Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

On the growth of solutions of certain higher-order linear differential equations

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Abstract

In this paper, we investigate the growth of meromorphic solutions of the equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0,$$
$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z),$$

where $A_0(z)$ ($\not\equiv 0$), $A_1(z), \dots, A_{k-1}(z)$ and $F(z)$ ($\not\equiv 0$) are entire functions of finite order. We find some conditions on the coefficients to guarantee that every nontrivial meromorphic solution of such equations is of infinite order.

MSC: 30D35; 34M10

Keywords: entire function; differential equation; order

1 Introduction and main results

It is assumed that the reader is familiar with the standard notations and the fundamental results of the Nevanlinna theory [1–3]. Let $f(z)$ be a nonconstant meromorphic function in the complex plane. We use notations $\sigma(f)$ and $\sigma_2(f)$ to denote the order of growth and the hyper-order of f, which are defined by

$$\sigma(f) = \varlimsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}, \qquad \sigma_2(f) = \varlimsup_{r\to\infty} \frac{\log^+ \log^+ T(r,f)}{\log r},$$

respectively.

Consider the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where $A_0(z)$ ($\not\equiv 0$), $A_1(z), \dots, A_{k-1}(z)$ are entire functions, if the coefficients $A_j(z)$ ($j = 0, \dots, k-1$) are polynomials, then all nontrivial solutions of (1.1) are of finite order (see [2, 4]). If s is the largest integer such that $A_s(z)$ is transcendental, it is well known that (1.1) has at most s linearly independent finite-order solutions. Thus when at least one of the coefficients $A_j(z)$ is transcendental, most of the solutions of (1.1) are of infinite order. In the case when

$$\max_{1\le j\le k-1}\{\sigma(A_j)\} < \sigma(A_0) < +\infty,$$

Chen and Yang [5] proved that all nontrivial solutions of (1.1) are of infinite order. In the case when

$$\max_{j\neq d}\{\sigma(A_j)\} < \sigma(A_d) \leq \frac{1}{2},$$

Hellerstein *et al.* [6] proved that all transcendental solutions of (1.1) are of infinite order. While in the case when $\sigma(A_0) = \sigma(A_1) = \cdots = \sigma(A_{k-1})$, (1.1) may have a solution of finite order.

Example 1.1 The equation

$$f''' + (e^z - 1)f'' - e^{2z}f' + (e^{2z} - e^z)f = 0$$

has a solution $f(z) = e^z$ of $\sigma(f) = 1$. Here $\sigma(e^z - 1) = \sigma(e^{2z}) = \sigma(e^{2z} - e^z) = 1$.

Thus a natural question is what conditions on $A_j(z)$ when $\sigma(A_0) = \sigma(A_1) = \cdots = \sigma(A_{k-1})$ will guarantee that all nontrivial solutions of (1.1) are of infinite order. Concerning this question, we recall the following results as for the special case of $k = 2$.

Theorem A (see [7]) *Let $P(z) = a_nz^n + \cdots$, $Q(z) = b_nz^n + \cdots$ be polynomials with degree n (≥ 1), $h_0(z)$ $(\not\equiv 0)$, $h_1(z)$ be entire functions of order less than n. If $\arg a_n \neq \arg b_n$ or $a_n = cb_n$ $(0 < c < 1)$, then every nontrivial solution f of the equation*

$$f'' + h_1(z)e^{P(z)}f' + h_0(z)e^{Q(z)}f = 0$$

satisfies $\sigma(f) = \infty$ and $\sigma_2(f) \geq n$.

Theorem B (see [8]) *Let $h_j(z)$ $(\not\equiv 0)$ $(j = 0, 1)$ be entire functions of order less than 1, a, b be nonzero complex numbers such that $\arg a \neq \arg b$ or $a = cb$ $(c > 1)$. Then every nontrivial solution f of the equation*

$$f'' + h_1(z)e^{az}f' + h_0(z)e^{bz}f = 0 \tag{1.2}$$

satisfies $\sigma(f) = \infty$.

From Theorems A and B, we obtain that if $a \neq b$, then every nontrivial solution f of Eq. (1.2) is of infinite order. When the coefficient of f is of the form $h_{01}(z)e^{a_1z} + h_{02}(z)e^{a_2z}$, where $a_1 \neq a_2$, Peng and Chen obtained the following result.

Theorem C (see [9]) *Let $h_{0j}(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions of order less than 1, a_1, a_2 be nonzero complex numbers and $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every nontrivial solution f of the equation*

$$f'' + e^{-z}f' + (h_{01}(z)e^{a_1z} + h_{02}(z)e^{a_2z})f = 0 \tag{1.3}$$

satisfies $\sigma(f) = \infty$ and $\sigma_2(f) = 1$.

In this paper, we continue to investigate the growth of solutions of higher linear differential equations, which has the same form as Eq. (1.3), and obtain the following results which generalize Theorem C.

Theorem 1.1 *Suppose that* $h_{01}(z)$ $(\not\equiv 0)$, $h_{02}(z)$, $h_j(z)$ $(j = 1,\dots,k-1)$ *are entire functions of order less than* n, $P_i(z) = a_{in}z^n + \cdots + a_{i0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(i = 1,2; j = 1,\dots,k-1)$ *are polynomials with degree* n (≥ 1), *where* a_{il}, b_{jl} $(i = 1,2; j = 1,\dots,k-1; l = 0,\dots,n)$ *are complex constants. If* $a_{1n} \neq a_{2n}$, b_{jn} $(j = 1,\dots,k-1)$ *satisfies the following conditions*:

(i) *there exists some* $s \in \{1,\dots,k-1\}$ *such that* $\arg b_{sn} \neq \arg a_{1n}$;

(ii) $b_{jn} = c_j a_{1n}$ $(0 < c_j < 1)$ *for* $j \in I_1$ *and* $b_{jn} = d_j b_{sn}$ $(0 < d_j < 1)$ *for* $j \in I_2$, *where* $I_1 \cup I_2 = \{1,\dots,k-1\} \setminus \{s\}$, $I_1 \cap I_2 = \emptyset$,

then every solution f $(\not\equiv 0)$ *of the equation*

$$f^{(k)} + h_{k-1}(z)e^{Q_{k-1}(z)}f^{(k-1)} + \cdots + h_1(z)e^{Q_1(z)}f' + \left(h_{01}(z)e^{P_1(z)} + h_{02}(z)e^{P_2(z)}\right)f = 0 \qquad (1.4)$$

satisfies $\sigma(f) = \infty$ *and* $\sigma_2(f) = n$.

From the proof of Theorem 1.1, we also obtain the following results.

Corollary 1.1 *Suppose that* $h_{01}(z)$ $(\not\equiv 0)$, $h_{02}(z)$, $h_j(z)$ $(j = 1,\dots,k-1)$ *are entire functions of order less than* n, $P_i(z) = a_{in}z^n + \cdots + a_{i0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(i = 1,2; j = 1,\dots,k-1)$ *are polynomials with degree* n (≥ 1). *If* $a_{1n} \neq a_{2n}$ *and there exists some* $s \in \{1,\dots,k-1\}$ *such that* $\arg b_{sn} \neq \arg a_{1n}$, $b_{jn} = d_j b_{sn}$ $(0 < d_j < 1, 1 \leq j \leq k-1, j \neq s)$, *then every solution* f $(\not\equiv 0)$ *of* (1.4) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) = n$.

Corollary 1.2 *Suppose that* $h_{01}(z)$ $(\not\equiv 0)$, $h_{02}(z)$, $h_j(z)$ $(j = 1,\dots,k-1)$ *are entire functions of order less than* n, $P_i(z) = a_{in}z^n + \cdots + a_{i0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(i = 1,2; j = 1,\dots,k-1)$ *are polynomials with degree* n (≥ 1). *If* $a_{1n} \neq a_{2n}$ *and* $b_{jn} = c_j a_{1n}$ $(0 < c_j < 1, 1 \leq j \leq k-1)$, *then every solution* f $(\not\equiv 0)$ *of* (1.4) *satisfies* $\sigma(f) = \infty$ *and* $\sigma_2(f) = n$.

Remark 1.1 Corollaries 1.1 and 1.2 are extensions of Theorem C, because Theorem C is just the case for $k = 2$, $n = 1$, $b_{1n} = -1$ in Corollaries 1.1 and 1.2.

Remark 1.2 From Theorem 1.1 and Corollaries 1.1, 1.2, we obtain that every solution f $(\not\equiv 0)$ of the equation

$$f^{(k)} + h_{k-1}(z)e^{Q_{k-1}(z)}f^{(k-1)} + \cdots + h_1(z)e^{Q_1(z)}f' + B(z)\cos\bigl(P(z)\bigr)f = 0$$

or

$$f^{(k)} + h_{k-1}(z)e^{Q_{k-1}(z)}f^{(k-1)} + \cdots + h_1(z)e^{Q_1(z)}f' + B(z)\sin\bigl(P(z)\bigr)f = 0$$

satisfies $\sigma(f) = \infty$ and $\sigma_2(f) = n$, where $P(z) = -iP_1$, $B(z)$ is an entire function of order less than n.

Recently, Wang and Laine investigated the growth of solutions of the non-homogeneous linear differential equation

$$f'' + h_1(z)e^{az}f' + h_0(z)e^{bz}f = F(z) \tag{1.5}$$

corresponding to (1.2) and obtained the following result.

Theorem D (see [10]) *Suppose that* h_0 $(\not\equiv 0)$, h_1 $(\not\equiv 0)$, F $(\not\equiv 0)$ *are entire functions of order less than* 1, *and the complex constants* a, b *satisfy* $ab \neq 0$ *and* $a \neq b$. *Then every nontrivial solution* f *of Eq.* (1.5) *is of infinite order.*

Thus the other purpose of this paper is to investigate the growth of solutions of the non-homogeneous linear differential equation

$$f^{(k)} + h_{k-1}(z)e^{Q_{k-1}(z)}f^{(k-1)} + \cdots + h_1(z)e^{Q_1(z)}f' + \bigl(h_{01}(z)e^{P_1(z)} + h_{02}(z)e^{P_2(z)}\bigr)f = F(z) \tag{1.6}$$

corresponding to (1.4). We obtain the following results.

Theorem 1.2 *Under the hypotheses of Theorem* 1.1, *if* $b_{sn} \neq a_{2n}$, $F(z)$ $(\not\equiv 0)$ *is an entire function of order less than* n, *then every solution* f *of Eq.* (1.6) *satisfies* $\sigma(f) = \infty$.

From the proof of Theorem 1.2, we also obtain the following results.

Corollary 1.3 *Suppose that* $h_{01}(z)$ $(\not\equiv 0)$, $h_{02}(z)$, $h_j(z)$ $(j = 1, \ldots, k-1)$, $F(z)$ $(\not\equiv 0)$ *are entire functions of order less than* n, $P_i(z) = a_{in}z^n + \cdots + a_{i0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(i = 1, 2;\ j = 1, \ldots, k-1)$ *are polynomials with degree* n (≥ 1). *If* $a_{1n} \neq a_{2n}$, *and there exists some* $s \in \{1, \ldots, k-1\}$ *such that* $\arg b_{sn} \neq \arg a_{1n}$, $b_{sn} \neq a_{2n}$, $b_{jn} = d_j b_{sn}$ $(0 < d_j < 1, 1 \leq j \leq k-1, j \neq s)$, *then every solution* f *of Eq.* (1.6) *satisfies* $\sigma(f) = \infty$.

Corollary 1.4 *Suppose that* $h_{01}(z)$ $(\not\equiv 0)$, $h_{02}(z)$, $h_j(z)$ $(j = 1, \ldots, k-1)$, $F(z)$ $(\not\equiv 0)$ *are entire functions of order less than* n, $P_i(z) = a_{in}z^n + \cdots + a_{i0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(i = 1, 2;\ j = 1, \ldots, k-1)$ *are polynomials with degree* n (≥ 1). *If* $a_{1n} \neq a_{2n}$ *and* $b_{jn} = c_j a_{1n}$ $(0 < c_j < 1, 1 \leq j \leq k-1)$, *then every solution* f *of Eq.* (1.6) *satisfies* $\sigma(f) = \infty$.

Remark 1.3 When $\sigma(F) \geq n$, (1.6) may have a solution of finite order.

Example 1.2 The equation

$$f''' + e^{-z}f'' + e^{z}f' + \bigl(e^{2z} - 2e^{-z}\bigr)f = \bigl(e^{2z} + 2ze^{z} + 4z^2e^{-z} + 8z^3 + 12z\bigr)e^{z^2}$$

has a solution $f(z) = e^{z^2}$ of $\sigma(f) = 2$. Here $P_1(z) = 2z$, $P_2(z) = -z$, $Q_1(z) = z$, $Q_2(z) = -z = Q_s(z)$ satisfy the hypotheses of Theorem 1.1, and $\sigma(F) = 2 > n = 1$.

Example 1.3 The equation

$$f^{(4)} + e^{-2z}f''' - e^{-z}f'' + e^{z}f' + \bigl(e^{2z} + e^{-z}\bigr)f = e^{3z} + e^{2z} + e^{z} + e^{-z}$$

has a solution $f(z) = e^{z}$ of $\sigma(f) = 1$. Here $P_1(z) = 2z$, $P_2(z) = -z$, $Q_1(z) = z$, $Q_2(z) = -z$, $Q_3(z) = -2z = Q_s(z)$ satisfy the hypotheses of Theorem 1.1, and $\sigma(F) = 1 = n$.

2 Lemmas

Lemma 2.1 ([11]) *Let f_j ($j = 1, \dots, n$) and g_j ($j = 1, \dots, n$) be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$,

(ii) *the order of f_j is less than the order of $e^{g_h - g_k}$ for $n \geq 2$ and $1 \leq j \leq n$, $1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.2 ([11]) *Let f_j ($j = 1, \dots, n+1$) and g_j ($j = 1, \dots, n$) be entire functions such that*

(i) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv f_{n+1}(z)$,

(ii) *the order of f_j is less than the order of e^{g_k} for $1 \leq j \leq n+1$, $1 \leq k \leq n$; and, furthermore, the order of f_j is less than the order of $e^{g_h - g_k}$ for $n \geq 2$ and $1 \leq j \leq n+1$, $1 \leq h < k \leq n$.*

Then $f_j(z) \equiv 0$ ($j = 1, \dots, n+1$).

Lemma 2.3 ([12]) *Let $f(z)$ be a transcendental meromorphic function of $\sigma(f) < \infty$, k, j ($k > j \geq 0$) be integers. Then, for any given $\varepsilon > 0$, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0, 2\pi) \setminus E$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\sigma(f)-1+\varepsilon)}.$$

Lemma 2.4 ([8, 13]) *Let $P(z) = (\alpha + i\beta)z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) be a polynomial with degree n (≥ 1), $A(z)$ ($\not\equiv 0$) be an entire function with $\sigma(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then, for any given $\varepsilon > 0$, there is a set $E_0 \subset [0, 2\pi)$ that has linear measure zero such that for any $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1)$ and sufficiently large r, we have*

(i) *if $\delta(P, \theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} \leq \left|g(re^{i\theta})\right| \leq \exp\{(1+\varepsilon)\delta(P,\theta)r^n\}; \tag{2.1}$$

(ii) *if $\delta(P, \theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} \leq \left|g(re^{i\theta})\right| \leq \exp\{(1-\varepsilon)\delta(P,\theta)r^n\}, \tag{2.2}$$

where $E_1 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.5 ([12]) *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $H \subset (1, \infty)$ that has a finite logarithmic measure and a constant $B > 0$ depending only on α and k, j ($k > j \geq 0$) such that for all z with $|z| = r \notin [0, 1] \cup H$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq B\left(\frac{T(\alpha r, f)}{r}\left(\log^{\alpha} r\right)\log T(\alpha r, f)\right)^{k-j}.$$

Lemma 2.6 *Let $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ ($j = 1, 2, 3$) be polynomials with degree n (≥ 1), where b_{jl} ($j = 1, 2, 3$; $l = 0, \dots, n$) are complex constants. Set $z = re^{i\theta}$, $\arg b_{jn} = \varphi_j \in [0, 2\pi)$, $\delta(Q_j, \theta) = |b_{jn}| \cos(\varphi_j + n\theta)$ ($j = 1, 2, 3$). If φ_j ($j = 1, 2, 3$) are distinct, then there exist two real numbers θ_1, θ_2 ($\theta_1 < \theta_2$) such that for each $\theta \in (\theta_1, \theta_2)$, we have*

$$\delta(Q_1, \theta)\delta(Q_2, \theta) < 0 \quad \textit{and} \quad \delta(Q_3, \theta) < 0.$$

Proof Set

$$\Omega_{jk} = \{z = re^{i\theta} : \theta_{jk} < \theta < \theta_{j(k+1)}\} \quad (k = 0,1,\ldots,2n-1, j = 1,2,3),$$

where $\theta_{jk} = (2k-1)\frac{\pi}{2n} - \frac{\varphi_j}{n}$. Then the following results hold.

(i) $\delta(Q_j,\theta_{jk}) = 0$ ($k = 0,1,\ldots,2n-1, j = 1,2,3$).

(ii) $\delta(Q_j,\theta) > 0$ holds for $z \in \Omega_{jk}$ and k is even, $\delta(Q_j,\theta) < 0$ holds for $z \in \Omega_{jk}$ and k is odd.

(iii) $\theta_{j(k+1)} - \theta_{jk} = \frac{\pi}{n}$ ($k = 0,1,\ldots,2n-1, j = 1,2,3$).

Next we discuss the following four cases.

Case 1. $|\varphi_i - \varphi_j| \neq \pi$ ($1 \leq i < j \leq 3$). By (ii), among the angular domains Ω_{1k}, we may take an angular domain Ω_{1k_0} such that for $z \in \Omega_{1k_0}$, $\delta(Q_1,\theta) > 0$. Hence when $z \in \Omega_{1(k_0-1)} \cup \Omega_{1(k_0+1)}$, we have $\delta(Q_1,\theta) < 0$. For the sake of convenience, we write

$$\Omega_{1k_0} = \{z = re^{i\theta} : \alpha < \theta < \beta\}.$$

Then by (iii) we write

$$\Omega_{1(k_0-1)} = \left\{z = re^{i\theta} : \alpha - \frac{\pi}{n} < \theta < \alpha\right\},$$
$$\Omega_{1(k_0+1)} = \left\{z = re^{i\theta} : \beta < \theta < \beta + \frac{\pi}{n}\right\}.$$

Since $|\varphi_i - \varphi_j| \neq \pi$, by (i) and (iii) we know that there exist two distinct rays

$$\arg z = \gamma_2 \in \Omega_{1k_0}, \qquad \arg z = \gamma_3 \in \Omega_{1k_0}$$

such that

$$\delta(Q_2,\gamma_2) = 0, \qquad \delta(Q_3,\gamma_3) = 0,$$

respectively. Hence by (ii) we obtain that $\delta(Q_i,\theta')\delta(Q_i,\theta'') < 0$ for any $\theta' \in (\alpha,\gamma_i)$, $\theta'' \in (\gamma_i,\beta)$ and $i = 2,3$. Without loss of generality, we assume that $\gamma_2 < \gamma_3$. Now we discuss the following four subcases.

Subcase 1. If $\delta(Q_2,\theta) < 0$ when $\theta \in (\alpha,\gamma_2)$ and $\delta(Q_3,\theta) < 0$ when $\theta \in (\alpha,\gamma_3)$, then we take $\theta_1 = \alpha$, $\theta_2 = \gamma_2$. Hence when $\theta \in (\theta_1,\theta_2)$, we have $\delta(Q_1,\theta) > 0$, $\delta(Q_2,\theta) < 0$ and $\delta(Q_3,\theta) < 0$.

Subcase 2. If $\delta(Q_2,\theta) < 0$ when $\theta \in (\alpha,\gamma_2)$ and $\delta(Q_3,\theta) > 0$ when $\theta \in (\alpha,\gamma_3)$, then we take $\theta_1 = \beta$, $\theta_2 = \gamma_2 + \frac{\pi}{n}$. Hence when $\theta \in (\theta_1,\theta_2)$, we have $\delta(Q_1,\theta) < 0$, $\delta(Q_2,\theta) > 0$ and $\delta(Q_3,\theta) < 0$.

Subcase 3. If $\delta(Q_2,\theta) > 0$ when $\theta \in (\alpha,\gamma_2)$ and $\delta(Q_3,\theta) < 0$ when $\theta \in (\alpha,\gamma_3)$, then we take $\theta_1 = \gamma_3 - \frac{\pi}{n}$, $\theta_2 = \alpha$. Hence when $\theta \in (\theta_1,\theta_2)$, we have $\delta(Q_1,\theta) < 0$, $\delta(Q_2,\theta) > 0$ and $\delta(Q_3,\theta) < 0$.

Subcase 4. If $\delta(Q_2,\theta) > 0$ when $\theta \in (\alpha,\gamma_2)$ and $\delta(Q_3,\theta) > 0$ when $\theta \in (\alpha,\gamma_3)$, then we take $\theta_1 = \gamma_3$, $\theta_2 = \beta$. Hence when $\theta \in (\theta_1,\theta_2)$, we have $\delta(Q_1,\theta) > 0$, $\delta(Q_2,\theta) < 0$ and $\delta(Q_3,\theta) < 0$.

Case 2. $|\varphi_1 - \varphi_2| = \pi$. Since $\varphi_j \in [0,2\pi)$ and φ_j ($j = 1,2,3$) are distinct, we obtain that

$$|\varphi_3 - \varphi_1| \neq \pi, \qquad |\varphi_3 - \varphi_2| \neq \pi. \tag{2.3}$$

By (ii), among the angular domains Ω_{1k}, we may take an angular domain $\Omega_{1k_0} = \{z = re^{i\theta} : \theta_{1k_0} < \theta < \theta_{1(k_0+1)}\}$ such that for $z \in \Omega_{1k_0}$, $\delta(Q_1,\theta) > 0$. Note that

$$\delta(Q_2,\theta) = |b_{2n}|\cos(\varphi_2 + n\theta) = -|b_{2n}|\cos(\varphi_1 + n\theta),$$

so when $z \in \Omega_{1k_0}$, we have $\delta(Q_2,\theta) < 0$. By (2.3), (i) and (iii), we know that there exists a ray $\arg z = \gamma_3 \in \Omega_{1k_0}$ such that $\delta(Q_3,\gamma_3) = 0$. Hence by (ii) we obtain that $\delta(Q_3,\theta) < 0$ for $\theta \in (\theta_{1k_0},\gamma_3)$ or $\delta(Q_3,\theta) < 0$ for $\theta \in (\gamma_3,\theta_{1(k_0+1)})$. Hence we may take $(\theta_1,\theta_2) = (\theta_{1k_0},\gamma_3)$ or $(\theta_1,\theta_2) = (\gamma_3,\theta_{1(k_0+1)})$ such that for $\theta \in (\theta_1,\theta_2)$, $\delta(Q_1,\theta) > 0$, $\delta(Q_2,\theta) < 0$ and $\delta(Q_3,\theta) < 0$.

Case 3. $|\varphi_1 - \varphi_3| = \pi$. Then, by a similar reasoning to that of Case 2, we can prove the result.

Case 4. $|\varphi_2 - \varphi_3| = \pi$. Since $\varphi_j \in [0,2\pi)$ and φ_j $(j = 1,2,3)$ are distinct, we obtain that

$$|\varphi_1 - \varphi_2| \neq \pi, \qquad |\varphi_1 - \varphi_3| \neq \pi. \tag{2.4}$$

By (ii), among the angular domains Ω_{2k}, we may take an angular domain $\Omega_{2k_0} = \{z = re^{i\theta} : \theta_{2k_0} < \theta < \theta_{2(k_0+1)}\}$ such that for $z \in \Omega_{2k_0}$, $\delta(Q_2,\theta) > 0$. Note that

$$\delta(Q_3,\theta) = |b_{3n}|\cos(\varphi_3 + n\theta) = -|b_{3n}|\cos(\varphi_2 + n\theta),$$

so when $z \in \Omega_{2k_0}$, we have $\delta(Q_3,\theta) < 0$. By (2.4), (i) and (iii), we know that there exists a ray $\arg z = \gamma_1 \in \Omega_{2k_0}$ such that $\delta(Q_1,\gamma_1) = 0$. Hence by (ii) we obtain that $\delta(Q_1,\theta) < 0$ for $\theta \in (\theta_{2k_0},\gamma_1)$ or $\delta(Q_1,\theta) < 0$ for $\theta \in (\gamma_1,\theta_{2(k_0+1)})$. Hence we may take $(\theta_1,\theta_2) = (\theta_{2k_0},\gamma_1)$ or $(\theta_1,\theta_2) = (\gamma_1,\theta_{2(k_0+1)})$ such that for $\theta \in (\theta_1,\theta_2)$, $\delta(Q_1,\theta) < 0$, $\delta(Q_2,\theta) > 0$ and $\delta(Q_3,\theta) < 0$. □

Remark 2.1 From the proof of Lemma 2.6, we also obtain the following result.

Let $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$ $(j = 1,2)$ be polynomials with degree n (≥ 1), where b_{jl} $(j = 1,2;$ $l = 0,\ldots,n)$ are complex constants. Set $z = re^{i\theta}$, $\arg b_{jn} = \varphi_j \in [0,2\pi)$, $\delta(Q_j,\theta) = |b_{jn}|\cos(\varphi_j + n\theta)$. If $\varphi_1 \neq \varphi_2$, then there exist two real numbers θ_1, θ_2 $(\theta_1 < \theta_2)$ such that for each $\theta \in (\theta_1,\theta_2)$, we have

$$\delta(Q_1,\theta) > 0 \quad \text{and} \quad \delta(Q_2,\theta) < 0.$$

Lemma 2.7 ([2]) *Let $g : (0,\infty) \to R$ and $h : (0,\infty) \to R$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set H of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ holds for all $r > r_0$.*

Lemma 2.8 ([14]) *Let $A_j(z)$ $(j = 0,\ldots,k-1)$ be entire functions of finite order. If $f(z)$ $(\not\equiv 0)$ is a solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0,$$

then $\sigma_2(f) \leq \max_{0\leq j\leq k-1}\{\sigma(A_j)\}$.

Lemma 2.9 ([15]) *Let $f(z)$ be an entire function and $\beta > 0$. If $G(z) = \frac{\log^+ |f^{(k)}(z)|}{|z|^\beta}$ is unbounded on some ray $\arg z = \theta$, then there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ $(m = 1, 2, \ldots)$ with $r_m \to \infty$ such that*

$$G(z_m) \to \infty$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \le \frac{1}{(k-j)!}\bigl(1 + o(1)\bigr) r_m^{k-j} \quad (j = 0, \ldots, k-1)$$

as $m \to \infty$.

Lemma 2.10 ([15]) *Let $f(z)$ be an entire function of $\sigma(f) < \infty$. If there exists a set $E \subset [0, 2\pi)$ which has linear measure zero such that $\log^+ |f(re^{i\theta})| \le Mr^\alpha$ for any ray $\arg z = \theta \in [0, 2\pi) \setminus E$, where M is a positive constant depending on θ, while α is a positive constant independent of θ, then $\sigma(f) \le \alpha$.*

3 Proofs of the results

Proof of Theorem 1.1 Let f $(\not\equiv 0)$ be a solution of (1.4), then f is an entire function.

Step 1. We prove that $\sigma(f) \ge n$. Suppose that $\sigma(f) < n$, rewrite (1.4) in the form

$$\sum_l B_l e^{d_{j_l} Q_s} + h_s f^{(s)} e^{Q_s} + \sum_q B_q e^{c_{j_q} P_1} + h_{01} f e^{P_1} + h_{02} f e^{P_2} = -f^{(k)}, \tag{3.1}$$

where $c_{j_q} \in \{c_j : j \in I_1\}$ and c_{j_q} are distinct, $d_{j_l} \in \{d_j : j \in I_2\}$ and d_{j_l} are distinct, B_l, B_q are entire functions of order less than n. Now we discuss the relations between the coefficients of the term z^n of polynomials $d_{j_l} Q_s$, Q_s, $c_{j_q} P_1$, P_1 and P_2.

Case 1. There exists some l such that $\deg(d_{j_l} Q_s - P_2) < n$. Then merging the terms $B_l e^{d_{j_l} Q_s}$ and $h_{02} f e^{P_2}$, by (3.1) we get

$$\sum_l B'_l e^{d_{j_l} Q_s} + h_s f^{(s)} e^{Q_s} + \sum_q B_q e^{c_{j_q} P_1} + h_{01} f e^{P_1} = -f^{(k)},$$

where B'_l are entire functions of order less than n. By the hypothesis of Theorem 1.1, we know that the coefficients of the term z^n of polynomials $d_{j_l} Q_s$, Q_s, $c_{j_q} P_1$, P_1 are distinct. Then, by Lemma 2.1 or Lemma 2.2, we get $h_{01} f \equiv 0$. This is absurd.

Case 2. $\deg(Q_s - P_2) < n$. Then merging the terms $h_s f^{(s)} e^{Q_s}$ and $h_{02} f e^{P_2}$, by (3.1) we get

$$\sum_l B_l e^{d_{j_l} Q_s} + D_s e^{Q_s} + \sum_q B_q e^{c_{j_q} P_1} + h_{01} f e^{P_1} = -f^{(k)},$$

where D_s is an entire function of order less than n. By the hypothesis of Theorem 1.1, we know that the coefficients of the term z^n of polynomials $d_{j_l} Q_s$, Q_s, $c_{j_q} P_1$, P_1 are distinct. Then, by Lemma 2.1 or Lemma 2.2, we get $h_{01} f \equiv 0$. This is absurd.

Case 3. There exists some q such that $\deg(c_{j_q} P_1 - P_2) < n$. Then merging the terms $B_q e^{c_{j_q} P_1}$ and $h_{02} f e^{P_2}$, by (3.1) we get

$$\sum_l B_l e^{d_{j_l} Q_s} + h_s f^{(s)} e^{Q_s} + \sum_q B'_q e^{c_{j_q} P_1} + h_{01} f e^{P_1} = -f^{(k)},$$

where B'_q are entire functions of order less than n. By the hypothesis of Theorem 1.1, we know that the coefficients of the term z^n of polynomials $d_{j_l}Q_s$, Q_s, $c_{j_q}P_1$, P_1 are distinct. Then by Lemma 2.1 or Lemma 2.2, we get $h_{01}f \equiv 0$. This is absurd.

Step 2. We prove that $\sigma(f) = \infty$ and $\sigma_2(f) \geq n$. By Step 1 we know that f is transcendental. Then by Lemma 2.5 there exists a set $H \subset (1,\infty)$ that has a finite logarithmic measure, and a constant $B > 0$ such that for all z with $|z| = r \notin [0,1] \cup H$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\bigl(T(2r,f)\bigr)^{k+1} \quad (j = 1,\ldots,k-1). \tag{3.2}$$

Set

$$z = re^{i\theta}, \qquad \delta(P_i,\theta) = |a_{in}|\cos(\arg a_{in} + n\theta) \quad (i = 1,2),$$
$$\delta(Q_j,\theta) = |b_{jn}|\cos(\arg b_{jn} + n\theta) \quad (j = 1,\ldots,k-1).$$

Next we discuss the following three cases.

Case 1. $\arg a_{1n} \neq \arg a_{2n}$ and $\arg a_{2n} \neq \arg b_{sn}$. By the hypothesis of Theorem 1.1, we know that $\arg a_{1n}$, $\arg a_{2n}$, $\arg b_{sn}$ are distinct. Then by Lemma 2.6 there exist two real numbers θ_1, θ_2 ($\theta_1 < \theta_2$) such that for each $\theta \in (\theta_1,\theta_2)$, we have

$$\delta(P_1,\theta)\delta(P_2,\theta) < 0 \quad \text{and} \quad \delta(Q_s,\theta) < 0.$$

Without loss of generality, we assume that $\delta(P_1,\theta) > 0$, $\delta(P_2,\theta) < 0$. Let $c = \max_{j\in I_1}\{c_j\}$, then $c < 1$. By Lemma 2.4, for $\forall\varepsilon$ ($0 < 2\varepsilon < \frac{1-c}{1+c}$), there exists a set $E \subset [0,2\pi)$ of linear measure zero such that for $z = re^{i\theta}$ with $\theta \in (\theta_1,\theta_2) \setminus E$ and sufficiently large r, we have

$$\bigl|h_{01}(z)e^{P_1(z)}\bigr| \geq \exp\bigl\{(1-\varepsilon)\delta(P_1,\theta)r^n\bigr\}, \tag{3.3}$$
$$\bigl|h_{02}(z)e^{P_2(z)}\bigr| \leq \exp\bigl\{(1-\varepsilon)\delta(P_2,\theta)r^n\bigr\}, \tag{3.4}$$
$$\bigl|h_j(z)e^{Q_j(z)}\bigr| \leq \exp\bigl\{(1+\varepsilon)c_j\delta(P_1,\theta)r^n\bigr\} \quad (j \in I_1), \tag{3.5}$$
$$\bigl|h_j(z)e^{Q_j(z)}\bigr| \leq \exp\bigl\{(1-\varepsilon)d_j\delta(Q_s,\theta)r^n\bigr\} \quad \bigl(j \in I_2 \cup \{s\}, d_s = 1\bigr). \tag{3.6}$$

Then combining (3.2)-(3.6) and (1.4), for $z = re^{i\theta}$ with $\theta \in (\theta_1,\theta_2) \setminus E$ and sufficiently large $r \notin H$, we have

$$\begin{aligned}
\exp\bigl\{(1-\varepsilon)\delta(P_1,\theta)r^n\bigr\} &\leq \bigl|h_{01}(z)e^{P_1(z)}\bigr| \\
&\leq \left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=1}^{k-1}\bigl|h_j(z)e^{Q_j(z)}\bigr|\left|\frac{f^{(j)}(z)}{f(z)}\right| + \bigl|h_{02}(z)e^{P_2(z)}\bigr| \\
&\leq (k+1)B\bigl(T(2r,f)\bigr)^{k+1}\exp\bigl\{(1+\varepsilon)c\delta(P_1,\theta)r^n\bigr\}.
\end{aligned} \tag{3.7}$$

Then by Lemma 2.7 and (3.7) we get $\sigma(f) = \infty$ and $\sigma_2(f) \geq n$.

Case 2. $\arg a_{1n} \neq \arg a_{2n}$ and $\arg a_{2n} = \arg b_{sn}$. Since $\arg a_{1n} \neq \arg b_{sn}$, by Remark 2.1 there exist two real numbers θ_1, θ_2 ($\theta_1 < \theta_2$) such that for each $\theta \in (\theta_1,\theta_2)$, we have

$$\delta(P_1,\theta) > 0 \quad \text{and} \quad \delta(Q_s,\theta) < 0.$$

Then, using a similar proof to that of Case 1, we get $\sigma(f) = \infty$ and $\sigma_2(f) \geq n$.

Case 3. $\arg a_{1n} = \arg a_{2n}$. Since $a_{1n} \neq a_{2n}$, without loss of generality, we assume that $|a_{1n}| > |a_{2n}|$. Since $\arg a_{1n} \neq \arg b_{sn}$, by Remark 2.1 there exist two real numbers θ_1, θ_2 ($\theta_1 < \theta_2$) such that for each $\theta \in (\theta_1, \theta_2)$, we have

$$\delta(P_1, \theta) > 0 \quad \text{and} \quad \delta(Q_s, \theta) < 0.$$

Let $c = \max_{j \in I_1}\{c_j, \frac{|a_{2n}|}{|a_{1n}|}\}$, then $c < 1$. By Lemma 2.4, for $\forall \varepsilon$ ($0 < 2\varepsilon < \frac{1-c}{1+c}$), there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for $z = re^{i\theta}$ with $\theta \in (\theta_1, \theta_2) \setminus E$ and sufficiently large r, we have (3.3), (3.5), (3.6) and

$$\bigl|h_{02}(z)e^{P_2(z)}\bigr| \le \exp\bigl\{(1+\varepsilon)\delta(P_2, \theta)r^n\bigr\}. \tag{3.8}$$

By (3.2), (3.3), (3.5), (3.6), (3.8) and (1.4), for $z = re^{i\theta}$ with $\theta \in (\theta_1, \theta_2) \setminus E$ and sufficiently large $r \notin H$, we have

$$\begin{aligned} \exp\bigl\{(1-\varepsilon)\delta(P_1,\theta)r^n\bigr\} &\le \bigl|h_{01}(z)e^{P_1(z)}\bigr| \\ &\le \left|\frac{f^{(k)}(z)}{f(z)}\right| + \sum_{j=1}^{k-1}\bigl|h_j(z)e^{Q_j(z)}\bigr|\left|\frac{f^{(j)}(z)}{f(z)}\right| + \bigl|h_{02}(z)e^{P_2(z)}\bigr| \\ &\le (k+1)B\bigl(T(2r,f)\bigr)^{k+1}\exp\bigl\{(1+\varepsilon)c\delta(P_1,\theta)r^n\bigr\}. \end{aligned} \tag{3.9}$$

Then, by Lemma 2.7 and (3.9), we get $\sigma(f) = \infty$ and $\sigma_2(f) \ge n$.

Step 3. We prove that $\sigma_2(f) = n$. By Step 2, we get $\sigma_2(f) \ge n$. On the other hand, by Lemma 2.8 we get $\sigma_2(f) \le n$. Hence we obtain that $\sigma_2(f) = n$. □

Proof of Theorem 1.2 Let f be a solution of (1.6), then f is a nonzero entire function. Using a similar proof to Step 1 in the proof of Theorem 1.1, we obtain that $\sigma(f) \ge n$. Now we prove that $\sigma(f) = \infty$. Suppose that $\sigma(f) < \infty$, by Lemma 2.3, for any given $\varepsilon > 0$, there exists a set $E_0 \subset [0, 2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0, 2\pi) \setminus E_0$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le |z|^{(j-i)(\sigma(f)-1+\varepsilon)} \quad (0 \le i < j \le k). \tag{3.10}$$

Let α, β be two real numbers such that $\sigma(F) < \alpha < \beta < n \le \sigma(f)$, then

$$\bigl|F(z)\bigr| \le \exp\bigl\{r^\alpha\bigr\} \tag{3.11}$$

holds for sufficiently large $|z| = r$. Set

$$\begin{aligned} &\delta(P_i, \theta) = |a_{in}|\cos(\arg a_{in} + n\theta) \quad (i = 1, 2), \\ &\delta(Q_j, \theta) = |b_{jn}|\cos(\arg b_{jn} + n\theta) \quad (j = 1, \ldots, k-1). \end{aligned}$$

By Lemma 2.4, there exists a set $E_1 \subset [0, 2\pi)$ of linear measure zero such that whenever $\theta \in [0, 2\pi) \setminus E_1$, we have

$$\begin{aligned} &\delta(Q_s, \theta) \neq 0, \qquad \delta(P_1, \theta) \neq 0, \qquad \delta(P_2, \theta) \neq 0, \\ &\delta(P_1 - P_2, \theta) \neq 0, \qquad \delta(P_1 - Q_s, \theta) \neq 0, \end{aligned}$$

and for $z = re^{i\theta}$ with $\theta \in [0, 2\pi) \setminus E_1$ and sufficiently large r, $h_{01}(z)e^{P_1(z)}$, $h_{02}(z)e^{P_2(z)}$ and each $h_j(z)e^{Q_j(z)}$ satisfy either (2.1) or (2.2). Next we discuss the following two cases.

Case 1. $\arg a_{1n} = \arg a_{2n}$. Since $a_{1n} \neq a_{2n}$, without loss of generality, we assume that $|a_{1n}| > |a_{2n}|$. For any given $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1)$, we discuss the following three subcases.

Subcase 1. $\delta(Q_s, \theta) > 0$ and $\delta(P_1, \theta) > 0$. If $\delta(Q_s, \theta) > \delta(P_1, \theta)$, let $d = \max_{j \in I_2}\{\frac{\delta(P_1,\theta)}{\delta(Q_s,\theta)}, d_j\}$, then $d < 1$. Now we prove that

$$\frac{\log^+ |f^{(s)}(z)|}{|z|^\beta}$$

is bounded on the ray $\arg z = \theta$. Suppose that this is not the case, then by Lemma 2.9 there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ with $r_m \to \infty$ such that

$$\frac{\log^+ |f^{(s)}(z_m)|}{|z_m|^\beta} \to \infty \tag{3.12}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \leq \frac{1}{(s-j)!}\bigl(1 + o(1)\bigr) r_m^{s-j} \quad (j = 0, \ldots, s-1). \tag{3.13}$$

Combining with (3.11) and (3.12), we get

$$\left|\frac{F(z_m)}{f^{(s)}(z_m)}\right| \to 0. \tag{3.14}$$

Then by (1.6), (2.1), (2.2), (3.10), (3.13) and (3.14), for sufficiently large r_m, we get

$$\begin{aligned}
\exp\bigl\{(1-\varepsilon)\delta(Q_s, \theta)r_m^n\bigr\} &\leq \bigl|h_s(z_m)e^{Q_s(z_m)}\bigr| \\
&\leq \left|\frac{F(z_m)}{f^{(s)}(z_m)}\right| + \left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right| + \sum_{\substack{j=1 \\ j \neq s}}^{k-1} \left|h_j(z_m)e^{Q_j(z_m)}\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \\
&\quad + \bigl\{\bigl|h_{01}(z_m)e^{P_1(z_m)}\bigr| + \bigl|h_{02}(z_m)e^{P_2(z_m)}\bigr|\bigr\}\left|\frac{f(z_m)}{f^{(s)}(z_m)}\right| \\
&\leq (k+2)r_m^{k\sigma(f)} \exp\bigl\{(1+\varepsilon)d\delta(Q_s, \theta)r_m^n\bigr\}.
\end{aligned}$$

When $0 < \varepsilon < \frac{1-d}{1+d}$, the above inequality does not hold. Therefore $\log^+ |f^{(s)}(z)|/|z|^\beta$ is bounded, and we have $|f^{(s)}(z)| \leq \exp\{M|z|^\beta\}$ on the ray $\arg z = \theta$, where $M > 0$ is a real constant, not the same at each occurrence. By the same reasoning as that of [16, Lemma 3.1], we get

$$\bigl|f(z)\bigr| \leq |z|^s \exp\bigl\{M|z|^\beta\bigr\}\bigl(1 + o(1)\bigr) \tag{3.15}$$

on the ray $\arg z = \theta$. If $\delta(Q_s, \theta) < \delta(P_1, \theta)$, let $d' = \max_{j \in I_1}\{\frac{\delta(Q_s,\theta)}{\delta(P_1,\theta)}, \frac{|a_{2n}|}{|a_{1n}|}, c_j\}$, then $d' < 1$. Now we prove that

$$\frac{\log^+ |f(z)|}{|z|^\beta}$$

is bounded on the ray $\arg z = \theta$. Suppose that this is not the case, then by Lemma 2.9 there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ with $r_m \to \infty$ such that

$$\frac{\log^+ |f(z_m)|}{|z_m|^\beta} \to \infty.$$

Then combining with (3.11), we get

$$\left|\frac{F(z_m)}{f(z_m)}\right| \to 0. \tag{3.16}$$

Then by (1.6), (2.1), (2.2), (3.10) and (3.16), for sufficiently large r_m, we get

$$\begin{aligned}
&\exp\bigl\{(1-\varepsilon)\delta(P_1,\theta)r_m^n\bigr\} \\
&\quad \le \bigl|h_{01}(z_m)e^{P_1(z_m)}\bigr| \\
&\quad \le \left|\frac{F(z_m)}{f(z_m)}\right| + \left|\frac{f^{(k)}(z_m)}{f(z_m)}\right| + \sum_{j=1}^{k-1}\left|h_j(z_m)e^{Q_j(z_m)}\frac{f^{(j)}(z_m)}{f^{(}z_m)}\right| + \bigl|h_{02}(z_m)e^{P_2(z_m)}\bigr| \\
&\quad \le (k+2)r_m^{k\sigma(f)}\exp\bigl\{(1+\varepsilon)d'\delta(P_1,\theta)r_m^n\bigr\}.
\end{aligned}$$

When $0 < \varepsilon < \frac{1-d'}{1+d'}$, the above inequality does not hold. Therefore $\log^+ |f(z)|/|z|^\beta$ is bounded, and we have

$$\bigl|f(z)\bigr| \le \exp\bigl\{M|z|^\beta\bigr\} \tag{3.17}$$

on the ray $\arg z = \theta$.

Subcase 2. $\delta(Q_s,\theta) < 0$ and $\delta(P_1,\theta) < 0$. Now we prove that

$$\frac{\log^+ |f^{(k)}(z)|}{|z|^\beta}$$

is bounded on the ray $\arg z = \theta$. Suppose that this is not the case, then by Lemma 2.9 there exists an infinite sequence of points $z_m = r_m e^{i\theta}$ with $r_m \to \infty$ such that

$$\frac{\log^+ |f^{(k)}(z_m)|}{|z_m|^\beta} \to \infty \tag{3.18}$$

and

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \le \frac{1}{(k-j)!}\bigl(1+o(1)\bigr)r_m^{k-j} \quad (j = 0,\dots,k-1). \tag{3.19}$$

Combining with (3.11) and (3.18), we get

$$\left|\frac{F(z_m)}{f^{(k)}(z_m)}\right| \to 0. \tag{3.20}$$

Then by (1.6), (2.2), (3.19) and (3.20), for sufficiently large r_m, we get

$$\begin{aligned} 1 &\le \left|\frac{F(z_m)}{f^{(k)}(z_m)}\right| + \sum_{j=1}^{k-1}\left|h_j(z_m)e^{Q_j(z_m)}\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \\ &\quad + \left\{\left|h_{01}(z_m)e^{P_1(z_m)}\right| + \left|h_{02}(z_m)e^{P_2(z_m)}\right|\right\}\left|\frac{f(z_m)}{f^{(k)}(z_m)}\right| \\ &\to 0. \end{aligned}$$

This is absurd. Therefore $\log^+|f^{(k)}(z)|/|z|^\beta$ is bounded, and we have $|f^{(k)}(z)| \le \exp\{M|z|^\beta\}$ on the ray $\arg z = \theta$. By the same reasoning as that of [16, Lemma 3.1], we get

$$\left|f(z)\right| \le |z|^k \exp\left\{M|z|^\beta\right\}\left(1 + o(1)\right) \tag{3.21}$$

on the ray $\arg z = \theta$.

Subcase 3. $\delta(Q_s,\theta)\delta(P_1,\theta) < 0$. Using a similar argument to that of Subcase 1, we obtain that (3.15) or (3.17) holds on the ray $\arg z = \theta$.

Then by (3.15), (3.17) and (3.21), for any given $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1)$, we have

$$\left|f\left(re^{i\theta}\right)\right| \le r^k \exp\left\{Mr^\beta\right\}\left(1 + o(1)\right),$$

where $E_0 \cup E_1$ is of linear measure zero. Hence by Lemma 2.10 we get $\sigma(f) \le \beta < n$, a contradiction. Hence $\sigma(f) = \infty$.

Case 2. $\arg a_{1n} \ne \arg a_{2n}$. Since $b_{sn} \ne a_{2n}$, $E_2 = \{\theta \in [0, 2\pi) : \delta(P_2 - Q_s, \theta) = 0\}$ is a finite set. Let $\delta_1 = \max\{\delta(P_1,\theta), \delta(P_2,\theta), \delta(Q_s,\theta)\}$, then for any given $\theta \in [0, 2\pi) \setminus (E_0 \cup E_1 \cup E_2)$, $\delta(P_1,\theta)$, $\delta(P_2,\theta)$, $\delta(Q_s,\theta)$ are distinct, and we have $\delta_1 > 0$ or $\delta_1 < 0$. If $\delta_1 > 0$, using the same argument as that of Subcase 1 in Case 1, we obtain that (3.15) or (3.17) holds on the ray $\arg z = \theta$. If $\delta_1 < 0$, using the same argument as that of Subcase 2 in Case 1, we obtain that (3.21) holds on the ray $\arg z = \theta$. Hence we get $\sigma(f) = \infty$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors drafted the manuscript, read and approved the final manuscript.

Acknowledgements
This work is supported by the National Natural Science Foundation of China (No. 11201195, 11171119), the Natural Science Foundation of Jiangxi, China (No. 20132BAB201008, 20122BAB201012). The authors thank the referee for his/her valuable suggestions to improve the present article.

Received: 6 November 2013 Accepted: 8 January 2014 Published: 27 Jan 2014

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10.1186/1687-1847-2014-33
Cite this article as: Mao and Liu: On the growth of solutions of certain higher-order linear differential equations. *Advances in Difference Equations* 2014, 2014:33