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Jordan decomposition and geometric multiplicity for a class of non-symmetric Ornstein-Uhlenbeck operators

Yulei Rao¹, Jiying Wang¹ and Yong Chen^{2*}

*Correspondence: chenrong77@gmail.com
²Hunan University of Science and Technology, Xiangtan, 411201, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we calculate the Jordan decomposition for a class of non-symmetric Ornstein-Uhlenbeck operators with the drift coefficient matrix, being a Jordan block, and the diffusion coefficient matrix, being the identity multiplying a constant. For the 2-dimensional case, we present all the general eigenfunctions by mathematical induction. For the 3-dimensional case, we divide the calculation of the Jordan decomposition into three steps. The key step is to do the canonical projection onto the homogeneous Hermite polynomials, and then use the theory of systems of linear equations. Finally, we get the geometric multiplicity of the eigenvalue of the Ornstein-Uhlenbeck operator.

Keywords: Jordan decomposition; 2-dimensional non-symmetric Ornstein-Uhlenbeck operator; 3-dimensional non-symmetric Ornstein-Uhlenbeck operator

1 Introduction

For the symmetric Ornstein-Uhlenbeck operator, the eigenfunctions are the well-known Hermite polynomials [1]. The eigenfunctions of a type of finite-dimensional normal but non-symmetric Ornstein-Uhlenbeck operators have recently been found. They are the so-called complex Hermite polynomials [2] (or, say, the Hermite-Laguerre-Itô polynomials) where the idea is to proceed by means of a decomposition to the summation of series of up to a 2-dimensional normal Ornstein-Uhlenbeck operator [3]. But if the Ornstein-Uhlenbeck operator is not normal, the general eigenfunctions are still unknown up to now.

In the present paper, we consider the d -dimensional ($d \geq 2$) non-symmetric Ornstein-Uhlenbeck process

$$\begin{bmatrix} dX_1(t) \\ dX_2(t) \\ \dots \\ dX_d(t) \end{bmatrix} = \begin{bmatrix} -c & 1 & 0 & \dots & 0 \\ 0 & -c & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -c \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ \dots \\ X_d(t) \end{bmatrix} dt + \sqrt{2\sigma^2} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ \dots \\ dB_d(t) \end{bmatrix}. \quad (1.1)$$

The associated Ornstein-Uhlenbeck operator is

$$\begin{aligned}
 A_d = & (-cx_1 + x_2) \frac{\partial}{\partial x_1} + (-cx_2 + x_3) \frac{\partial}{\partial x_3} + \dots \\
 & - cx_d \frac{\partial}{\partial x_d} + \sigma^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_d^2} \right).
 \end{aligned} \tag{1.2}$$

Denote $B = -c\text{Id} + R$ with Id the identity and R the nilpotent. Clearly, A_d is a non-symmetric operator since B does not satisfy the reversible^a condition of Ornstein-Uhlenbeck [4]

$$BQ = QB^*,$$

where $Q = 2\sigma^2 \text{Id}$ is the diffusion coefficient matrix and B^* is the transpose matrix of B .

The associated Markov semigroup $(T(t))_{t \geq 0}$ on the Banach space of the bounded measurable functions is

$$(T(t)f)(x) = \frac{1}{(4\pi)^{3/2} (\det Q_t)^{1/2}} \int_{\mathbb{R}^3} e^{-(Q_t^{-1}y, y)/4} f(e^{tB}x - y) \, dy, \tag{1.3}$$

and

$$Q_t = \sigma^2 \int_0^t e^{sB} e^{sB^*} \, ds.$$

It is well known that $(T(t))_{t \geq 0}$ extends to a strongly continuous semigroup of positive contractions in the Hilbert space $L^2_\mu = L^2(\mathbb{R}^d, d\mu)$, where μ is the unique invariant measure [5, 6]. Still we denote by (A_d, D) the generator of $(T(t))_{t \geq 0}$ in L^2_μ ; it was shown [7] that the spectrum consists of eigenvalues of finite multiplicities, $\sigma(A_d) = \{-nc : n \in \mathbb{N}\}$, and all the generalized eigenfunctions are polynomials and form a complete system in L^2_μ . Let $\gamma = -nc$. It follows from [7, Theorem 4.1] that the algebraic multiplicity of γ is

$$k_{A_d}(\gamma) = \binom{n+d-1}{d-1}, \tag{1.4}$$

and it follows from [7, Proposition 4.3, Theorem 4.1] that $\nu_A(\gamma)$, the index of the eigenvalue γ , is

$$\nu_{A_d}(\gamma) = 1 + (d-1)n. \tag{1.5}$$

A natural question is what the geometric multiplicity of the eigenvalue γ is. In addition, since the spectral subspace associated with γ (i.e. $\text{Ker}(\gamma - A_d)^{\nu_{A_d}(\gamma)}$) is a finite-dimensional vector space over the real field \mathbb{R} , what is the Jordan decomposition (or the Jordan canonical form) [8, 9] of A_d restricted on the spectral subspace? That is to say, what are the integers $r > 0$, $0 < q_r \leq q_{r-1} \leq \dots \leq q_1 \leq q_0 \leq \nu_A(\gamma)$ and the generalized eigenfunctions $f_r, f_{r-1}, \dots, f_1, f_0$, such that

$$\{f_k, (\gamma - A_d)f_k, \dots, (\gamma - A_d)^{q_k-1}f_k : k = 0, 1, \dots, r\} \tag{1.6}$$

form the basis of the spectral subspace associated to γ , and

$$(\gamma - A_d)^{q_k} f_k = 0, \quad k = 0, 1, \dots, r? \tag{1.7}$$

The integers $(q_r, q_{r-1}, \dots, q_1, q_0)$ are also called Segre characteristic (or, say, Segre type, Segre notation). f_k is called a lead vector (or, say, a cyclic vector, a generator) of a Jordan chain $\{f_k, (\gamma - A_d)f_k, \dots, (\gamma - A_d)^{q_k-1}f_k\}$ by some authors [9, 10].

In the present paper, we present an approach to calculate the Jordan decomposition and the generalized eigenfunctions (see Theorems 2.1, 3.1) for $d = 2, 3$.^b The proof of Theorem 2.1 is by direct calculation. The main techniques of the proof of Theorem 3.1 are canonical projection and the theory of systems of linear equations. This approach is novel to the Jordan decomposition of differential operators as far as we know.

It is a difficult problem to get the geometric multiplicity of the eigenvalue of a differential operator from the perspective of functional analysis. It is well known that the spectrum of the symmetric Ornstein-Uhlenbeck operator is the starting point of stochastic analysis (more precisely: Malliavin calculus), thus the analogous results of the non-symmetric Ornstein-Uhlenbeck operator are interesting. We will treat the more general non-symmetric Ornstein-Uhlenbeck operator in the future.

2 In case of dimension 2

In this section, we treat the case of $d = 2$. Denote $\rho = \frac{\sigma^2}{c}$. The Hermite polynomials [1] are defined by the formula

$$H_n(x) = (-\rho)^n e^{x^2/2\rho} \frac{d^n}{dx^n} e^{-x^2/2\rho}, \quad n = 1, 2, \dots$$

Clearly, it has a power series expression,

$$H_n(x, \rho) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!! x^{n-k} (-\rho)^k,$$

and satisfies

$$\begin{aligned} \frac{d}{dx} H_n(x, \rho) &= n H_{n-1}(x, \rho), \\ H_{n+1}(x, \rho) &= x H_n(x, \rho) - n \rho H_{n-1}(x, \rho), \\ \left(-\rho \frac{d^2}{dx^2} + x \frac{d}{dx}\right) H_n(x, \rho) &= n H_n(x, \rho). \end{aligned}$$

Theorem 2.1 *The geometric multiplicity of the eigenvalue γ is 1. Set*

$$G_i(x) = \sum_{j=0}^{[i/2]} \frac{1}{2^j j! (i-2j)!} \left(-\frac{\rho}{2c^2}\right)^j H_{i-2j}(x), \quad i = 0, 1, \dots \tag{2.1}$$

Suppose $f = G_n$, then $\{f, (\gamma - A_2)f, (\gamma - A_2)^2f, \dots, (\gamma - A_2)^n f\}$ forms a basis of the spectral subspace associated to γ . Also

$$(\gamma - A_2)^k f = (-1)^k \sum_{i=0}^{n-k} \left(-\frac{\rho}{2c}\right)^{n-k-i} \binom{k}{n-k-i} G_i(x) H_{2k-n+i}(y), \tag{2.2}$$

where $k = 0, 1, \dots, n$ and $H_l(y) = 0$ when $l < 0$. In particular, $(\gamma - A_2)^n f = (-1)^n H_n(y)$ is the eigenfunction associated to γ .

Proof We need only prove Eq. (2.2). It is easy to check that $G_i(x)$ satisfies the recursion relation:

$$\begin{cases} (-ic - A_2)G_i(x) = -yC_{i-1}(x) + \frac{\rho}{2c}G_{i-2}(x), \\ G_0(x) = 1, \quad G_1(x) = x. \end{cases} \tag{2.3}$$

Clearly, for any differentiable function $h(x), g(y)$, we have

$$(\gamma - A_2)h(x)g(y) = g(y)(\gamma - A_2)h(x) + c \cdot h(x) \left(-\rho \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} \right) g(y).$$

Then by the property of the Hermite polynomials [1], we have

$$\begin{aligned} & (\gamma - A_2)G_i(x)H_{2k-n+i}(y) \\ &= H_{2k-n+i}(y) \left[-yG_{i-1}(x) + \frac{\rho}{2c}G_{i-2}(x) + (i-n)cG_i(x) \right] \\ & \quad + cG_i(x)(2k-n+i)H_{2k-n+i}(y) \quad (\text{by (2.3)}) \\ &= H_{2k-n+i}(y) \left[-yG_{i-1}(x) + \frac{\rho}{2c}G_{i-2}(x) + 2(i+k-n)cG_i(x) \right] \\ &= -G_{i-1}(x) \left[H_{2k-n+i+1}(y) + (2k-n+i)\rho H_{2k-n+i-1}(y) \right] \\ & \quad + H_{2k-n+i}(y) \left[\frac{\rho}{2c}G_{i-2}(x) + 2(i+k-n)cG_i(x) \right]. \end{aligned}$$

By mathematical induction, we have

$$\begin{aligned} & (-1)^{k+1}(\gamma - A_2)^{k+1}f \\ &= -\sum_{i=0}^{n-k} \left(-\frac{\rho}{2c} \right)^{n-k-i} \binom{k}{n-k-i} (\gamma - A_2)G_i(x)H_{2k-n+i}(y) \\ &= \sum_{i=1}^{n-k} \left(-\frac{\rho}{2c} \right)^{n-k-i} \binom{k}{n-k-i} G_{i-1}(x) \left[H_{2k-n+i+1}(y) + (2k-n+i)\rho H_{2k-n+i-1}(y) \right] \\ & \quad + \sum_{i=2}^{n-k} \left(-\frac{\rho}{2c} \right)^{n-k+1-i} \binom{k}{n-k-i} G_{i-2}(x)H_{2k-n+i}(y) \\ & \quad - \rho \sum_{i=0}^{n-k-1} \left(-\frac{\rho}{2c} \right)^{n-k-1-i} (n-i-k) \binom{k}{n-k-i} G_i(x)H_{2k-n+i}(y) \\ &= \sum_{i=0}^{n-k-1} \left(-\frac{\rho}{2c} \right)^{n-k-i-1} \binom{k+1}{n-k-i-1} G_i(x)H_{2(k+1)-n+i}(y). \quad \square \end{aligned}$$

3 In case of dimension 3

In this section, we treat the case of $d = 3$. For convenience, we give some notation firstly. Set \mathcal{P} for the space of all polynomials with variables (x, y, z) , \mathcal{P}_n the space of polynomials of

degree less than or equal to n , and \mathcal{H}_n the space of homogeneous polynomials of degree n . Then $\mathcal{P} = \bigcup_n \mathcal{P}_n$ and one has the usual direct sum decomposition of all polynomials [7],

$$\mathcal{P}_n = \bigoplus_{m=0}^n \mathcal{H}_m. \tag{3.1}$$

Notation 1 Set $\gamma = -nc$, $r = \lfloor \frac{n}{2} \rfloor$ and $\rho = \frac{\sigma^2}{c}$.

By the monomials property of the Hermite polynomials [1]

$$x^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (2k-1)!! \rho^k H_{n-2k}(x),$$

the Hermite polynomials are another basis of \mathcal{P} .

Let $\mathcal{H}'_m = \text{span}\{H_i(x)H_j(y)H_k(z), i+j+k = m\}$, then we have another direct sum decomposition of all polynomials, *i.e.*,

$$\mathcal{P} = \bigcup_n \mathcal{P}_n, \quad \mathcal{P}_n = \bigoplus_{m=0}^n \mathcal{H}'_m. \tag{3.2}$$

We denote by \mathcal{Q}_m the canonical projection [11] of \mathcal{P} onto \mathcal{H}'_m .

Theorem 3.1 *Let*

$$q_k = 2n + 1 - 4k, \quad k = 0, 1, 2, \dots, r. \tag{3.3}$$

Then there exist $\{f_k, k = 0, \dots, r\}$ so that

$$\{f_k, (\gamma - A_3)f_k, \dots, (\gamma - A_3)^{q_k-1}f_k : k = 0, \dots, r\}$$

forms a basis of the spectral subspace associated to γ . Set $h_k = (\gamma - A_3)^{q_k-1}f_k$. Then $\{h_k, k = 0, 1, 2, \dots, r\}$ are the basis of the eigenspace of the eigenvalue γ and satisfy

$$\mathcal{Q}_n h_k = \sum_{i=0}^k (-2)^{k-i} \binom{k}{i} H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z). \tag{3.4}$$

The proof of Theorem 3.1 is presented in Section 3.1. The following is a by-product.

Corollary 3.2 *The geometric multiplicity of the eigenvalue γ of the Ornstein-Uhlenbeck operator A_3 is $r + 1$.*

3.1 Proof of Theorem 3.1

Note that

$$A_3 = -c \left(-\rho \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} \right) - c \left(-\rho \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} \right) - c \left(-\rho \frac{\partial^2}{\partial z^2} + z \frac{\partial}{\partial z} \right) + y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}.$$

It follows from the property of the Hermite polynomials [1] that

$$\begin{aligned}
 &(\gamma - A_3)(H_i(x)H_j(y)H_k(z)) \\
 &= (m - n)cH_i(x)H_j(y)H_k(z) - iH_{i-1}(x)[H_{j+1}(y) + j\rho H_{j-1}(y)]H_k(z) \\
 &\quad - jH_i(x)H_{j-1}(y)[H_{k+1}(z) + k\rho H_{k-1}(z)].
 \end{aligned} \tag{3.5}$$

For convenience, Eq. (3.5) can be rewritten in the following way.

Proposition 3.3 *If $\varphi \in \mathcal{H}'_m$ then $(\gamma - A_3)\varphi = \mathcal{Q}_m(\gamma - A_3)\varphi + \mathcal{Q}_{m-2}(\gamma - A_3)\varphi$. In particular,*

$$\begin{aligned}
 (\gamma - A_3)(H_i(x)H_j(y)H_k(z)) &= \mathcal{Q}_m(\gamma - A_3)(H_i(x)H_j(y)H_k(z)) \\
 &\quad + \mathcal{Q}_{m-2}(\gamma - A_3)(H_i(x)H_j(y)H_k(z)),
 \end{aligned} \tag{3.6}$$

where $m = i + j + k$ and

$$\begin{aligned}
 \mathcal{Q}_m(\gamma - A_3)(H_i(x)H_j(y)H_k(z)) &= (m - n)cH_i(x)H_j(y)H_k(z) - iH_{i-1}(x)H_{j+1}(y)H_k(z) \\
 &\quad - jH_i(x)H_{j-1}(y)H_{k+1}(z),
 \end{aligned} \tag{3.7}$$

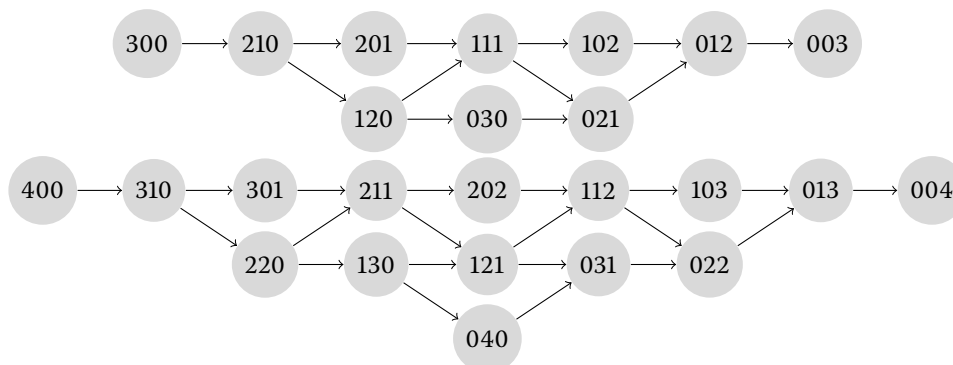
and

$$\begin{aligned}
 \mathcal{Q}_{m-2}(\gamma - A_3)(H_i(x)H_j(y)H_k(z)) \\
 = -ij\rho H_{i-1}(x)H_{j-1}(y)H_k(z) - jk\rho H_i(x)H_{j-1}(y)H_{k-1}(z).
 \end{aligned} \tag{3.8}$$

In particular, if $m = n$, then

$$\mathcal{Q}_n(\gamma - A_3)(H_i(x)H_j(y)H_k(z)) = -iH_{i-1}(x)H_{j+1}(y)H_k(z) - jH_i(x)H_{j-1}(y)H_{k+1}(z). \tag{3.9}$$

Remark 1 Equations (3.7)-(3.9) make us use the terminology of graph theory. In fact, by Eq. (3.9), we have a weighted and directed acyclic graph (which also can be seen as a Hasse diagram) of the evolution of basis of \mathcal{H}'_n operated by $\mathcal{Q}_n(\gamma - A_3)$. For example, when $n = 3$ and $n = 4$,^c the directed acyclic graphs are, respectively,



where we denote by the triple integers (i, j, k) the Hermite polynomial $H_i(x)H_j(y)H_k(z)$. It follows from Eq. (3.9) that the weights of the arrows between (i, j, k) and $(i, j - 1, k + 1)$, $(i - 1, j + 1, k)$ are $-j, -i$, respectively.

One can find many properties from the directed acyclic graph. For example, for the vertex (i, j, k) with $i + j + k = n$, the height (which is defined by the distance between vertices $(n, 0, 0)$ and (i, j, k) and thus is between $0 \sim 2n$) is $h = j + 2k$. For simplicity, in each height of the graph, we list the vertices (i, j, k) decreasingly by lexicographic order. Then the vertices (i, j, k) and (k, j, i) are symmetric about the n th height of the graph.

It follows from Eq. (1.4), Corollary 3.2, and Theorem 3.1 that the order of the graph (the numbers of vertices in the graph) is the algebraic multiplicity of γ , q_k are 1 plus the distance between vertices $(\lceil \frac{n}{2} \rceil + k, 0, \lceil \frac{n}{2} \rceil - k)$ and $(\lceil \frac{n}{2} \rceil - k, 0, \lceil \frac{n}{2} \rceil + k)$, and the numbers of vertices in the n th height of the graph is the geometric multiplicity of γ .

Proposition 3.4 *The spectral subspace associated to $\gamma = -nc$ belongs to $\bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{H}'_{n-2i}$.*

Proof Suppose that f is a generalized eigenfunction, i.e., there exists an integer $k \geq 1$ such that $(\lambda - A_3)^k f = 0$. It follows from Eq. (3.7) that if the degree of f is $m \neq n$, then $\mathcal{Q}_m(\lambda - A_3)^k f \neq 0$. This is a contradiction; then $f \in \mathcal{P}_n$ and the degree is exactly n .^d If there is an $i = 0, 1, \dots, \lfloor \frac{n-1}{2} \rfloor$ such that $\mathcal{Q}_{n-1-2i} f \neq 0$, then by Eq. (3.6), $\mathcal{Q}_{n-1-2i}(\lambda - A_3)^k f \neq 0$. This is a contradiction; then $f \in \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{H}'_{n-2i}$. \square

Lemma 3.5 *For any polynomial $g \in \mathcal{H}'_m$ with $m \neq n$, there exists a unique solution $f \in \mathcal{H}'_m$ to the equation $\mathcal{Q}_m(\gamma - A_3)f = g$.*

Proof Suppose that

$$g = \sum_{i+j+k=m} b_{ijk} H_i(x) H_j(y) H_k(z)$$

and $f = \sum_{i+j+k=m} a_{ijk} H_i(x) H_j(y) H_k(z)$. It follows from Eq. (3.7) that

$$\begin{aligned} & \mathcal{Q}_m(\gamma - A_3)f \\ &= \mathcal{Q}_m(\gamma - A_3) \sum_{i+j+k=m} a_{ijk} H_i(x) H_j(y) H_k(z) \\ &= \sum_{i+j+k=m} a_{ijk} [(m-n)c H_i(x) H_j(y) H_k(z) \\ & \quad - i H_{i-1}(x) H_{j+1}(y) H_k(z) - j H_i(x) H_{j-1}(y) H_{k+1}(z)] \\ &= \sum_{i+j+k=m} b_{ijk} H_i(x) H_j(y) H_k(z). \end{aligned}$$

By the linear independence of $\{H_i(x) H_j(y) H_k(z) : i + j + k = m\}$, we have a system of $\binom{m+1}{2}$ linear equations in $\binom{m+1}{2}$ unknowns.

We sort $\{H_i(x) H_j(y) H_k(z) : i + j + k = m\}$, which appear in the directed acyclic graph in Remark 1. The coefficient matrix of the linear equations is a lower triangle matrix with nonzero diagonal entry $(m-n)c$. Thus the linear equations have a unique solution. \square

Suppose that $f = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} f_{m-2s}$ with $f_s \in \mathcal{H}'_s$. It follows from Eq. (3.6) that

$$\begin{aligned} (\gamma - A_3)f &= (\gamma - A_3) \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} f_{m-2s} = \sum_{s=0}^{\lfloor \frac{m}{2} \rfloor} [\mathcal{Q}_{m-2s}(\gamma - A_3)f_{m-2s} + \mathcal{Q}_{m-2s-2}(\gamma - A_3)f_{m-2s}] \\ &= \mathcal{Q}_m(\gamma - A_3)f_m + \sum_{s=1}^{\lfloor \frac{m}{2} \rfloor} \mathcal{Q}_{m-2s}(\gamma - A_3)[f_{m-2s} + f_{m+2-2s}]. \end{aligned}$$

Thus the equation $(\gamma - A_3)f = g$ is equal to the system of equations

$$\mathcal{Q}_m(\gamma - A_3)f_m = g, \tag{3.10}$$

$$\mathcal{Q}_{m-2s}(\gamma - A_3)f_{m-2s} = -\mathcal{Q}_{m-2s}(\gamma - A)f_{m+2-2s}, \quad s = 1, 2, \dots, \left\lfloor \frac{m}{2} \right\rfloor. \tag{3.11}$$

It follows from Lemma 3.5 that when $m \neq n$, Eq. (3.10) has a unique solution $f_m \in \mathcal{H}'_m$ and when $m - 2s \neq n$, Eq. (3.11) has a unique solution $f_{m-2s} \in \mathcal{H}'_{m-2s}$.

Clearly, if f satisfies $(\gamma - A_3)f = g$, so does $f + h$ where h is any eigenfunction of A_3 associated to γ . Thus we have the following proposition.

Proposition 3.6 *For any $g \in \mathcal{P}$ with $\mathcal{Q}_{n+2s}g = 0, s = 0, 1, 2, \dots$, there exist solutions $f \in \mathcal{P}$ to the equation $(\gamma - A_3)f = g$. In addition, if f is the same degree polynomial to g then there exists one and only one solution.*

Proposition 3.7 *Set $\psi_k = \sum_{i=0}^k (-2)^{k-i} \binom{k}{i} H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z), k = 0, 1, \dots, r$. Then it satisfies the equation $\mathcal{Q}_n(\gamma - A_3)\psi_k = 0$.*

Proof Suppose that $\psi_k = \sum_{i=0}^k a_i H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z)$. By Eq. (3.7), we have

$$\begin{aligned} \mathcal{Q}_n(\gamma - A_3)\psi_k &= \mathcal{Q}_n(\gamma - A_3) \sum_{i=0}^k a_i H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z) \\ &= \sum_{i=0}^k -a_i [(k-i)H_{k-i-1}(x)H_{2i+1}(y)H_{n-k-i}(z) \\ &\quad + 2iH_{k-i}(x)H_{2i-1}(y)H_{n-k-i+1}(z)] \\ &= 0. \end{aligned}$$

By the linear independence of $\{H_i(x)H_j(y)H_k(z)\}$, we have a system of k linear homogeneous equations in $k + 1$ unknowns. The coefficient matrix is

$$M_k = - \begin{bmatrix} k & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & k-1 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & k-2 & 6 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & 2(k-1) & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2k \end{bmatrix}.$$

Clearly, the solution is 1-dimensional and $a_i = (-2)^{k-i} \binom{k}{i}, i = 0, \dots, k$, is a solution. □

Now suppose that $h = \psi + \phi$ with $\psi \in \mathcal{H}'_n$ and $\phi \in \mathcal{P}_{n-2}$; then by Eq. (3.6),

$$\begin{aligned} (\gamma - A_3)h &= (\gamma - A_3)\psi + (\gamma - A_3)\phi \\ &= \mathcal{Q}_n(\gamma - A_3)\psi + [\mathcal{Q}_{n-2}(\gamma - A_3)\psi + (\gamma - A_3)\phi]. \end{aligned}$$

Therefore, the equation $(\gamma - A_3)h = 0$ is equivalent to two equations:

$$\mathcal{Q}_n(\gamma - A_3)\psi = 0, \quad \psi \in \mathcal{H}'_n, \tag{3.12}$$

$$(\gamma - A_3)\phi = -\mathcal{Q}_{n-2}(\gamma - A_3)\psi, \quad \phi \in \mathcal{P}_{n-2}. \tag{3.13}$$

By Proposition 3.7, Eq. (3.12) has $1 + r$ independent solutions. It follows from Proposition 3.6 that there exists a unique ϕ satisfying Eq. (3.13) for each ψ . Thus we have the following corollary.

Corollary 3.8 *The geometric multiplicity of the eigenvalue $\gamma = -nc$ is greater than or equal to $1 + r$ (i.e., there are at least $1 + r$ independent solutions to the equation $(\gamma - A_3)h = 0$).*

Denote by $h_k, k = 0, 1, \dots, r$ the solutions to the equation $(\gamma - A_3)h = 0$ given by Corollary 3.8. Clearly, $h_k = \psi_k + (\text{Id} - \mathcal{Q}_n)h_k$, where $\psi_k \in \mathcal{H}'_n$ is the same as in Proposition 3.7.

Suppose that $f_k = \varphi_k + g_k$ with $\varphi_k \in \mathcal{H}'_n$ and $g_k \in \mathcal{P}_{n-2}$, then

$$\begin{aligned} (\gamma - A_3)^{q_k-1}f_k &= (\gamma - A_3)^{q_k-1}(\varphi_k + g_k) \\ &= \mathcal{Q}_n(\gamma - A_3)^{q_k-1}\varphi_k + \mathcal{Q}_{n-2}(\gamma - A_3)^{q_k-1}\varphi_k + (\gamma - A_3)^{q_k-1}g_k. \end{aligned}$$

Therefore, the equation $(\gamma - A_3)^{q_k-1}f_k = h_k$ is equivalent to two equations:

$$\mathcal{Q}_n(\gamma - A_3)^{q_k-1}\varphi_k = \psi_k, \tag{3.14}$$

$$(\gamma - A_3)^{q_k-1}g_k = (\text{Id} - \mathcal{Q}_n)h_k - \mathcal{Q}_{n-2}(\gamma - A_3)^{q_k-1}\varphi_k. \tag{3.15}$$

Note that $\psi_k = \sum_{i=0}^k b_i H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z)$. Set $\varphi_k = \sum_{i=0}^k a_i H_{n-k-i}(x)H_{2i}(y)H_{k-i}(z)$, then the l.h.s. of Eq. (3.14) is a linear mapping from the span $\{H_{n-k-i}(x)H_{2i}(y)H_{k-i}(z) : i = 0, \dots, k\}$ to the span $\{H_{k-i}(x)H_{2i}(y)H_{n-k-i}(z) : i = 0, \dots, k\}$. The linear mapping is the evolution from the $2k$ -height to the $2(n - k)$ -height of the directed acyclic graph in Remark 1, which is represented in the natural basis by a $(k + 1)$ -square matrix S_{r-k} (it is the multiplication of some matrices; for details, please refer to Section 3.2). By Proposition 3.10, the matrix S_{r-k} is nonsingular, which implies that Eq. (3.14) has a solution. Since $g_k, (\text{Id} - \mathcal{Q}_n)h_k \in \mathcal{P}_{n-2}$, it follows from Proposition 3.6 that Eq. (3.15) has a solution. Then we have the following proposition.

Proposition 3.9 *There exists an $f_k \in \mathcal{P}_n$ such that $(\gamma - A)^{q_k-1}f_k = h_k$.*

Proof of Theorem 3.1 Note that $\psi_k = \mathcal{Q}_n h_k$ in Proposition 3.7 are linear independent, so are the eigenfunctions h_k . Let $\{f_k, k = 0, 1, \dots, r\}$ be as in Proposition 3.9. Then the generalized eigenfunctions $\{(\gamma - A)^j f_k : j = 0, 1, \dots, q_k - 1, k = 0, 1, \dots, r\}$ are linear independent (please refer to the proof of [12, p.264, Theorem 6.2]).

Note that $q_k = 2n + 1 - 4k$; then the algebraic multiplicity of the eigenvalue γ can be decomposed to

$$\binom{n+2}{2} = \sum_{k=0}^r (2n + 1 - 4k) = \sum_{k=0}^r q_k. \tag{3.16}$$

Thus $\{(\gamma - A)^j f_k : j = 0, 1, \dots, q_k - 1, k = 0, 1, \dots, r\}$ forms a basis of the spectral subspace associated to γ . Together with Proposition 3.8, we see that the geometric multiplicity of the eigenvalue γ is equal to $r + 1$ (otherwise, the algebraic multiplicity should be greater than $\binom{n+1}{2}$). Then $\{h_k, k = 0, \dots, r\}$ forms the basis of the eigenspace of γ . Equation (3.4) is exactly the conclusion of Proposition 3.7. \square

3.2 Linear mapping represented by the multiplication of some matrices

For example, by Eq. (3.9), when n is odd, the evolution from the $(n - 1)$ -height to the n -height (the n -height to the $(n + 1)$ -height) of the directed acyclic graph is

$$\begin{aligned} \mathcal{Q}_n(\gamma - A_3) & \sum_{i=0}^r a_i H_{r+1-i}(x) H_{2i}(y) H_{r-i}(z) \\ & = \sum_{i=0}^r a_i (-(r + 1 - i) H_{r-i}(x) H_{1+2i}(y) H_{r-i}(z) - 2i H_{r+1-i}(x) H_{2i-1}(y) H_{r+1-i}(z)) \\ & = \sum_{i=0}^r (-(r + 1 - i) a_i - (2 + 2i) a_{i+1}) H_{r-i}(x) H_{1+2i}(y) H_{r-i}(z) \quad (\text{where } a_{r+1} = 0), \\ \mathcal{Q}_n(\gamma - A_3) & \sum_{i=0}^r b_i H_{r-i}(x) H_{1+2i}(y) H_{r-i}(z) \\ & = \sum_{i=0}^r (-(r + 1 - i) b_{i-1} - (1 + 2i) b_i) H_{r-i}(x) H_{2i}(y) H_{r+1-i}(z) \quad (\text{where } b_{-1} = 0). \end{aligned}$$

Then the matrices associated to the linear mappings are $-D_0, -A_0$ (see below). When n is even, the evolution from the $(n - 1)$ -height to the n -height (the n -height to the $(n + 1)$ -height) of the directed acyclic graph is

$$\begin{aligned} \mathcal{Q}_n(\gamma - A_3) & \sum_{i=0}^{r-1} c_i H_{r-i}(x) H_{2i+1}(y) H_{r-1-i}(z) \\ & = \sum_{i=0}^r (-(r + 1 - i) c_{i-1} - (1 + 2i) c_i) H_{r-i}(x) H_{2i}(y) H_{r-i}(z) \quad (\text{where } c_{-1} = c_r = 0), \\ \mathcal{Q}_n(\gamma - A_3) & \sum_{i=0}^r d_i H_{r-i}(x) H_{2i}(y) H_{r-i}(z) \\ & = \sum_{i=0}^{r-1} (-(r - i) d_i - (2 + 2i) d_{i+1}) H_{r-i-1}(x) H_{2i+1}(y) H_{r-i}(z). \end{aligned}$$

Then the matrices associated to the linear mappings are $-B_1, -C_1$ (see below).

The others are similar. In general, the matrix S_{r-k} (see Proposition 3.9) associated to the linear mapping of the evolution from the $(2k)$ -height to the $2(n-k)$ -height of the directed acyclic graph is as follows.

Proposition 3.10 *Let $r = \lfloor \frac{n}{2} \rfloor$. If n is odd, then suppose that S_k is an $(r+1-k)$ -square matrix given by*

$$\begin{cases} S_0 = D_0 A_0, \\ S_k = D_k C_k S_{k-1} B_k A_k, \quad k = 1, 2, \dots, r, \end{cases} \tag{3.17}$$

where D_k, A_k are $(r+1-k)$ -square matrices,

$$A_k = \begin{bmatrix} r+k+1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r+k & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & r+k-1 & 6 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2k+2 & n-(2k+1) \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2k+1 \end{bmatrix}, \tag{3.18}$$

and

$$D_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ r-k & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r-k-1 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & n-2k-2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & n-2k \end{bmatrix}, \tag{3.19}$$

and B_k, C_k are $(r+2-k) \times (r+1-k), (r+1-k) \times (r+2-k)$ matrices, respectively,

$$B_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ r+k & 3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r+k-1 & 5 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2k+2 & n-2k-2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2k+1 & n-2k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 2k \end{bmatrix}, \tag{3.20}$$

and

$$C_k = \begin{bmatrix} r-k+1 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & r-k & 4 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & r-k-1 & 6 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & n-2k-1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & n-2k+1 \end{bmatrix}. \tag{3.21}$$

If n is even, then suppose that S_k is an $(r + 1 - k)$ -square matrix given by

$$\begin{cases} S_0 = \text{Id}_{r+1}, \\ S_k = D_k C_k S_{k-1} B_k A_k, \quad k = 1, 2, \dots, r, \end{cases} \quad (3.22)$$

where D_k, A_k are $r + 1 - k$ order matrices,

$$A_k = \begin{bmatrix} r+k & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & r+k-1 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & r+k-2 & 6 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 2k+1 & n-2k \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2k \end{bmatrix}, \quad (3.23)$$

and

$$D_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ r-k & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & r-k-1 & 5 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & n-2k-1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & n+1-2k \end{bmatrix}, \quad (3.24)$$

and B_k, C_k are $(r + 2 - k) \times (r + 1 - k), (r + 1 - k) \times (r + 2 - k)$ matrices, respectively,

$$B_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ r+k-1 & 3 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & r+k-2 & 5 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2k+2 & n-2k-1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 2k & n+1-2k \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 2k-1 \end{bmatrix}, \quad (3.25)$$

and

$$C_k = \begin{bmatrix} r-k+1 & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & r-k & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & r-k-1 & 6 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 & n-2k & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & n-2k+2 \end{bmatrix}. \quad (3.26)$$

Then the matrices S_k are nonsingular.

Remark 2 Set the column vector $\vec{u}_k = [(-2)^k, \binom{k}{k-1}(-2)^{k-1}, \dots, \binom{k}{2}(-2)^2, \binom{k}{1}(-2), 1]'$, where $k = 0, 1, \dots, r$. We conjecture that the column vector \vec{u}_{r-k} is the eigenvector of S_k associated

to the eigenvalue λ_k , which is defined as follows: when n is odd,

$$\begin{cases} \lambda_0 = 1, \\ \lambda_k = 2k(2k + 1)(4k - 1)(4k + 1)\lambda_{k-1}, \quad k = 1, 2, \dots, r, \end{cases} \tag{3.27}$$

when n is even,

$$\begin{cases} \lambda_0 = 1, \\ \lambda_k = 2k(2k - 1)(4k - 3)(4k - 1)\lambda_{k-1}, \quad k = 1, 2, \dots, r. \end{cases}$$

If the conjecture is valid, then we can characterize the leader vectors f_k more clearly, *i.e.*,

$$\mathcal{Q}_n f_k = \sum_{i=0}^k (-2)^{k-i} \binom{k}{i} H_{n-k-i}(x) H_{2i}(y) H_{k-i}(z).$$

Apply the notation in [10, 13]. Let A be an $p \times q$ matrix, $\alpha = \{i_1, \dots, i_s\}$, and $\beta = \{j_1, \dots, j_t\}$, $1 \leq i_1 < \dots < i_s \leq p$, $1 \leq j_1 < \dots < j_t \leq q$. Denote by $A[i_1, \dots, i_s; j_1, \dots, j_t]$, or simply $A[\alpha, \beta]$, the submatrix of A consisting of the entries in rows i_1, \dots, i_s and columns j_1, \dots, j_t . Let $A(i_1, \dots, i_s | j_1, \dots, j_t)$ be the submatrix of A obtained by deleting rows i_1, \dots, i_s and columns j_1, \dots, j_t . For convenience, $A(i_1, \dots, i_s |)$ ($A(| j_1, \dots, j_t)$) means deleting rows (columns) only.

Proof We divide the proof into three steps.

Claim 1: All the minors (*i.e.*, the determinant of the square submatrix) of the matrices A_k, B_k, C_k, D_k , $k = 1, \dots, r$, are nonnegative. Since $A_k (B_k)$ and $D_k^T (C_k^T)$ have the same types, we only need to prove the cases of A_k, B_k . The square submatrix of B_k is the same as that of a certain $B_k(i)$, $i = 1, \dots, r + 2 - k$, which is a direct sum of two matrices [13] with the same type of D_k or A_k . Thus we only need to prove the case of A_k . In fact, $A[i_1, \dots, i_s; j_1, \dots, j_s]$ is a direct sum [13] of several nonnegative triangular matrices, which implies that $A[i_1, \dots, i_s; j_1, \dots, j_s]$ has a nonnegative determinant.

Claim 2: All the minors of S_k are nonnegative. In fact, we have the following type of Binet-Cauchy formula:

$$\det(AB)[\alpha, \beta] = \sum_{\kappa} \det A[\alpha, \kappa] \times \det A[\kappa, \beta], \tag{3.28}$$

where $A (B)$ is $m \times n$ ($n \times m$) matrix, $m \leq n$, κ runs over all the sequences of $\{1, \dots, n\}$ with the same length of α, β . By successively applying Eq. (3.28), we have

$$\begin{aligned} \det S_k[\alpha, \beta] &= \det(D_k C_k S_{k-1} B_k A_k)[\alpha, \beta] \\ &= \sum_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} \det D_k[\alpha, \kappa_1] \\ &\quad \times \det C_k[\kappa_1, \kappa_2] \det S_{k-1}[\kappa_2, \kappa_3] \det B_k[\kappa_3, \kappa_4] \det A_k[\kappa_4, \beta]. \end{aligned}$$

Together with Claim 1 and by mathematical induction, we have $\det S_k[\alpha, \beta] \geq 0$.

Claim 3: $\det S_k > 0$. Clearly, $\det S_0 > 0$. Since $\det A_k, \det D_k > 0$, we need only prove that $\det(C_k S_{k-1} B_k) > 0$. By the Binet-Cauchy formula, we have

$$\det(C_k S_{k-1} B_k) = \sum_{i,j} \det C_k(i) \det S_{k-1}(i|j) \det B_k(j).$$

Clearly, $\det C_k(i), \det B_k(j) > 0$. By the induction assumption $\det S_{k-1} \neq 0$, Claim 2 implies that there exists at least one $\det S_{k-1}(i|j) > 0$. This ends the proof. \square

Competing interests

The authors did not provide this information.

Authors' contributions

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Author details

¹Business School of Central South University, Changsha, 410083, P.R. China. ²Hunan University of Science and Technology, Xiangtan, 411201, P.R. China.

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Endnotes

- ^a It is well known that the reversibility is equivalent to the symmetric property of the generator for a Markov process with invariant distribution.
- ^b We think that the approach is also able to solve the same problems for $d \geq 4$.
- ^c The directed acyclic graph can easily be given for any integer n , and we ignore it for convenience.
- ^d The reader can also refer to [7, Proposition 3.1].

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