0^{\xi_2} x(s)\,ds, \tag{1.3}$$

where $\varepsilon_1 = -(\mu_1/\mu_2)$ and $\varepsilon_2 = -(\nu_1/\nu_2)$. Note that the nonlocal conditions (1.2) and (1.3) do not contain values of an unknown function x on the left-hand side and the right-hand side of boundary points $t = 0$ and $t = T$, respectively.

Fractional differential equations have been shown to be very useful in the study of models of many phenomena in various fields of science and engineering, such as physics, chemistry, biology, signal and image processing, biophysics, blood flow phenomena, control theory, economics, aerodynamics and fitting of experimental data. For examples and recent development of the topic, see [1–13] and the references cited therein.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [14]. For some new developments on the fractional Langevin equation, see, for example, [15–24].

In the present paper several new existence and uniqueness results are proved by using a variety of fixed point theorems (such as Banach's contraction principle, Krasnoselskii's fixed point theorem, Leray-Schauder's nonlinear alternative and Leray-Schauder's degree theory).

The rest of the paper is organized as follows. In Section 2 we recall some preliminary facts that we need in the sequel. In Section 3 we present our existence and uniqueness results. Examples illustrating the obtained results are presented in Section 4.

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

Definition 2.1 For an at least n-times differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$${}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s)\,ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2 The Riemann-Liouville fractional integral of order q is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.1 *For $q > 0$, the general solution of the fractional differential equation ${}^cD^q u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n-1$ ($n = [q]+1$).

In view of Lemma 2.1, it follows that

$$I^{q\,c}D^q u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n-1$ ($n = [q]+1$).

In the following, for the sake of convenience, we set constants

$$\Omega_1 = \sum_{i=1}^{m} \mu_i \frac{\eta_i^{q+\alpha_i}}{\Gamma(\alpha_i+q+1)}, \qquad \Psi_1 = \sum_{i=1}^{m} \mu_i \frac{\eta_i^{\alpha_i}}{\Gamma(\alpha_i+1)},$$
$$\Omega_2 = \sum_{j=1}^{n} \nu_j \frac{\xi_j^{q+\beta_j}}{\Gamma(\beta_j+q+1)}, \qquad \Psi_2 = \sum_{j=1}^{n} \nu_j \frac{\xi_j^{\beta_i}}{\Gamma(\beta_j+1)},$$

and $\Delta = \Omega_1\Psi_2 - \Omega_2\Psi_1$.

Lemma 2.2 *Let $\Delta \neq 0$, $0 < p,q \le 1$, $1 < p+q \le 2$, $\alpha_i, \beta_j > 0$, $\mu_i, \nu_j, \lambda, \sigma_1, \sigma_2 \in \mathbb{R}$, $\eta_i, \xi_j \in (0,T)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, and $y \in C([0,T],\mathbb{R})$. Then the nonlinear fractional Caputo-Langevin equation*

$$D^p\big(D^q + \lambda\big)x(t) = y(t), \tag{2.1}$$

subject to the nonlocal Riemann-Liouville fractional integral conditions

$$\sum_{i=1}^{m} \mu_i I^{\alpha_i} x(\eta_i) = \sigma_1, \qquad \sum_{j=1}^{n} \nu_j I^{\beta_j} x(\xi_j) = \sigma_2, \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} x(t) &= I^{q+p}y(t) - \lambda I^q x(t) \\ &\quad + \frac{\Psi_2 t^q - \Omega_2\Gamma(q+1)}{\Delta\Gamma(q+1)}\left(\sigma_1 - \sum_{i=1}^{m}\mu_i I^{\alpha_i+q+p}y(\eta_i) + \lambda\sum_{i=1}^{m}\mu_i I^{\alpha_i+q}x(\eta_i)\right) \\ &\quad - \frac{\Psi_1 t^q - \Omega_1\Gamma(q+1)}{\Delta\Gamma(q+1)}\left(\sigma_2 - \sum_{j=1}^{n}\nu_j I^{\beta_j+q+p}y(\xi_j) + \lambda\sum_{j=1}^{n}\nu_j I^{\beta_j+q}x(\xi_j)\right). \end{aligned} \tag{2.3}$$

Proof The general solution of equation (2.1) is expressed as the following integral equation:

$$x(t) = I^{q+p}y(t) - \lambda I^q x(t) + c_0\frac{t^q}{\Gamma(q+1)} + c_1, \tag{2.4}$$

where c_0 and c_1 are arbitrary constants. By taking the Riemann-Liouville fractional integral of order $\alpha_i > 0$ for (2.4), we get

$$I^{\alpha_i}x(t) = I^{\alpha_i+q+p}y(t) - \lambda I^{\alpha_i+q}x(t) + c_0\left(\frac{t^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right) + c_1\left(\frac{t^{\alpha_i}}{\Gamma(\alpha_i+1)}\right).$$

In particular, for $t = \eta_i$, we have

$$I^{\alpha_i} x(\eta_i) = I^{\alpha_i+q+p} y(\eta_i) - \lambda I^{\alpha_i+q} x(\eta_i) + c_0 \left(\frac{\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \right) + c_1 \left(\frac{\eta_i^{\alpha_i}}{\Gamma(\alpha_i+1)} \right).$$

Repeating the above process for the Riemann-Liouville fractional integral of order $\beta_j > 0$, substituting $t = \xi_j$ and applying the nonlocal condition (2.2), we obtain the following system of linear equations:

$$\begin{aligned} c_0\Omega_1 + c_1\Psi_1 &= \sigma_1 - \sum_{i=1}^{m} \mu_i I^{\alpha_i+q+p} y(\eta_i) + \lambda \sum_{i=1}^{m} \mu_i I^{\alpha_i+q} x(\eta_i), \\ c_0\Omega_2 + c_1\Psi_2 &= \sigma_2 - \sum_{j=1}^{n} \nu_j I^{\beta_j+q+p} y(\xi_j) + \lambda \sum_{j=1}^{n} \nu_j I^{\beta_j+q} x(\xi_j). \end{aligned} \tag{2.5}$$

Solving the linear system of equations in (2.5) for constants c_0, c_1, we have

$$\begin{aligned} c_0 &= \frac{\Psi_2}{\Delta} \left(\sigma_1 - \sum_{i=1}^{m} \mu_i I^{\alpha_i+q+p} y(\eta_i) + \lambda \sum_{i=1}^{m} \mu_i I^{\alpha_i+q} x(\eta_i) \right) \\ &\quad - \frac{\Psi_1}{\Delta} \left(\sigma_2 - \sum_{j=1}^{n} \nu_j I^{\beta_j+q+p} y(\xi_j) + \lambda \sum_{j=1}^{n} \nu_j I^{\beta_j+q} x(\xi_j) \right), \\ c_1 &= \frac{\Omega_1}{\Delta} \left(\sigma_2 - \sum_{j=1}^{n} \nu_j I^{\beta_j+q+p} y(\xi_j) + \lambda \sum_{j=1}^{n} \nu_j I^{\beta_j+q} x(\xi_j) \right) \\ &\quad - \frac{\Omega_2}{\Delta} \left(\sigma_1 - \sum_{i=1}^{m} \mu_i I^{\alpha_i+q+p} y(\eta_i) + \lambda \sum_{i=1}^{m} \mu_i I^{\alpha_i+q} x(\eta_i) \right). \end{aligned}$$

Substituting c_0 and c_1 into (2.4), we obtain solution (2.3). □

3 Main results

Throughout this paper, for convenience, the expression $I^x \phi(y)$ means

$$I^x \phi(y) = \frac{1}{\Gamma(x)} \int_0^y (y-s)^{x-1} \phi(s)\,ds \quad \text{for } y \in [0,T].$$

Let $\mathcal{C} = C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\| = \sup_{t\in[0,T]} |u(t)|$. As in Lemma 2.2, we define an operator $\mathcal{K}: \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned} \mathcal{K}x(t) &= I^{q+p} f\big(s,x(s)\big)(t) - \lambda I^q x(t) \\ &\quad + \frac{\Psi_2 t^q - \Omega_2 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\sigma_1 - \sum_{i=1}^{m} \mu_i I^{\alpha_i+q+p} f\big(s,x(s)\big)(\eta_i) + \lambda \sum_{i=1}^{m} \mu_i I^{\alpha_i+q} x(\eta_i) \right) \\ &\quad - \frac{\Psi_1 t^q - \Omega_1 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\sigma_2 - \sum_{j=1}^{n} \nu_j I^{\beta_j+q+p} f\big(s,x(s)\big)(\xi_j) + \lambda \sum_{j=1}^{n} \nu_j I^{\beta_j+q} x(\xi_j) \right). \end{aligned} \tag{3.1}$$

It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{K}$ has fixed points.

In the following subsections, we prove existence, as well as existence and uniqueness results, for problem (1.1) by using a variety of fixed point theorems.

3.1 Existence and uniqueness result via Banach's fixed point theorem

Theorem 3.1 *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(H_1) *there exists a constant $L>0$ such that $|f(t,x)-f(t,y)|\le L|x-y|$ for each $t\in[0,T]$ and $x,y\in\mathbb{R}$.*

If

$$L\Lambda_1+\Lambda_2<1, \tag{3.2}$$

where constants Λ_1, Λ_2 are defined by

$$\begin{aligned}\Lambda_1 :={}& \frac{T^{q+p}}{\Gamma(q+p+1)}+\left(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)}\\ &+\left(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Lambda_2 :={}& |\lambda|\left(\frac{T^q}{\Gamma(q+1)}+\left(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right.\\ &\left.+\left(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right),\end{aligned} \tag{3.4}$$

then problem (1.1) *has a unique solution on* $[0,T]$.

Proof Problem (1.1) is equivalent to a fixed point problem by defining the operator $\mathcal{K}$ as in (3.1), which yields $x=\mathcal{K}x$. Using the Banach contraction mapping principle, we will show that problem (1.1) has a unique solution. Setting $\sup_{t\in[0,T]}|f(t,0)|=M<\infty$, we define a set $B_r=\{x\in\mathcal{C}:\|x\|\le r\}$,

$$r\ge\frac{M\Lambda_1+\Phi}{1-(L\Lambda_1+\Lambda_2)},$$

where

$$\Phi:=|\sigma_1|\left(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)+|\sigma_2|\left(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right). \tag{3.5}$$

For any $x\in B_r$, we have

$$\begin{aligned}\bigl|\mathcal{K}x(t)\bigr| \le{}& I^{q+p}\bigl|f\bigl(s,x(s)\bigr)\bigr|(t)+|\lambda|I^q\bigl|x(s)\bigr|(t)\\ &+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(|\sigma_1|+\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q+p}\bigl|f\bigl(s,x(s)\bigr)\bigr|(\eta_i)\right.\\ &\left.+|\lambda|\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q}\bigl|x(s)\bigr|(\eta_i)\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\Biggl(|\sigma_2|+\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q+p}\bigl|f\bigl(s,x(s)\bigr)\bigr|(\xi_j)\\
&+|\lambda|\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q}\bigl|x(s)\bigr|(\xi_j)\Biggr)\\
\le{}& I^{q+p}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr|+\bigl|f(s,0)\bigr|\bigr)(t)+|\lambda|I^q\bigl|x(s)\bigr|(t)\\
&+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\Biggl(|\sigma_1|+\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q+p}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr|\\
&+\bigl|f(s,0)\bigr|\bigr)(\eta_i)+|\lambda|\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q}\bigl|x(s)\bigr|(\eta_i)\Biggr)\\
&+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\Biggl(|\sigma_2|+\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q+p}\bigl(\bigl|f\bigl(s,x(s)\bigr)-f(s,0)\bigr|\\
&+\bigl|f(s,0)\bigr|\bigr)(\xi_j)+|\lambda|\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q}\bigl|x(s)\bigr|(\xi_j)\Biggr)\\
\le{}&(Lr+M)\Biggl[\frac{T^{q+p}}{\Gamma(q+p+1)}+\biggl(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)}\\
&+\biggl(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)}\Biggr]\\
&+r|\lambda|\Biggl[\frac{T^q}{\Gamma(q+1)}+\biggl(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&+\biggl(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Biggr]\\
&+|\sigma_1|\biggl(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)+|\sigma_2|\biggl(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\biggr)\\
={}&(Lr+M)\Lambda_1+r\Lambda_2+\Phi\le r,
\end{aligned}
$$

which implies that $\mathcal{K}B_r\subset B_r$. Next, we need to show that $\mathcal{K}$ is a contraction mapping. Let $x,y\in\mathcal{C}$. Then, for $t\in[0,T]$, we have

$$
\begin{aligned}
&\bigl|\mathcal{K}x(t)-\mathcal{K}y(t)\bigr|\\
&\quad\le I^{q+p}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|(t)+|\lambda|I^q\bigl|x(s)-y(s)\bigr|(t)\\
&\qquad+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\Biggl(\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q+p}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|(\eta_i)\\
&\qquad+|\lambda|\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q}\bigl|x(s)-y(s)\bigr|(\eta_i)\Biggr)\\
&\qquad+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\Biggl(\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q+p}\bigl|f\bigl(s,x(s)\bigr)-f\bigl(s,y(s)\bigr)\bigr|(\xi_j)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |\lambda| \sum_{j=1}^{n} |\nu_j| I^{\beta_j+q} \bigl|x(s) - y(s)\bigr|(\xi_j)\Bigg) \\
&\leq \|x-y\| \Bigg[\frac{LT^{p+q}}{\Gamma(1+p+q)} + |\lambda| \frac{T^q}{\Gamma(1+q)} \\
&\quad + \frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Bigg(\sum_{i=1}^{m} \frac{L|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+p+q+1)} + |\lambda| \sum_{i=1}^{m} \frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \Bigg) \\
&\quad + \frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Bigg(\sum_{j=1}^{n} \frac{L|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)} + |\lambda| \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \Bigg) \Bigg] \\
&\leq (L\Lambda_1 + \Lambda_2)\|x-y\|,
\end{aligned}
$$

which leads to $\|\mathcal{K}x - \mathcal{K}y\| \leq (L\Lambda_1 + \Lambda_2)\|x-y\|$. Since $(L\Lambda_1 + \Lambda_2) < 1$, $\mathcal{K}$ is a contraction mapping. Therefore $\mathcal{K}$ has only one fixed point, which implies that problem (1.1) has a unique solution. □

3.2 Existence and uniqueness result via Banach's fixed point theorem and Hölder's inequality

Now we give another existence and uniqueness result for problem (1.1) by using Banach's fixed point theorem and Hölder's inequality. For $\sigma \in (0,1)$, we set

$$
\begin{aligned}
\Lambda_3 := &\Bigg[\left(\frac{1-\sigma}{q+p-\sigma} \right)^{1-\sigma} \frac{T^{q+p-\sigma}}{\Gamma(q+p)} \\
&+ \frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Bigg(\sum_{i=1}^{m} \left(\frac{1-\sigma}{\alpha_i+q+p-\sigma} \right)^{1-\sigma} \frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p)} \Bigg) \\
&+ \frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Bigg(\sum_{j=1}^{n} \left(\frac{1-\sigma}{\beta_j+q+p-\sigma} \right)^{1-\sigma} \frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p)} \Bigg) \Bigg]. \qquad (3.6)
\end{aligned}
$$

Theorem 3.2 *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. In addition we assume that*

(H_2) $|f(t,x) - f(t,y)| \leq \delta(t)|x-y|$ *for each* $t \in [0,T]$, $x, y \in \mathbb{R}$, *where* $\delta \in L^{\sigma}([0,T],\mathbb{R}^+)$, $\sigma \in (0,1)$.

Denote $\|\delta\| = (\int_0^T \delta^{\frac{1}{\sigma}}(s)\,ds)^{\sigma}$.
If

$$
\Lambda_3\|\delta\| + \Lambda_2 < 1, \qquad (3.7)
$$

where Λ_2 *and* Λ_3 *are defined by* (3.4) *and* (3.6), *respectively, then problem* (1.1) *has a unique solution.*

Proof For $x, y \in \mathcal{C}$ and each $t \in [0,T]$, by Hölder's inequality, we have

$$
\begin{aligned}
&\bigl|\mathcal{K}x(t) - \mathcal{K}y(t)\bigr| \\
&\quad \leq I^{q+p}\delta(s)\bigl|x(s) - y(s)\bigr|(t) + |\lambda| I^q \bigl|x(s) - y(s)\bigr|(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q+p}\delta(s)\big|x(s)-y(s)\big|(\eta_i)\right.\\
&\left.+|\lambda|\sum_{i=1}^{m}|\mu_i|I^{\alpha_i+q}\big|x(s)-y(s)\big|(\eta_i)\right)\\
&+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q+p}\delta(s)\big|x(s)-y(s)\big|(\xi_j)\right.\\
&\left.+|\lambda|\sum_{j=1}^{n}|\nu_j|I^{\beta_j+q}\big|x(s)-y(s)\big|(\xi_j)\right)\\
\le{}& \|x-y\|\left[\int_0^t(t-s)^{q+p-1}\delta(s)\,ds+|\lambda|\frac{t^q}{\Gamma(q+1)}\right.\\
&+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(\sum_{i=1}^{m}|\mu_i|\int_0^{\eta_i}(\eta_i-s)^{\alpha_i+q+p-1}\delta(s)\,ds\right.\\
&\left.+|\lambda|\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right)\\
&+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|}\left(\sum_{j=1}^{n}|\nu_j|\int_0^{\xi_j}(\xi_j-s)^{\beta_j+q+p-1}\delta(s)\,ds\right.\\
&\left.\left.+|\lambda|\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\right]\\
\le{}& \|x-y\|\left[\left(\int_0^t(t-s)^{\frac{q+p-1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t\delta(s)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}+|\lambda|\frac{t^q}{\Gamma(q+1)}\right.\\
&+\frac{|\Psi_2|t^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(\sum_{i=1}^{m}|\mu_i|\left(\int_0^{\eta_i}(\eta_i-s)^{\frac{\alpha_i+q+p-1}{1-\sigma}}\,ds\right)^{1-\sigma}\right.\\
&\left.\times\left(\int_0^{\eta_i}\delta(s)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}+|\lambda|\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\gamma(\alpha_i+q+1)}\right)\\
&+\frac{|\Psi_1|t^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\left(\sum_{j=1}^{n}|\nu_j|\left(\int_0^{\xi_j}(\xi_j-s)^{\frac{\beta_j+q+p-1}{1-\sigma}}\,ds\right)^{1-\sigma}\right.\\
&\left.\left.\times\left(\int_0^{\xi_j}\delta(s)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}+|\lambda|\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\right]\\
\le{}& (\Lambda_3\|\delta\|+\Lambda_2)\|x-y\|.
\end{aligned}
$$

Therefore,

$$\|\mathcal{K}x-\mathcal{K}y\|\le\big(\Lambda_3\|\delta\|+\Lambda_2\big)\|x-y\|.$$

Hence, from (3.7), $\mathcal{K}$ is a contraction mapping. Banach's fixed point theorem implies that $\mathcal{K}$ has a unique fixed point, which is the unique solution of problem (1.1). This completes the proof. □

3.3 Existence result via Krasnoselskii's fixed point theorem

Lemma 3.1 (Krasnoselskii's fixed point theorem [25]) *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be operators such that* (a) $Ax + Bx \in M$ *whenever $x, y \in M$*; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

Theorem 3.3 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Moreover, we assume that*

(H_3) $|f(t,x)| \le \phi(t)$, $\forall(t,x) \in [0,T] \times \mathbb{R}$ *and* $\phi \in C([0,T], \mathbb{R}^+)$.

Then problem (1.1) *has at least one solution on* $[0, T]$ *if*

$$\Lambda_2 < 1, \tag{3.8}$$

where Λ_2 is defined by (3.4).

Proof We define the operators $\mathcal{A}$ and $\mathcal{B}$ on B_r by

$$\begin{aligned}
\mathcal{A}x(t) &= I^{q+p} f\big(s, x(s)\big)(t) + \frac{\Psi_2 t^q - \Omega_2 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\sigma_1 - \sum_{i=1}^{m} \mu_i I^{\alpha_i + q + p} f\big(s, x(s)\big)(\eta_i) \right) \\
&\quad - \frac{\Psi_1 t^q - \Omega_1 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\sigma_2 - \sum_{j=1}^{n} \nu_j I^{\beta_j + q + p} f\big(s, x(s)\big)(\xi_j) \right), \\
\mathcal{B}x(t) &= -\lambda I^{q+p} x(s)(t) + \frac{\Psi_2 t^q - \Omega_2 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\lambda \sum_{i=1}^{m} \mu_i I^{\alpha_i + q} x(s)(\eta_i) \right) \\
&\quad - \frac{\Psi_1 t^q - \Omega_1 \Gamma(q+1)}{\Delta \Gamma(q+1)} \left(\lambda \sum_{j=1}^{n} \nu_j I^{\beta_j + q} x(s)(\xi_j) \right),
\end{aligned}$$

where the ball B_r is defined by $B_r = \{x \in \mathcal{C}, \|x\| \le r\}$ for some suitable r such that

$$r \ge \frac{\Lambda_1 \|\phi\| + \Phi}{1 - \Lambda_2},$$

with $\|\phi\| = \sup_{t \in [0,T]} |\phi(t)|$ and Λ_1, Λ_2 and Φ are defined by (3.3), (3.4) and (3.5), respectively. To show that $\mathcal{A}x + \mathcal{B}y \in B_r$, we let $x, y \in B_r$. Then we have

$$\begin{aligned}
&\big|\mathcal{A}x(t) + \mathcal{B}y(t)\big| \\
&\quad \le I^{q+p} \big| f\big(s, x(s)\big)\big|(t) + \frac{|\Psi_2| t^q + |\Omega_2| \Gamma(q+1)}{|\Delta| \Gamma(q+1)} \\
&\qquad \times \left(|\sigma_1| + \sum_{i=1}^{m} |\mu_i| I^{\alpha_i + q + p} \big| f\big(s, x(s)\big)\big|(\eta_i) \right) \\
&\qquad + \frac{|\Psi_1| t^q + |\Omega_1| \Gamma(q+1)}{|\Delta| \Gamma(q+1)} \left(|\sigma_2| + \sum_{j=1}^{n} |\nu_j| I^{\beta_j + q + p} \big| f\big(s, x(s)\big)\big|(\xi_j) \right) \\
&\qquad + |\lambda| I^{q+p} \big| y(s)\big|(t) + \frac{|\Psi_2| t^q + |\Omega_2| \Gamma(q+1)}{|\Delta| \Gamma(q+1)} \\
&\qquad \times \left(|\lambda| \sum_{i=1}^{m} |\mu_i| I^{\alpha_i + q} \big| y(s)\big|(\eta_i) \right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|\Psi_1|t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \left(|\lambda| \sum_{j=1}^{n} |\nu_j| I^{\beta_j+q} \big|y(s)\big|(\xi_j) \right) \\
&\le \Lambda_1 \|\phi\| + \Phi + r\Lambda_2 \le r.
\end{aligned}
$$

It follows that $\mathcal{A}x + \mathcal{B}y \in B_r$, and thus condition (a) of Lemma 3.1 is satisfied. For $x, y \in \mathcal{C}$, we have $\|\mathcal{B}x - \mathcal{B}y\| \le \Lambda_2 \|x - y\|$. Since $\Lambda_2 < 1$, the operator $\mathcal{B}$ is a contraction mapping. Therefore, condition (c) of Lemma 3.1 is satisfied.

The continuity of f implies that the operator $\mathcal{A}$ is continuous. For $x \in B_r$, we obtain

$$
\|\mathcal{A}x\| \le \Lambda_1 \|\phi\| + \Phi.
$$

This means that the operator $\mathcal{A}$ is uniformly bounded on B_r. Next we show that $\mathcal{A}$ is equicontinuous. We set $\sup_{t\in[0,T]} f(t, x(t)) = \bar{f}$, and consequently we get

$$
\begin{aligned}
&\big|\mathcal{A}x(t_2) - \mathcal{A}x(t_1)\big| \\
&\le \frac{1}{\Gamma(q+p+1)} \bigg| \int_0^{t_1} \big[(t_2 - s)^{q+p-1} - (t_1 - s)^{q+p-1}\big] f\big(s, x(s)\big)\, ds \\
&\quad + \int_{t_1}^{t_2} (t_2 - s)^{q+p-1} f\big(s, x(s)\big)\, ds \bigg| \\
&\quad + \frac{|\Psi_2||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \left(|\sigma_1| + \sum_{i=1}^{m} |\mu_i| I^{\alpha_i+q+p} \big|f\big(s, x(s)\big)\big|(\eta_i) \right) \\
&\quad + \frac{|\Psi_1||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \left(|\sigma_2| + \sum_{j=1}^{n} |\nu_j| I^{\beta_j+q+p} \big|f\big(s, x(s)\big)\big|(\xi_j) \right) \\
&\le \frac{\bar{f}}{\Gamma(q+p+1)} \big|t_2^{q+p} - t_1^{q+p}\big| + \frac{\bar{f}|\Psi_2||t_2^q - t_1^q|}{|\Delta|} \left(|\sigma_1| + \sum_{i=1}^{m} \frac{|\mu_i| \eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i + q + p + 1)} \right) \\
&\quad + \frac{\bar{f}|\Psi_1||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \left(|\sigma_2| + \sum_{j=1}^{n} \frac{|\nu_j| \xi_j^{\beta_j+q+p}}{\Gamma(\beta_j + q + p + 1)} \right),
\end{aligned}
$$

which is independent of x and tends to zero as $t_1 \to t_2$. Then $\mathcal{A}$ is equicontinuous. So $\mathcal{A}$ is relatively compact on B_r, and by the Arzelá-Ascoli theorem, $\mathcal{A}$ is compact on B_r. Thus condition (b) of Lemma 3.1 is satisfied. Hence the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy the hypotheses of Krasnoselskii's fixed point theorem; and consequently, problem (1.1) has at least one solution on $[0, T]$. □

3.4 Existence result via Leray-Schauder's nonlinear alternative

Theorem 3.4 (Nonlinear alternative for single-valued maps [26]) *Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $\mathcal{A} : \overline{U} \to C$ is a continuous, compact* (*that is, $\mathcal{F}(\overline{U})$ is a relatively compact subset of* C) *map. Then either*

(i) *$\mathcal{A}$ has a fixed point in $\overline{U}$, or*

(ii) *there is $x \in \partial U$* (*the boundary of U in C*) *and $\lambda \in (0,1)$ with $x = \lambda\mathcal{A}(x)$.*

Theorem 3.5 *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(H_4) *there exists a continuous nondecreasing function denoted by* $\psi : [0,\infty) \to (0,\infty)$ *and a function* $g \in C([0,T],\mathbb{R}^+)$ *such that*

$$\left|f(t,u)\right| \le g(t)\psi\left(\|x\|\right) \quad \textit{for each } (t,x) \in [0,T] \times \mathbb{R};$$

(H_5) *there exists a constant* $M > 0$ *such that*

$$\frac{M}{\psi(M)\|g\|\Lambda_1 + M\Lambda_2 + \Phi} > 1,$$

where Λ_1, Λ_2 *and* Φ *are defined by* (3.3), (3.4) *and* (3.5), *respectively.*

Then problem (1.1) *has at least one solution on* $[0,T]$.

Proof Let the operator $\mathcal{K}$ be defined by (3.1). Firstly, we shall show that $\mathcal{K}$ maps *bounded sets* (*balls*) *into bounded sets in* $\mathcal{C}$. For a number $r > 0$, let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a bounded ball in $\mathcal{C}$. Then, for $t \in [0,T]$, we have

$$\begin{aligned}
\left|\mathcal{K}x(t)\right| &\le I^{q+p}\left|f\left(s,x(s)\right)\right|(t) + |\lambda| I^q \left|x(s)\right|(t) \\
&\quad + \frac{|\Psi_2| t^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \left(|\sigma_1| + \sum_{i=1}^m |\mu_i| I^{\alpha_i+q+p}\left|f\left(s,x(s)\right)\right|(\eta_i) \right. \\
&\quad \left. + |\lambda| \sum_{i=1}^m |\mu_i| I^{\alpha_i+q}\left|x(s)\right|(\eta_i) \right) \\
&\quad + \frac{|\Psi_1| t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \left(|\sigma_2| + \sum_{j=1}^n |\nu_j| I^{\beta_j+q+p}\left|f\left(s,x(s)\right)\right|(\xi_j) \right. \\
&\quad \left. + |\lambda| \sum_{j=1}^n |\nu_j| I^{\beta_j+q}\left|x(s)\right|(\xi_j) \right) \\
&\le \psi(r)\|g\|\Lambda_1 + r\Lambda_2 + \Phi,
\end{aligned}$$

and consequently,

$$\|\mathcal{K}x\| \le \psi(r)\|g\|\Lambda_1 + r\Lambda_2 + \Phi.$$

Next, we will show that $\mathcal{K}$ *maps bounded sets into equicontinuous sets of* $\mathcal{C}$. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ and $x \in B_r$. Then we have

$$\begin{aligned}
&\left|\mathcal{K}x(t_2) - \mathcal{K}x(t_1)\right| \\
&\quad \le \frac{1}{\Gamma(q+p+1)} \left| \int_0^{t_1} \left[(t_2 - s)^{q+p-1} - (t_1 - s)^{q+p-1}\right] f\left(s,x(s)\right) ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} (t_2 - s)^{q+p-1} f\left(s,x(s)\right) ds \right| + \frac{|\Psi_2||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \left(|\sigma_1| + \sum_{i=1}^m |\mu_i| I^{\alpha_i+q+p}\left|f\left(s,x(s)\right)\right|(\eta_i) \right. \\
&\qquad \left. + |\lambda| \sum_{i=1}^m |\mu_i| I^{\alpha_i+q}\left|x(s)\right|(\eta_i) \right) + \frac{|\Psi_1||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \left(|\sigma_2| + \sum_{j=1}^n |\nu_j| I^{\beta_j+q+p}\left|f\left(s,x(s)\right)\right|(\xi_j) \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + |\lambda| \sum_{j=1}^{n} |\nu_j| I^{\beta_j+q} \bigl|x(s)\bigr|(\xi_j)\Biggr) \\
&\leq \frac{\psi(r)\|g\|}{\Gamma(1+q+p)} \bigl|t_2^{q+p} - t_1^{q+p}\bigr| \\
&\quad + \frac{|\Psi_2|t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \Biggl(|\sigma_1| + \sum_{i=1}^{m} \frac{\psi(r)\|g\||\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)} + \sum_{i=1}^{m} \frac{r|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\Biggr) \\
&\quad + \frac{|\Psi_1||t_2^q - t_1^q|}{|\Delta|\Gamma(q+1)} \Biggl(|\sigma_2| + \sum_{j=1}^{n} \frac{\psi(r)\|g\||\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)} + \sum_{j=1}^{n} \frac{r|\nu_j|xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Biggr).
\end{aligned}
$$

As $t_2 - t_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore, by the Arzelá-Ascoli theorem, the operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

Let x be a solution. Then, for $t \in [0,T]$, and following similar computations as in the first step, we have

$$
\bigl|x(t)\bigr| \leq \psi\bigl(\|x\|\bigr)\|g\|\Lambda_1 + \|x\|\Lambda_2 + \Phi,
$$

which leads to

$$
\frac{\|x\|}{\psi(\|x\|)\|g\|\Lambda_1 + \|x\|\Lambda_2 + \Phi} \leq 1.
$$

By (H_5) there is M such that $\|x\| \neq M$. Let us set

$$
U = \bigl\{x \in \mathcal{C} : \|x\| < M\bigr\}.
$$

We see that the operator $\mathcal{K} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \nu\mathcal{K}x$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{K}$ has a fixed point $x \in \overline{U}$ which is a solution of problem (1.1). This completes the proof. □

3.5 Existence result via Leray-Schauder's degree theory

Theorem 3.6 *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Suppose that*

(H_6) *there exist constants* $0 \leq \gamma < (1 - \Lambda_2)\Lambda_1^{-1}$ *and* $M > 0$ *such that*

$$
\bigl|f(t,x)\bigr| \leq \gamma|x| + M \quad \textit{for all } (t,x) \in [0,T] \times \mathbb{R},
$$

where Λ_1, Λ_2 *are defined by* (3.3) *and* (3.4), *respectively.*

Then problem (1.1) *has at least one solution on* $[0,T]$.

Proof We define an operator $\mathcal{K} : \mathcal{C} \to \mathcal{C}$ as in (3.1) and consider the fixed point equation

$$
x = \mathcal{K}x.
$$

We shall prove that there exists a fixed point $x \in \mathcal{C}$ satisfying (1.1).

Set a ball $B_r \subset \mathcal{C}$ as

$$B_r = \left\{ x \in \mathcal{C} : \sup_{t\in[0,T]} \left|x(t)\right| < r \right\},$$

where a constant radius $r > 0$. Hence, we show that $\mathcal{K} : \overline{B}_r \to \mathcal{C}$ satisfies the condition

$$x \neq \theta \mathcal{K}x, \quad \forall x \in \partial B_r, \forall \theta \in [0,1]. \tag{3.9}$$

We define

$$H(\theta, x) = \theta \mathcal{K}x, \quad x \in \mathcal{C}, \theta \in [0,1].$$

As shown in Theorem 3.5, the operator $\mathcal{K}$ is continuous, uniformly bounded and equicontinuous. Then, by the Arzelá-Ascoli theorem, a continuous map h_θ defined by $h_\theta(x) = x - H(\theta, x) = x - \theta\mathcal{K}x$ is completely continuous. If (3.9) holds, then the following Leray-Schauder degrees are well defined, and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned} \deg(h_\theta, B_r, 0) &= \deg(I - \theta\mathcal{K}, B_r, 0) = \deg(h_1, B_r, 0) \\ &= \deg(h_0, B_r, 0) = \deg(I, B_r, 0) = 1 \neq 0, \quad 0 \in B_r, \end{aligned}$$

where I denotes the unit operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \mathcal{K}x = 0$ for at least one $x \in B_r$. Let us assume that $x = \theta\mathcal{K}x$ for some $\theta \in [0,1]$ and for all $t \in [0,T]$ so that

$$\begin{aligned} \left|x(t)\right| &= \left|\theta(\mathcal{K}x)(t)\right| \\ &\leq I^{q+p}\left|f\bigl(s,x(s)\bigr)\right|(t) + |\lambda| I^q \left|x(s)\right|(t) \\ &\quad + \frac{|\Psi_2|t^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Biggl(|\sigma_1| + \sum_{i=1}^m |\mu_i| I^{\alpha_i+q+p}\left|f\bigl(s,x(s)\bigr)\right|(\eta_i) \\ &\quad + |\lambda| \sum_{i=1}^m |\mu_i| I^{\alpha_i+q}\left|x(s)\right|(\eta_i) \Biggr) \\ &\quad + \frac{|\Psi_1|t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \Biggl(|\sigma_2| + \sum_{j=1}^n |\nu_j| I^{\beta_j+q+p}\left|f\bigl(s,x(s)\bigr)\right|(\xi_j) \\ &\quad + |\lambda| \sum_{j=1}^n |\nu_j| I^{\beta_j+q}\left|x(s)\right|(\xi_j) \Biggr) \\ &\leq \bigl(\gamma\left|x(t)\right| + M\bigr) \Biggl(\frac{t^{q+p}}{\Gamma(q+p+1)} + \biggl(\frac{|\Psi_2|t^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \biggr) \sum_{i=1}^m \frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)} \\ &\quad + \biggl(\frac{|\Psi_1|t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \biggr) \sum_{j=1}^n \frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)} \Biggr) \\ &\quad + |\lambda|\left|x(t)\right| \Biggl(\frac{t^q}{\Gamma(q+1)} + \biggl(\frac{|\Psi_2|t^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \biggr) \sum_{i=1}^m \frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \end{aligned}$$

$$+ \left(\frac{|\Psi_1|t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right) \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\Bigg)$$
$$+ |\sigma_1|\left(\frac{|\Psi_2|t^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right) + |\sigma_2|\left(\frac{|\Psi_1|t^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right).$$

Taking norm $\sup_{t\in[0,T]} |x(t)| = \|x\|$, we get

$$\|x\| \le (\gamma\|x\| + M)\Lambda_1 + \|x\|\Lambda_2 + \Phi.$$

Solving the above inequality for $\|x\|$ yields

$$\|x\| \le \frac{M\Lambda_1 + \Phi}{1 - \gamma\Lambda_1 - \Lambda_2}.$$

If $r = \frac{M\Lambda_1+\Phi}{1-\gamma\Lambda_1-\Lambda_2} + 1$, then inequality (3.9) holds. This completes the proof. □

4 Examples

Example 4.1 Consider the following fractional Caputo-Langevin equation with Riemann-Liouville fractional integral conditions:

$$\begin{cases} D^{\frac{7}{10}}(D^{\frac{2}{5}} + \frac{1}{10})x(t) = \frac{1-e^{-t}}{4(t+1)^2}\frac{|x|}{2|x|+3} + \frac{2}{3}, \quad t \in (0,1), \\ \frac{1}{2}I^{\frac{9}{10}}x(\frac{1}{10}) + \frac{3}{10}I^{\frac{4}{5}}x(\frac{3}{20}) + \frac{1}{5}I^{\frac{7}{10}}x(\frac{1}{5}) = 5, \\ \frac{2}{5}I^{\frac{3}{10}}x(\frac{4}{5}) + \frac{2}{5}I^{\frac{1}{5}}x(\frac{17}{20}) + \frac{1}{5}I^{\frac{1}{10}}x(\frac{9}{10}) = 20. \end{cases} \tag{4.1}$$

Here $p = 7/10$, $q = 2/5$, $\lambda = 1/10$, $T = 1$, $m = 3$, $n = 3$, $\mu_1 = 1/2$, $\alpha_1 = 9/10$, $\eta_1 = 1/10$, $\mu_2 = 3/10$, $\alpha_2 = 4/5$, $\eta_2 = 3/20$, $\mu_3 = 1/5$, $\alpha_3 = 7/10$, $\eta_3 = 1/5$, $\sigma_1 = 5$, $\nu_1 = 2/5$, $\beta_1 = 3/10$, $\xi_1 = 4/5$, $\nu_2 = 2/5$, $\beta_2 = 1/5$, $\xi_2 = 17/20$, $\nu_3 = 1/5$, $\beta_3 = 1/10$, $\xi_3 = 9/10$, $\sigma_2 = 20$ and $f(t,x) = ((1-e^{-t})|x|/(4(t+1)^2(2|x|+3))) + 2/3$. Since $|f(t,x) - f(t,y)| \le (1/4)|x-y|$, then (H_1) is satisfied with $L = 1/4$. We can find that

$$\Lambda_1 = \frac{T^{q+p}}{\Gamma(q+p+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)}$$
$$+ \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)} \approx 2.849024,$$

$$\Lambda_2 = |\lambda|\left(\frac{T^q}{\Gamma(q+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right.$$
$$\left.+ \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right) \approx 0.052040.$$

Therefore, we have

$$L\Lambda_1 + \Lambda_2 \approx 0.764296 < 1.$$

Hence, by Theorem 3.1, problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following fractional Caputo-Langevin equation with Riemann-Liouville fractional integral conditions:

$$\begin{cases} D^{\frac{3}{5}}(D^{\frac{3}{7}} + \frac{1}{50})x(t) = \frac{xe^{-10t}\sin^{\frac{1}{2}}(t/2)-x}{4-t} + \frac{1}{(4+t)(|x|+1)} + 5, \quad t \in (0,\pi), \\ \frac{7}{2}I^{\frac{5}{2}}x(\frac{3\pi}{5}) - \frac{1}{5}I^{\frac{14}{11}}x(\frac{4\pi}{15}) - \frac{3}{7}I^{\frac{17}{12}}x(\frac{3\pi}{8}) - \frac{2}{7}I^{\frac{4}{3}}x(\frac{2\pi}{9}) + \frac{11}{15}I^{\frac{4}{5}}x(\frac{\pi}{5}) = \frac{3}{7}, \\ \frac{5}{3}I^{\frac{10}{3}}x(\frac{3\pi}{11}) - \frac{5}{2}I^{\frac{19}{10}}x(\frac{\pi}{5}) + \frac{6}{5}I^{\frac{13}{18}}x(\frac{\pi}{3}) - \frac{17}{13}I^{\frac{13}{7}}x(\frac{\pi}{4}) + \frac{3}{17}I^{\frac{6}{5}}x(\frac{\pi}{16}) = \frac{5}{6}. \end{cases} \tag{4.2}$$

Here $p = 3/5$, $q = 3/7$, $\lambda = 1/50$, $T = \pi$, $m = 5$, $n = 5$, $\mu_1 = 7/2$, $\alpha_1 = 5/2$, $\eta_1 = 3\pi/5$, $\mu_2 = -1/5$, $\alpha_2 = 14/11$, $\eta_2 = 4\pi/15$, $\mu_3 = -3/7$, $\alpha_3 = 17/12$, $\eta_3 = 3\pi/8$, $\mu_4 = -2/7$, $\alpha_4 = 4/3$, $\eta_4 = 2\pi/9$, $\mu_5 = 11/15$, $\alpha_5 = 4/5$, $\eta_5 = \pi/5$, $\sigma_1 = 3/7$, $\nu_1 = 5/3$, $\beta_1 = 10/3$, $\xi_1 = 3\pi/11$, $\nu_2 = -5/2$, $\beta_2 = 19/10$, $\xi_2 = \pi/5$, $\nu_3 = 6/5$, $\beta_3 = 13/18$, $\xi_3 = \pi/3$, $\nu_4 = -17/13$, $\beta_4 = 13/7$, $\xi_4 = \pi/4$, $\nu_5 = 3/17$, $\beta_5 = 6/5$, $\xi_5 = \pi/16$, $\sigma_2 = 5/6$ and $f(t,x) = ((xe^{-10t}\sin^{\frac{1}{2}}(t/2) - x)/(4-t)) + (1/((4+t)(|x|+1))) + 5$. Since $|f(t,x) - f(t,y)| \le e^{-10t}\sin^{\frac{1}{2}}(t/2)|x-y|$, then ($H_2$) is satisfied with $\delta(t) = e^{-10t}\sin^{\frac{1}{2}}(t/2)$ such that $\delta \in L^{1/2}([0,\pi],\mathbb{R}^+)$. We can find that

$$\begin{aligned} \Lambda_2 &= |\lambda| \left(\frac{T^q}{\Gamma(q+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{i=1}^{m} \frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \right. \\ &\quad \left. + \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right) \approx 0.350711, \\ \Lambda_3 &= \left[\left(\frac{1-\sigma}{q+p-\sigma} \right)^{1-\sigma} \frac{T^{q+p-\sigma}}{\Gamma(q+p)} \right. \\ &\quad + \frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \left(\sum_{i=1}^{m} \left(\frac{1-\sigma}{\alpha_i+q+p-\sigma} \right)^{1-\sigma} \frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p)} \right) \\ &\quad \left. + \frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \left(\sum_{j=1}^{n} \left(\frac{1-\sigma}{\beta_j+q+p-\sigma} \right)^{1-\sigma} \frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p)} \right) \right] \\ &\approx 17.417544, \end{aligned}$$

and $\|\delta\| \approx 0.035344$. Therefore, we have

$$\Lambda_3\|\delta\| + \Lambda_2 \approx 0966321 < 1.$$

Hence, by Theorem 3.2, problem (4.2) has a unique solution on $[0,\pi]$.

Example 4.3 Consider the following fractional Caputo-Langevin equation with Riemann-Liouville fractional integral conditions:

$$\begin{cases} D^{\frac{2}{3}}(D^{\frac{3}{4}} - \frac{3}{35})x(t) = (\frac{t^2+(1+|x|)t+|x|}{t+|x|+1})\ln(t+1), \quad t \in (0,e-1), \\ \frac{3}{11}I^{\frac{1}{6}}x(\frac{1}{10}) + \frac{1}{5}I^{\frac{1}{5}}x(\frac{10e-11}{10}) + \frac{3}{7}I^{\frac{3}{8}}x(\frac{2e}{9}) + \frac{4}{3}I^{\frac{1}{7}}x(\frac{\sqrt{2}e}{5}) = 15, \\ \frac{2}{5}I^{\frac{1}{2}}x(\frac{3}{8}) - \frac{1}{5}I^{\frac{6}{25}}x(\frac{3e-2}{4}) + \frac{1}{10}I^{\frac{1}{4}}x(\frac{5e-6}{5}) = -5. \end{cases} \tag{4.3}$$

Here $p = 2/3$, $q = 3/4$, $\lambda = -3/35$, $T = e-1$, $m = 4$, $n = 3$, $\mu_1 = 3/11$, $\alpha_1 = 1/6$, $\eta_1 = 1/10$, $\mu_2 = 1/5$, $\alpha_2 = 1/5$, $\eta_2 = (10e-11)/10$, $\mu_3 = 3/7$, $\alpha_3 = 3/8$, $\eta_3 = 2e/9$, $\mu_4 = 4/3$, $\alpha_4 = 1/7$, $\eta_4 =$

$\sqrt{2}e/5$, $\sigma_1 = 15$, $\nu_1 = 2/5$, $\beta_1 = 1/2$, $\xi_1 = 3/8$, $\nu_2 = -1/5$, $\beta_2 = 6/25$, $\xi_2 = (3e-2)/4$, $\nu_3 = 1/10$, $\beta_3 = 1/4$, $\xi_3 = (5e-6)/5$, $\sigma_2 = -5$ and $f(t,x) = ((t^2 + (1+|x|)t + |x|)/(t+|x|+1))\ln(t+1)$.

Since $|f(t,x)| \leq \ln(t+1) + e$, then (H_3) is satisfied. We can find that

$$\Lambda_2 = |\lambda| \left(\frac{T^q}{\Gamma(q+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{i=1}^{m} \frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right) \approx 0.980502.$$

This means that $\Lambda_2 < 1$. By Theorem 3.3 problem (4.3) has as least one solution on $[0, e-1]$.

Example 4.4 Consider the following fractional Caputo-Langevin equation with Riemann-Liouville fractional integral conditions:

$$\begin{cases} D^{\frac{5}{6}}(D^{\frac{2}{3}} - \frac{1}{50})x(t) = \frac{e^{-t}x^2+40\sin t}{40|x|+60t}(\frac{\sin(2t/3)}{(t+1)^2}), \quad t \in (0, \frac{\pi}{2}), \\ \frac{\sqrt{2}}{2} I^{\frac{3}{8}} x(\frac{\pi}{18}) - 2I^{\frac{1}{10}} x(\frac{4\pi}{9}) + \sqrt{3} I^{\frac{1}{4}} x(1) - I^{\frac{2}{3}} x(\frac{3}{4}) = 1, \\ \sqrt{\frac{2}{3}} I^{\frac{2}{7}} x(\frac{\pi}{6}) - 3I^{\frac{3}{10}} x(\frac{\pi}{3}) + \frac{2\sqrt{2}}{7} I^{\frac{11}{20}} x(\frac{5}{4}) - \frac{2}{5} I^{\frac{7}{9}} x(\frac{1}{2}) = 0. \end{cases} \tag{4.4}$$

Here $p = 5/6$, $q = 2/3$, $\lambda = -1/50$, $m = 4$, $n = 4$, $\mu_1 = \sqrt{2}/2$, $\alpha_1 = 3/8$, $\eta_1 = \pi/18$, $\mu_2 = -2$, $\alpha_2 = 1/10$, $\eta_2 = 4\pi/9$, $\mu_3 = \sqrt{3}$, $\alpha_3 = 1/4$, $\eta_3 = 1$, $\mu_4 = -1$, $\alpha_4 = 2/3$, $\eta_4 = 3/4$, $\sigma_1 = 1$, $\nu_1 = \sqrt{2/3}$, $\beta_1 = 2/7$, $\xi_1 = \pi/6$, $\nu_2 = -3$, $\beta_3 = 3/10$, $\xi_2 = \pi/3$, $\nu_3 = 2\sqrt{2}/7$, $\beta_3 = 11/20$, $\xi_3 = 5/4$, $\nu_4 = -2/5$, $\beta_4 = 7/9$, $\xi_4 = 1/2$, $\sigma_2 = 0$, and $f(t,x) = ((e^{-t}x^2 + 40\sin t)/(40|x| + 60t))(\sin(2t/3)/(t+1)^2)$. Then we can find that

$$\Lambda_1 = \frac{T^{q+p}}{\Gamma(q+p+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{i=1}^{m} \frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)} + \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)} \approx 18.473830,$$

$$\Lambda_2 = |\lambda| \left(\frac{T^q}{\Gamma(q+1)} + \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{i=1}^{m} \frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} + \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \sum_{j=1}^{n} \frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} \right) \approx 0.531365,$$

$$\Phi = |\sigma_1| \left(\frac{|\Psi_2|T^q + |\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) + |\sigma_2| \left(\frac{|\Psi_1|T^q + |\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)} \right) \approx 3.515304.$$

Clearly,

$$\left|f(t,x)\right| = \left| \frac{e^{-t}x^2(t) + 40\sin t}{40|x| + 60t} \left(\frac{\sin(2t/3)}{(t+1)^2} \right) \right| \leq \frac{1}{40}\left(|x|+1\right)\left|\sin(2t/3)\right|.$$

By choosing $\psi(|x|) = |x| + 1$ and $g(t) = |\sin(2t/3)|/40$, we can show that

$$\frac{M}{\psi(M)\|g\|\Lambda_1 + M\lambda_2 + \Phi} > 1,$$

which implies $M > 57.019704$. By Theorem 3.5, problem (4.4) has at least one solution on $[0,\pi/2]$.

Example 4.5 Consider the following fractional Caputo-Langevin equation with Riemann-Liouville fractional integral conditions:

$$\begin{cases} D^{\frac{4}{5}}(D^{\frac{3}{10}}+\frac{1}{80})x(t)=\frac{t^2e^{-2t^2}}{4\pi}\sin(x+\frac{\pi}{3}), \quad t\in(0,2\pi),\\ \frac{3}{2}I^{\frac{2}{5}}x(\frac{8}{5})-\frac{1}{3}I^{\frac{3}{10}}x(\frac{4}{3})-\frac{2}{11}I^{\frac{2}{15}}x(\frac{9}{2})+\frac{1}{7}I^{\frac{1}{3}}x(\frac{3}{2})=1,\\ \frac{5}{2}I^{\frac{1}{6}}x(6)+\frac{1}{2}I^{\frac{1}{20}}x(\frac{5}{2})-\frac{4}{5}I^{\frac{2}{9}}x(\frac{3}{20})-\frac{4}{3}I^{\frac{2}{7}}x(\frac{1}{2})-\frac{2}{17}I^{\frac{3}{5}}x(\frac{1}{16})=-1. \end{cases} \tag{4.5}$$

Here $p=4/5$, $q=3/10$, $\lambda=1/80$, $m=4$, $n=5$, $\mu_1=3/2$, $\alpha_1=2/5$, $\eta_1=8/5$, $\mu_2=-1/3$, $\alpha_2=3/10$, $\eta_2=4/3$, $\mu_3=-2/11$, $\alpha_3=2/15$, $\eta_3=9/2$, $\mu_4=1/7$, $\alpha_4=1/3$, $\eta_4=3/2$, $\sigma_1=1$, $\nu_1=5/2$, $\beta_1=1/6$, $\xi_1=6$, $\nu_2=1/2$, $\beta_2=1/20$, $\xi_2=5/2$, $\nu_3=-4/5$, $\beta_3=2/9$, $\xi_3=3/20$, $\nu_4=-4/3$, $\beta_4=2/7$, $\xi_4=1/2$, $\nu_5=-2/17$, $\beta_5=3/5$, $\xi_5=1/16$, $\sigma_2=-1$, and $f(t,x)=t^2e^{-2t^2}\sin(x+\pi/3)/4\pi$. By a direct computation, we have

$$\begin{aligned} \Lambda_1 &= \frac{T^{q+p}}{\Gamma(q+p+1)}+\left(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q+p}}{\Gamma(\alpha_i+q+p+1)}\\ &\quad+\left(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q+p}}{\Gamma(\beta_j+q+p+1)}\approx 39.248431,\\ \Lambda_2 &= |\lambda|\left(\frac{T^q}{\Gamma(q+1)}+\left(\frac{|\Psi_2|T^q+|\Omega_2|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{i=1}^{m}\frac{|\mu_i|\eta_i^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\right.\\ &\quad\left.+\left(\frac{|\Psi_1|T^q+|\Omega_1|\Gamma(q+1)}{|\Delta|\Gamma(q+1)}\right)\sum_{j=1}^{n}\frac{|\nu_j|\xi_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}\right)\approx 0.214034. \end{aligned}$$

Choosing $\gamma=0.02<(1-\Lambda_2)\Lambda_1^{-1}\approx 0.020025$ and $M=0.016$, we can show that

$$\begin{aligned} \bigl|f\bigl(t,x(t)\bigr)\bigr| &\le \left|\frac{t^2e^{-2t^2}}{4\pi}\right|\left|\sin\left(x+\frac{\pi}{3}\right)\right|\\ &\le \left(\frac{t^2e^{-2t^2}}{4\pi}\right)|x|+\left(\frac{t^2e^{-2t^2}}{12}\right)\\ &\le 0.014637|x|+0.015328\\ &\le \gamma|x|+M, \end{aligned}$$

which satisfies (H_6). By Theorem 3.6, problem (4.5) has at least one solution on $[0,2\pi]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

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