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Periodic solutions for stochastic shunting inhibitory cellular neural networks with distributed delays

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Abstract
In this paper, by using an integral inequality, we establish some sufficient conditions ensuring the existence and p-exponential stability of periodic solutions for a class of stochastic shunting inhibitory cellular neural networks (SICNNs) with distributed delays. Moreover, we present an example to illustrate the feasibility of our theoretical results.

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Keywords: stochastic SICNNs; distributed delays; periodic solution; p-exponential stability

1 Introduction

The shunting inhibitory cellular neural networks (SICNNs) have been described as new cellular neural networks by Bouzerdout and Pinter in [1–3]. The layers in SICNNs are arranged into two-dimensional arrays of processing units called cells, where each cell is coupled to its neighboring units only. The interactions among cells within a single layer are mediated via the biophysical mechanism of recurrent shunting inhibition, where the shunting conductance of each cell is modulated by the voltages of neighboring cells.

Recently, due to its wide applications in image and signal processing, vision, pattern recognition, and optimization, SICNNs received much attention from many scholars. In particular, many authors devoted much effort to the existence and global exponential stability of periodic or almost periodic solutions of SICNNs (see [4–9]). For example, in [10–12], the authors considered the existence and stability of almost periodic solutions for SCINNs; in [13, 14], the authors considered the existence and stability of periodic solutions for SCINNs; in [15–17], the authors considered the existence and stability of anti-periodic solutions for impulsive SCINNs; in [18, 19], the authors obtained some sufficient conditions for the existence and stability of an equilibrium point.

As pointed out in [20], in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Neural networks could be stabilized or destabilized by some stochastic inputs [21]. Therefore, it is significant and of prime importance to consider stochastic effects on the dynamic behavior of neural networks, which are called stochastic neural networks. With respect to stochastic neural networks, there are many works on the stability of considered systems (see [22–30] and references therein).

However, it is well known that studies of neural dynamical systems not only involve a discussion of the stability properties, but they also involve many other dynamic behaviors such as periodic oscillatory behavior and so on. Therefore, it is significant to study the existence and stability of periodic solutions for stochastic neural networks (see [31]). But to the best of our knowledge, there is no paper published on the existence and stability of periodic solutions of stochastic shunting inhibitory cellular neural networks.

Motivated by the above discussion, our main purpose of this paper is by using an integral inequality to obtain some sufficient conditions for the existence and p-exponential stability of periodic solutions in the case of the following stochastic shunting inhibitory cellular neural network with distributed delays:

$$\begin{aligned} \mathrm{d}x_{ij}(t) = \Bigg[&-a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\big(x_{kl}\big(t-\delta(t)\big)\big)x_{ij}(t) \\ &- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_{t-\tau_{ij}(t)}^{t} k_{ij}(t-u)g\big(x_{kl}(u)\big)\,\mathrm{d}u\,x_{ij}(t) + L_{ij}(t)\Bigg]\,\mathrm{d}t \\ &+ \sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(t)\sigma_{ij}\big(x_{ij}(t)\big)\,\mathrm{d}w_{ij}(t), \quad t\ge t_0, \end{aligned} \tag{1.1}$$

where $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$; C_{ij} denotes the cell at the (i,j) position of the lattice, the r-neighborhood $N_r(i,j)$ is given as

$$N_r(i,j) = \big\{C_{kl} : \max\big(|k-i|,|l-j|\big) \le r, 1\le k\le m, 1\le l\le n\big\},$$

$N_q(i,j)$ and $N_p(i,j)$ are similarly specified; x_{ij} is the activity of the cell C_{ij} at time t; L_{ij} is the external input to C_{ij} at time t; $a_{ij}(t) > 0$ represents the passive decay rate of the cell activity at time t; $C_{ij}^{kl}(t)\ge 0$, $B_{ij}^{kl}(t)\ge 0$ and $D_{ij}^{kl}(t)\ge 0$ are the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij} at time t; the activity functions $f,g\in C(R,R)$ represent the output or firing rate of the cell C_{kl}; $\tau_{ij}(t) > 0$ and $\delta(t) > 0$ correspond to the transmission delays at time t; $k_{ij}(\cdot)$ is the kernel function determining the distributed delays at cells (i,j); $w(t) = (w_{11}(t), w_{12}(t),\ldots,w_{mn}(t))^T$ is $m\times n$-dimensional Brownian motions defined on a complete probability space; $\sigma_{ij}\in C(R,R)$ is a Borel measurable function and $\sigma = (\sigma_{ij})_{mn\times mn}$ is a diffusion coefficient matrix, where $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

In the following, we introduce some notation. Let $R^n(R^n_+)$ be the space of n-dimensional (nonnegative) real column vectors and R^{mn} be the space of $m\times n$-dimensional real column vectors. We denote $(\Omega, F, \{F_t\}_{t\ge 0}, P)$ by a complete probability space with a filtration $\{F_t\}_{t\ge 0}$, where F is a σ-algebra on a given set Ω, P is the probability measure and the filtration $\{F_t\}_{t\ge 0}$ satisfies the usual conditions, that is, $\{F_t\}_{t\ge 0}$ is right continuous and F_0 contains all P-null sets. Denote by $BC^b_{F_0}(R,R^{mn})$ the family of bounded F_0-measurable, R^{mn}-valued random variables $x(t)$, that is, the value of $x(t)$ is an $m\times n$-dimensional real vector and can be decided from the values of $w(s)$ for $s\le 0$. Then $BC^b_{F_0}(R,R^{mn})$ is a Banach space with the norm $\|x\| = \sup_{0\le t\le\omega}(E|x(t)|_1^p)^{\frac{1}{p}}$, where $p > 1$ is an integer, $|x(t)|_1 = \max_{(i,j)}|x_{ij}(t)|$, and $E(\cdot)$ stands for the correspondent expectation operator with respect to the given probability measure P. For convenience, for an ω-periodic continuous function $f : R\to R$, denote $\overline{f} = \max_{0\le t\le\omega}|f(t)|$, $\underline{f} = \min_{0\le t\le\omega}|f(t)|$; for any $\phi\in BC^b_{F_0}([-\tau,0],R^{mn})$, denote

$[\phi(t)]_\tau^+ = (|\phi_{11}|_\tau, |\phi_{12}|_\tau, \ldots, |\phi_{mn}|_\tau)^T$, where $|\phi_{ij}|_\tau = \sup_{-\tau \le s \le 0} |\phi_{ij}(t+s)|$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

The initial value of (1.1) is

$$x_{ij}(s) = \varphi_{ij}(s), \quad s \in [t_0 - \tau, t_0], \tag{1.2}$$

where $\varphi_{ij}(s) \in BC_{F_0}^b([t_0 - \tau, t_0], R)$, $\tau = \max\{\max_{(i,j)} \max_{t\in[0,\omega]} \tau_{ij}(t), \max_{t\in[0,\omega]} \delta(t)\}$, $t_0 \in R$. The main aim of this paper is to obtain some sufficient conditions on the existence and p-exponential stability of periodic solutions for (1.1) with initial condition (1.2).

Throughout this paper, we assume that

(H_1) $a_{ij}(t)$, $C_{ij}^{kl}(t)$, $B_{ij}^{kl}(t)$, $D_{ij}^{kl}(t)$, $L_{ij}(t)$ are all periodic continuous functions with period ω for $t \in R$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$;

(H_2) $f, g, \sigma_{ij} \in C(R, R)$ are all Lipschitz-continuous with positive Lipchitz constants L_f, L_g and l_{ij}, respectively and there exist positive constants M_f and M_g such that $|f(u)| \le M_f$, $|g(u)| \le M_g$, for all $u \in R$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$;

(H_3) $k_{ij} \in C(R, (0, \infty))$ satisfies $\int_0^{\overline{\tau}_{ij}} k_{ij}(s)\,\mathrm{d}s \le \overline{k}_{ij}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

This paper is organized as follows: In Section 2, we introduce some definitions and state some preliminary results which are needed in later sections. In Section 3, we establish some sufficient conditions for the existence of periodic solutions of (1.1). In Section 4, we prove that the periodic solution obtained in Section 3 is p-exponentially stable. In Section 5, we give an example to illustrate the feasibility of our results obtained in the previous sections.

2 Preliminaries

In this section, we recall some definitions and make some preparations.

Definition 2.1 (Definition 2.2 [31]) A stochastic process $x(t)$ is said to be periodic with period ω if its finite-dimensional distributions are periodic with period ω, that is, for any positive integer m and any moments of time $t_1, t_2, \ldots, t_m$, the joint distribution of the random variables $x(t_1 + k\omega), x(t_2 + k\omega), \ldots, x(t_m + k\omega)$ are independent of k, $k = \pm 1, \pm 2, \ldots$.

Lemma 2.1 (p.43 [32]) *If $x(t)$ is an ω-periodic stochastic process, then its mathematical expectation and variance are ω-periodic.*

Definition 2.2 A function $x(t) = (x_{11}(t), x_{12}(t), \ldots, x_{mn}(t))^T$ defined on $[t_0 - \tau, \infty]$ is said to be a solution of (1.1) with initial condition (1.2) if

(i) $x_{ij}(t)$ is absolutely continuous on $[t_0 - \tau, \infty]$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$;

(ii) $x_{ij}(t)$ satisfies (1.1) for almost everywhere $t \in [t_0, \infty)$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$;

(iii) $x_{ij}(s) = \varphi_{ij}(s)$, $s \in [t_0 - \tau, t_0]$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Throughout this paper, we assume that there exists a unique solution of (1.1) with initial condition (1.2). In the following, we denote the solution of (1.1) by $x(t) = x(t, t_0, \varphi)$ for all $\varphi \in BC_{F_0}^b([t_0 - \tau, t_0], R^{mn})$ and $t_0 \in R$.

Definition 2.3 (Definition 2.5 [31]) The solution $x(t, t_0, \varphi)$ of (1.1) is said to be

(i) p-uniformly bounded, if for each $\alpha > 0$, $t_0 \in R$, there exists a positive constant $\theta = \theta(\alpha)$ which is independent of t_0 such that $\|\varphi\|^p \leq \alpha$ implies $E(\|x(t,t_0,\varphi)\|^p) \leq \theta$, $t \geq t_0$;

(ii) p-point dissipative, if there exists a constant $N > 0$ such that for any point $\varphi \in BC^b_{F_0}([-\tau,0],R^n)$, there exists $T(t_0,\varphi)$ such that $E(\|x(t,t_0,\varphi)\|^p) \leq N$, $t \geq t_0 + T(t_0,\varphi)$.

Lemma 2.2 (Theorem 3.5 [33]) *Under conditions* (H_1)-(H_3), *assume that the solution of* (1.1) *is p-uniformly bounded and p-point dissipative for* $p > 2$, *then* (1.1) *has an* ω*-periodic solution.*

Lemma 2.3 (Lemma 2.2 [34]) *For any* $x \in R^n_+$ *and* $p > 0$,

$$|x|^p \leq n^{(\frac{p}{2}-1)\vee 0}\sum_{i=1}^n x_i^p, \qquad \left(\sum_{i=1}^n x_i\right)^p \leq n^{(p-1)\vee 0}\sum_{i=1}^n x_i^p.$$

Definition 2.4 (Definition 2.4 [31]) The periodic solution $x(t,t_0,\varphi)$ with initial value $\varphi \in BC^b_{F_0}([-\tau,0],R^n)$ of (1.1) is said to be p-exponentially stable, if there are constants $\lambda > 0$ and $M > 1$ such that for any solution $y(t,t_0,\varphi_1)$ with initial value $\varphi_1 \in BC^b_{F_0}([-\tau,0],R^n)$ of (1.1) satisfies

$$E\bigl(|x-y|_1^p\bigr) \leq M\|\varphi-\varphi_1\|^p \mathrm{e}^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Lemma 2.4 (Lemma 2.5 [31]) *Let* $u(t) \in C(R,R^n_+)$ *be a solution of the delay integral inequality*

$$\begin{cases} u(t) \leq M_1\mathrm{e}^{-\delta(t-t_0)}[\varphi]^+_\tau + \int_{t_0}^t \mathrm{e}^{-C_1(t-s)}A_1u(s)\,\mathrm{d}s \\ \qquad + \int_{t_0}^t \mathrm{e}^{-C_1(t-s)}B_1[u(s)]^+_\tau\,\mathrm{d}s + J_1, \quad t \geq t_0, \\ u(t) \leq \varphi(t), \quad \forall t \in [t_0-\tau,t_0], \end{cases} \tag{2.1}$$

where $A_1,B_1,C_1,M_1 \in R^{n\times n}_+$, $J_1 \geq 0$ *is a constant vector,* $\varphi(t) \in C([t_0-\tau,t_0],R^n_+)$. *If* $\rho(\Pi) < 1$, *where* $\Pi = C_1^{-1}(A_1+B_1)$, *then there are constants* $0 < \lambda \leq \delta$ *and* $N \geq 1$ *such that*

$$u(t) \leq Nz\mathrm{e}^{-\lambda(t-t_0)} + (I-\Pi)^{-1}J_1, \quad t \geq t_0,$$

where z satisfies $[\varphi]^+_\tau \leq z$.

Lemma 2.5 (Corollary 2.1 [31]) *Assume that all conditions of Lemma* 2.4 *hold. If* $J_1 = 0$, *then all solutions of inequality of* (2.1) *exponentially convergent to zero.*

By Lemma 2.4 and Lemma 2.5, we have the following corollary.

Corollary 2.1 *Let* $u(t) \in C(R,R_+)$ *be a solution of the delay integral inequality*

$$\begin{cases} u(t) \leq M_1\mathrm{e}^{-\delta(t-t_0)}[\varphi]^+_\tau + \int_{t_0}^t \mathrm{e}^{-C_1(t-s)}A_1u(s)\,\mathrm{d}s \\ \qquad + \int_{t_0}^t \mathrm{e}^{-C_1(t-s)}B_1[u(s)]^+_\tau\,\mathrm{d}s + J_1, \quad t \geq t_0, \\ u(t) \leq \varphi(t), \quad \forall t \in [t_0-\tau,t_0], \end{cases} \tag{2.2}$$

where $A_1, B_1, C_1, M_1 \in R_+$, $J_1 \geq 0$ *is a constant*, $\varphi(t) \in C([t_0 - \tau, t_0], R_+)$. *If* $\frac{A_1+B_1}{C_1} < 1$, *then there are constants* $0 < \lambda \leq \delta$ *and* $N \geq 1$ *such that*

$$u(t) \leq Nze^{-\lambda(t-t_0)} + \left(1 - \frac{A_1 + B_1}{C_1}\right)^{-1} J_1, \quad t \geq t_0,$$

where z satisfies $[\varphi]_\tau^+ \leq z$. *Moreover, if* $J_1 = 0$, *then all solutions of the inequality of* (2.2) *are exponentially convergent to zero.*

3 Existence of periodic solution

In this section, we will state and prove the existence of periodic solutions of (1.1).

Theorem 3.1 *Let* (H_1)-(H_3) *hold. Suppose further that*

(H_4) *there exists an integer* $p > 2$ *such that* $\epsilon\theta^{-1} < 1$, *where* $\theta = \min_{(i,j)}\{\underline{a}_{ij}\}$, $l_p = (\frac{p(p-1)}{2})^{\frac{p}{2}}$,

$$\epsilon = \max_{(i,j)}\left\{5^{p-1}\left((\underline{a}_{ij})^{1-p}\left[\left(M_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\right)^p + \left(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\right)^p\right] + l_p(mn)^{\frac{p}{2}}\left(\frac{2\underline{a}_{ij}(p-1)}{p-2}\right)^{1-\frac{p}{2}}\left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\right)^p\right)\right\}.$$

Then (1.1) *has an* ω*-periodic solution.*

Proof By the method of variation parameter, for $t \geq t_0$, from (1.1), we have the following:

$$\begin{aligned} x_{ij}(t) &= x_{ij}(t_0)e^{-\int_{t_0}^t a_{ij}(\vartheta)\,d\vartheta} - \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}\Bigg[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)f\big(x_{kl}(s-\delta(s))\big)x_{ij}(s) \\ &\quad + \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)g\big(x_{kl}(u)\big)\,du\,x_{ij}(s) - L_{ij}(s)\Bigg]\,ds \\ &\quad + \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}\sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\sigma_{ij}\big(x_{ij}(s)\big)\,dw_{ij}(s), \quad i = 1,2,\ldots,m, j = 1,2,\ldots,n. \end{aligned}$$

For $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, denote

$$\begin{aligned} F_{ij}^{(1)} &= x_{ij}(t_0)e^{-\int_{t_0}^t a_{ij}(\vartheta)\,d\vartheta}, \\ F_{ij}^{(2)} &= \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)f\big(x_{kl}(s-\delta(s))\big)x_{ij}(s)\,ds, \\ F_{ij}^{(3)} &= \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)g\big(x_{kl}(u)\big)\,du\,x_{ij}(s)\,ds, \\ F_{ij}^{(4)} &= \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}L_{ij}(s)\,ds, \qquad F_{ij}^{(5)} = \int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta}\sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\sigma_{ij}\big(x_{ij}(s)\big)\,dw_{ij}(s). \end{aligned}$$

Taking expectations and using Lemma 2.3, we have

$$E\big|x_{ij}(t)\big|^p \leq 5^{p-1}E\big(\big|F_{ij}^{(1)}\big|^p + \big|F_{ij}^{(2)}\big|^p + \big|F_{ij}^{(3)}\big|^p + \big|F_{ij}^{(4)}\big|^p + \big|F_{ij}^{(5)}\big|^p\big). \tag{3.1}$$

For $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we evaluate the first term of (3.1) as follows:

$$E\left|F_{ij}^{(1)}\right|^p = E\left|x_{ij}(t_0)e^{-\int_{t_0}^t a_{ij}(\vartheta)\,d\vartheta}\right|^p \le E\left|x_{ij}(t_0)e^{-\underline{a}_{ij}(t-t_0)}\right|^p \le e^{-p\underline{a}_{ij}(t-t_0)}E\left|x_{ij}(t_0)\right|^p.$$

For the second term of (3.1), by the Hölder inequality, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
E\left|F_{ij}^{(2)}\right|^p &= E\left|\int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s) f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)x_{ij}(s)\,ds\right|^p \\
&\le E\left(\int_{t_0}^t e^{-\underline{a}_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\left|f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)\right|\left|x_{ij}(s)\right|\,ds\right)^p \\
&\le E\left(\int_{t_0}^t e^{-\underline{a}_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f\left|x_{ij}(s)\right|\,ds\right)^p \\
&= E\left(\int_{t_0}^t \bigl(e^{-\underline{a}_{ij}(t-s)}\bigr)^{\frac{p-1}{p}} \bigl(e^{-\underline{a}_{ij}(t-s)}\bigr)^{\frac{1}{p}} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f\left|x_{ij}(s)\right|\,ds\right)^p \\
&\le E\left(\left(\int_{t_0}^t e^{-\underline{a}_{ij}(t-s)}\,ds\right)^{p-1} \int_{t_0}^t e^{-\underline{a}_{ij}(t-s)}\left(\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f\left|x_{ij}(s)\right|\right)^p ds\right) \\
&\le (\underline{a}_{ij})^{1-p}\int_{t_0}^t e^{-\underline{a}_{ij}(t-s)}\left(M_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\right)^p E\left|x_{ij}(s)\right|^p\,ds.
\end{aligned}$$

As to the third term of (3.1), by the Hölder inequality, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we also have

$$\begin{aligned}
E\left|F_{ij}^{(3)}\right|^p &= E\left|\int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta} \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)g\bigl(x_{kl}(u)\bigr)\,du\,x_{ij}(s)\,ds\right|^p \\
&\le E\left(\int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta} \sum_{C_{kl}\in N_q(i,j)} \left|B_{ij}^{kl}(s)\right|\left|\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)g\bigl(x_{kl}(u)\bigr)\,du\right|\left|x_{ij}(s)\right|\,ds\right)^p \\
&\le E\left(\int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta} \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} M_g \overline{k}_{ij}\left|x_{ij}(s)\right|\,ds\right)^p \\
&\le E\left(\left(\int_{t_0}^t e^{-a_{ij}(t-s)}\,ds\right)^{p-1} \int_{t_0}^t e^{-a_{ij}(t-s)}\left(\sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} M_g \overline{k}_{ij}\left|x_{ij}(s)\right|\right)^p ds\right) \\
&\le (\underline{a}_{ij})^{1-p}\int_{t_0}^t e^{-a_{ij}(t-s)}\left(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\right)^p E\left|x_{ij}(s)\right|^p\,ds.
\end{aligned}$$

For the fourth term of (3.1), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$E\left|F_{ij}^{(4)}\right|^p = E\left|\int_{t_0}^t e^{-\int_s^t a_{ij}(\vartheta)\,d\vartheta} L_{ij}(s)\,ds\right|^p \le E\left|\int_{t_0}^t e^{-a_{ij}(t-s)} L_{ij}(s)\,ds\right|^p \le \left(\frac{\overline{L}_{ij}}{\underline{a}_{ij}}\right)^p.$$

As to the last term of (3.1), using Proposition 1.9 in [35] and the Hölder inequality, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
&E\left|F_{ij}^{(5)}\right|^p \\
&= E\left|\int_{t_0}^t \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\sigma_{ij}\big(x_{ij}(s)\big)\,\mathrm{d}w_{ij}(s)\right|^p \\
&\le l_p\left[\int_{t_0}^t \left(\mathrm{e}^{-p\underline{a}_{ij}(t-s)} E\left|\sum_{C_{kl}\in N_p(i,j)} \big(D_{ij}^{kl}(s)\big)^2 \sigma_{ij}^2\big(x_{ij}(s)\big)\right|^{\frac{p}{2}}\right)^{\frac{2}{p}} \mathrm{d}s\right]^{\frac{p}{2}} \\
&\le l_p(mn)^{\frac{p}{2}}\left[\int_{t_0}^t \left(\mathrm{e}^{-p\underline{a}_{ij}(t-s)} E\left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\big|x_{ij}(s)\big|\right)^p\right)^{\frac{2}{p}} \mathrm{d}s\right]^{\frac{p}{2}} \\
&= l_p(mn)^{\frac{p}{2}}\left[\int_{t_0}^t \left(\mathrm{e}^{-(p-1)\underline{a}_{ij}(t-s)} \mathrm{e}^{-\underline{a}_{ij}(t-s)} E\left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\big|x_{ij}(s)\big|\right)^p\right)^{\frac{2}{p}} \mathrm{d}s\right]^{\frac{p}{2}} \\
&\le l_p(mn)^{\frac{p}{2}}\left(\int_{t_0}^t \mathrm{e}^{-\frac{2\underline{a}_{ij}(p-1)}{p-2}(t-s)}\,\mathrm{d}s\right)^{\frac{p}{2}-1} \left(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} E\left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\big|x_{ij}(s)\big|\right)^p \mathrm{d}s\right) \\
&\le l_p(mn)^{\frac{p}{2}}\left(\frac{2\underline{a}_{ij}(p-1)}{p-2}\right)^{1-\frac{p}{2}} \left(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} E\left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\big|x_{ij}(s)\big|\right)^p \mathrm{d}s\right) \\
&\le l_p(mn)^{\frac{p}{2}}\left(\frac{2\underline{a}_{ij}(p-1)}{p-2}\right)^{1-\frac{p}{2}} \left(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} \left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\right)^p E\big|x_{ij}(s)\big|^p \,\mathrm{d}s\right).
\end{aligned}$$

Therefore, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned}
E\big|x_{ij}(t)\big|^p \le 5^{p-1}\Bigg\{ & \mathrm{e}^{-p\underline{a}_{ij}(t-t_0)} E\big|x_{ij}(t_0)\big|^p + \left(\frac{\overline{L}_{ij}}{\underline{a}_{ij}}\right)^p \\
& + (\underline{a}_{ij})^{1-p} \int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} \left[\left(M_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\right)^p + \left(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl} \overline{k}_{ij}\right)^p\right] \\
& \times E\big|x_{ij}(s)\big|^p \,\mathrm{d}s + l_p(mn)^{\frac{p}{2}} \left(\frac{2\underline{a}_{ij}(p-1)}{p-2}\right)^{1-\frac{p}{2}} \\
& \times \left(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} \left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\right)^p E\big|x_{ij}(s)\big|^p \,\mathrm{d}s\right)\Bigg\}. \qquad (3.2)
\end{aligned}$$

Set $V(t) = (V_{11}(t), V_{12}(t), \ldots, V_{mn}(t))^T$, where $V_{ij}(t) = E|x_{ij}(t)|^p$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. By (3.2), we have

$$V_{ij}(t) \le 5^{p-1}\mathrm{e}^{-\theta(t-t_0)} V_{ij}(t_0) + \int_{t_0}^t \mathrm{e}^{-\theta(t-s)} \epsilon V_{ij}(s)\,\mathrm{d}s + J,$$

where $J = \max_{(i,j)}\{(\frac{\overline{L}_{ij}}{\underline{a}_{ij}})^p\}$. By ($H_4$) and Lemma 2.4, the solutions of (1.1) are p-uniformly bounded and it also show that the family of all solutions of (1.1) is p-point dissipative. Then it follows from Lemma 2.2 that (1.1) has an ω-periodic solution. This completes the proof of Theorem 3.1. □

4 p-Exponential stability of periodic solution

In this section, we will study the p-exponential stability of periodic solutions of (1.1).

Theorem 4.1 *Let* (H_1)-(H_3) *hold. Suppose further that*

(H_5) *there exists an integer* $p > 2$ *such that* $(\epsilon_1 + \epsilon_2)\theta^{-1} < 1$, *where*

$$\epsilon_1 = \max_{(i,j)}\Bigg\{6^{p-1}\bigg((\underline{a}_{ij})^{1-p}\bigg[\bigg(M_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\bigg)^p + \bigg(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p + \bigg(L_g \sum_{C_{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p\bigg] + l_p(mn)^{\frac{p}{2}}\bigg(\frac{2\underline{a}_{ij}(p-1)}{p-2}\bigg)^{1-\frac{p}{2}}\bigg(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\bigg)^p\bigg)\Bigg\}$$

and

$$\epsilon_2 = \max_{(i,j)}\Bigg\{6^{p-1}(\underline{a}_{ij})^{1-p}N\bigg(L_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\bigg)^p\Bigg\}.$$

Then the periodic solution of (1.1) *is* p*-exponentially stable.*

Proof It is obvious that if (H_5) holds, then (H_4) must hold. By Theorem 3.1, (1.1) has an ω-periodic solution $x^*(t) = \{x_{ij}^*(t)\}$ with initial condition $\varphi(t) = \{\varphi_{ij}(t)\}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. It follows that $x^*(t)$ is p-uniform, that is, there exists a positive constant N such that $E|x_{ij}^*(t)|^p < N$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Suppose that $x(t) = \{x_{ij}(t)\}$ is an arbitrary solution of (1.1) with the initial condition $\psi(t) = \{\psi_{ij}(t)\}$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Denote $y(t) = \{y_{ij}(t)\}$, where $y_{ij}(t) = x_{ij}(t) - x_{ij}^*(t)$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. Then from (1.1), for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and $t \geq t_0$, we have

$$\begin{aligned} \mathrm{d}y_{ij}(t) = {}& \bigg[-a_{ij}(t)y_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)\big(f\big(x_{kl}\big(t-\delta(t)\big)\big)x_{ij}(t) - f\big(x_{kl}^*\big(t-\delta(t)\big)\big)x_{ij}^*(t)\big) \\ & - \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_{t-\tau_{ij}(t)}^{t} k_{ij}(t-u)\big(g\big(x_{kl}(u)\big)x_{ij}(t) - g\big(x_{kl}^*(u)\big)x_{ij}^*(t)\big)\,\mathrm{d}u\bigg]\,\mathrm{d}t \\ & + \sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(t)\big(\sigma_{ij}\big(x_{ij}(t)\big) - \sigma_{ij}\big(x_{ij}^*(t)\big)\big)\,\mathrm{d}w_{ij}(t). \end{aligned} \tag{4.1}$$

The initial condition of (4.1) is

$$\phi_{ij}(s) = \psi_{ij}(s) - \varphi_{ij}(s), \quad s \in [-\tau, t_0], i = 1,2,\ldots,m, j = 1,2,\ldots,n.$$

By the method of variation parameter, for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$ and $t \geq t_0$, we have the following:

$$\begin{aligned} y_{ij}(t) = {}& y_{ij}(t_0)\mathrm{e}^{-\int_{t_0}^{t} a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \\ & - \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\bigg[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\big(f\big(x_{kl}\big(s-\delta(s)\big)\big)x_{ij}(s) - f\big(x_{kl}^*\big(s-\delta(s)\big)\big)x_{ij}^*(s)\big) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^{s} k_{ij}(s-u)\bigl(g\bigl(x_{kl}(u)\bigr)x_{ij}(s)-g\bigl(x_{kl}^*(u)\bigr)x_{ij}^*(s)\bigr)\,\mathrm{d}u\Bigg]\,\mathrm{d}s \\
&+\int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\bigl(\sigma_{ij}\bigl(x_{ij}(s)\bigr)-\sigma_{ij}\bigl(x_{ij}^*(s)\bigr)\bigr)\,\mathrm{d}w_{ij}(s) \\
={}& y_{ij}(t_0)\mathrm{e}^{-\int_{t_0}^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}-\int_{t_0}^{t}\mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\Bigg[\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\bigl(f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)y_{ij}(s) \\
&+\bigl(f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)-f\bigl(x_{kl}^*\bigl(s-\delta(s)\bigr)\bigr)\bigr)x_{ij}^*(s)\bigr) \\
&+\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^{s} k_{ij}(s-u)\bigl(g\bigl(x_{kl}(u)\bigr)y_{ij}(s) \\
&+\bigl(g\bigl(x_{kl}(u)\bigr)-g\bigl(x_{kl}^*(u)\bigr)\bigr)x_{ij}^*(s)\bigr)\,\mathrm{d}u\Bigg]\,\mathrm{d}s \\
&+\int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\bigl(\sigma_{ij}\bigl(x_{ij}(s)\bigr)-\sigma_{ij}\bigl(x_{ij}^*(s)\bigr)\bigr)\,\mathrm{d}w_{ij}(s).
\end{aligned}
$$

For $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, denote $H_{ij}^{(1)} = y_{ij}(t_0)\mathrm{e}^{-\int_{t_0}^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}$,

$$
\begin{aligned}
H_{ij}^{(2)} &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)y_{ij}(s)\,\mathrm{d}s, \\
H_{ij}^{(3)} &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\bigl(f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)-f\bigl(x_{kl}^*\bigl(s-\delta(s)\bigr)\bigr)\bigr)x_{ij}^*(s)\,\mathrm{d}s, \\
H_{ij}^{(4)} &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^{s} k_{ij}(s-u)g\bigl(x_{kl}(u)\bigr)\,\mathrm{d}u y_{ij}(s)\,\mathrm{d}s, \\
H_{ij}^{(5)} &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^{s} k_{ij}(s-u)\bigl(g\bigl(x_{kl}(u)\bigr)-g\bigl(x_{kl}^*(u)\bigr)\bigr)\,\mathrm{d}u x_{ij}^*(s)\,\mathrm{d}s, \\
H_{ij}^{(6)} &= \int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\bigl(\sigma_{ij}\bigl(x_{ij}(t)\bigr)-\sigma_{ij}\bigl(x_{ij}^*(t)\bigr)\bigr)\,\mathrm{d}w_{ij}(s).
\end{aligned}
$$

Taking expectations and using Lemma 2.3, for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, we have

$$
E\bigl|y_{ij}(t)\bigr|^p \le 6^{p-1}E\bigl(\bigl|H_{ij}^{(1)}\bigr|^p+\bigl|H_{ij}^{(2)}\bigr|^p+\bigl|H_{ij}^{(3)}\bigr|^p+\bigl|H_{ij}^{(4)}\bigr|^p+\bigl|H_{ij}^{(5)}\bigr|^p+\bigl|H_{ij}^{(6)}\bigr|^p\bigr). \tag{4.2}
$$

Proceeding as in the proof of Theorem 3.1, we evaluate the first term of (4.2) as follows:

$$
E\bigl|H_{ij}^{(1)}\bigr|^p \le \mathrm{e}^{-p\underline{a}_{ij}(t-t_0)}E\bigl|y_{ij}(t_0)\bigr|^p, \quad i = 1,2,\dots,m, j = 1,2,\dots,n.
$$

For the second term of (4.2), for $i = 1,2,\dots,m$, $j = 1,2,\dots,n$, we have

$$
\begin{aligned}
E\bigl|H_{ij}^{(2)}\bigr|^p &= E\biggl|\int_{t_0}^{t} \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta}\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)f\bigl(x_{kl}\bigl(s-\delta(s)\bigr)\bigr)y_{ij}(s)\,\mathrm{d}s\biggr|^p \\
&\le (\underline{a}_{ij})^{1-p}\int_{t_0}^{t} \mathrm{e}^{-\underline{a}_{ij}(t-s)}\biggl(M_f\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\biggr)^p E\bigl|y_{ij}(s)\bigr|^p\,\mathrm{d}s.
\end{aligned}
$$

As to the third term of (4.2), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we also have

$$\begin{aligned} E\big|H_{ij}^{(3)}\big|^p &= E\bigg|\int_{t_0}^t \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(s)\big(f\big(x_{kl}\big(s-\delta(s)\big)\big) \\ &\quad - f\big(x_{kl}^*\big(s-\delta(s)\big)\big)\big)x_{ij}^*(s)\,\mathrm{d}s\bigg|^p \\ &\le E\bigg(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} N L_f \big|y_{kl}\big(s-\delta(s)\big)\big|\,\mathrm{d}s\bigg)^p \\ &\le (\underline{a}_{ij})^{1-p}\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)}\bigg(NL_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\bigg)^p E\big|y_{kl}\big(s-\delta(s)\big)\big|^p\,\mathrm{d}s. \end{aligned}$$

For the fourth term of (4.2), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned} E\big|H_{ij}^{(4)}\big|^p &= E\bigg|\int_{t_0}^t \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)g\big(x_{kl}(u)\big)\,\mathrm{d}u\, y_{ij}(s)\,\mathrm{d}s\bigg|^p \\ &\le (\underline{a}_{ij})^{1-p}\int_{t_0}^t \mathrm{e}^{-a_{ij}(t-s)}\bigg(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p E\big|y_{ij}(s)\big|^p\,\mathrm{d}s. \end{aligned}$$

As to the fifth term of (4.2), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned} E\big|H_{ij}^{(5)}\big|^p &= E\bigg|\int_{t_0}^t \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(s)\int_{s-\tau_{ij}(s)}^s k_{ij}(s-u)\big(g\big(x_{kl}(u)\big) \\ &\quad - g\big(x_{kl}^*(u)\big)\big)\,\mathrm{d}u\, x_{ij}^*(s)\,\mathrm{d}s\bigg|^p \\ &\le (\underline{a}_{ij})^{1-p}\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)}\bigg(NL_g \sum_{C_{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p E\big|y_{kl}(s)\big|^p\,\mathrm{d}s. \end{aligned}$$

As to the last term of (4.2), for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned} E\big|H_{ij}^{(6)}\big|^p &= E\bigg|\int_{t_0}^t \mathrm{e}^{-\int_s^t a_{ij}(\vartheta)\,\mathrm{d}\vartheta} \sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(s)\big(\sigma_{ij}\big(x_{ij}(t)\big) - \sigma_{ij}\big(x_{ij}^*(t)\big)\big)\,\mathrm{d}w_{ij}(s)\bigg|^p \\ &\le l_p(mn)^{\frac{p}{2}}\bigg(\frac{2\underline{a}_{ij}(p-1)}{p-2}\bigg)^{1-\frac{p}{2}}\bigg(\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)}\bigg(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\bigg)^p E\big|y_{ij}(s)\big|^p\,\mathrm{d}s\bigg). \end{aligned}$$

Therefore, for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, we have

$$\begin{aligned} E\big|y_{ij}(t)\big|^p &\le 6^{p-1}\bigg\{\mathrm{e}^{-p\underline{a}_{ij}(t-t_0)}E\big|y_{ij}(t_0)\big|^p + (\underline{a}_{ij})^{1-p}\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)}\bigg[\bigg(M_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\bigg)^p \\ &\quad + \bigg(M_g \sum_{C_{kl}\in N_q(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p + \bigg(L_g \sum_{C_{kl}\in N_r(i,j)} \overline{B}_{ij}^{kl}\overline{k}_{ij}\bigg)^p\bigg]E\big|y_{ij}(s)\big|^p\,\mathrm{d}s \\ &\quad + (\underline{a}_{ij})^{1-p}N\int_{t_0}^t \mathrm{e}^{-\underline{a}_{ij}(t-s)}\bigg(L_f \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl}\bigg)^p E\big|y_{kl}\big(s-\delta(s)\big)\big|^p\,\mathrm{d}s \end{aligned}$$

$$+ l_p(mn)^{\frac{p}{2}} \left(\frac{2\underline{a}_{ij}(p-1)}{p-2}\right)^{1-\frac{p}{2}} \times \left(\int_{t_0}^{t} \mathrm{e}^{-\underline{a}_{ij}(t-s)} \left(\sum_{C_{kl}\in N_p(i,j)} \overline{D}_{ij}^{kl} l_{ij}\right)^{p} E\left|y_{ij}(s)\right|^{p} \,\mathrm{d}s\right)\Bigg\}. \tag{4.3}$$

Define $U(t) = (U_{11}(t), U_{12}(t), \ldots, U_{mn}(t))^T$, where $U_{ij}(t) = E|y_{ij}(t)|^p$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$. By (4.3), we have

$$U_{ij}(t) \le 6^{p-1}\mathrm{e}^{-\theta(t-t_0)}U_{ij}(t_0) + \int_{t_0}^{t} \mathrm{e}^{-\theta(t-s)}\epsilon_1 U_{ij}(s)\,\mathrm{d}s + \int_{t_0}^{t} \mathrm{e}^{-\theta(t-s)}\epsilon_2\left|U_{ij}(s)\right|_{\tau}^{+}\,\mathrm{d}s.$$

By (H_5) and Lemma 2.5, the periodic solution $x^*(t)$ of (1.1) is p-exponentially stable. This completes the proof of Theorem 4.1. □

5 An example

In this section, we will give an example to illustrate the feasibility of our results.

Example 5.1 Let $n = m = 3$. Consider the following stochastic SICNNs:

$$\begin{aligned}\mathrm{d}x_{ij}(t) = \Bigg[&-a_{ij}(t)x_{ij}(t) - \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f\big(x_{kl}\big(t-\delta(t)\big)\big)x_{ij}(t) \\ &- \sum_{C_{kl}\in N_q(i,j)} B_{ij}^{kl}(t)\int_{t-\tau_{ij}(t)}^{t} k_{ij}(t-u)g\big(x_{kl}(u)\big)\,\mathrm{d}u\,x_{ij}(t) + L_{ij}(t)\Bigg]\mathrm{d}t \\ &+ \sum_{C_{kl}\in N_p(i,j)} D_{ij}^{kl}(t)\sigma_{ij}\big(x_{ij}(t)\big)\,\mathrm{d}w_{ij}(t),\end{aligned} \tag{5.1}$$

where

$$\big(a_{ij}(t)\big)_{3\times 3} = \begin{pmatrix} 0.4+0.02\sin\frac{\pi}{2}t & 0.5+0.01\sin\frac{\pi}{2}t & 0.3+0.02\sin\frac{\pi}{2}t \\ 0.5+0.02\sin\frac{\pi}{2}t & 0.6+0.01\sin\frac{\pi}{2}t & 0.4+0.02\cos\frac{\pi}{2}t \\ 0.3+0.02\cos\frac{\pi}{2}t & 0.4+0.03\cos\frac{\pi}{2}t & 0.2+0.03\cos\frac{\pi}{2}t \end{pmatrix},$$

$$\big(C_{ij}(t)\big)_{3\times 3} = \begin{pmatrix} 0.01|\sin\frac{\pi}{4}t| & 0.02|\sin\frac{\pi}{4}t| & 0.01|\sin\frac{\pi}{4}t| \\ 0.01|\sin\frac{\pi}{4}t| & 0.01|\sin\frac{\pi}{4}t| & 0.01|\sin\frac{\pi}{4}t| \\ 0.01|\sin\frac{\pi}{4}t| & 0.01|\sin\frac{\pi}{4}t| & 0.01|\sin\frac{\pi}{4}t| \end{pmatrix},$$

$$\big(B_{ij}(t)\big)_{3\times 3} = \big(D_{ij}(t)\big)_{3\times 3} = \begin{pmatrix} 0.01|\cos\frac{\pi}{4}t| & 0.01|\cos\frac{\pi}{4}t| & 0.02|\cos\frac{\pi}{4}t| \\ 0.01|\cos\frac{\pi}{4}t| & 0.01|\cos\frac{\pi}{4}t| & 0.01|\cos\frac{\pi}{4}t| \\ 0.01|\cos\frac{\pi}{4}t| & 0.01|\cos\frac{\pi}{4}t| & 0.01|\cos\frac{\pi}{4}t| \end{pmatrix},$$

$$\big(L_{ij}(t)\big)_{3\times 3} = \begin{pmatrix} 0.2\sin\frac{\pi}{2}t & 0.3\sin\frac{\pi}{2}t & 0.1\cos\frac{\pi}{2}t \\ 0.6\sin\frac{\pi}{2}t & 0.3\cos\frac{\pi}{2}t & 0.6\cos\frac{\pi}{2}t \\ 0.3\cos\frac{\pi}{2}t & 0.2\cos\frac{\pi}{2}t & 0.5\cos\frac{\pi}{2}t \end{pmatrix},$$

$$\big(\sigma_{ij}(u)\big)_{3\times 3} = \begin{pmatrix} 0.1\cos u & 0.2\sin u & 0.4\cos u \\ 0.6\sin u & 0.3\sin u & 0.6\cos u \\ 0.3\cos u & 0.2\cos u & 0.5\cos u \end{pmatrix},$$

$$f(u) = g(u) = 0.2\cos u, \qquad k_{ij}(u) = e^{-4u}, \quad i,j = 1,2,3.$$

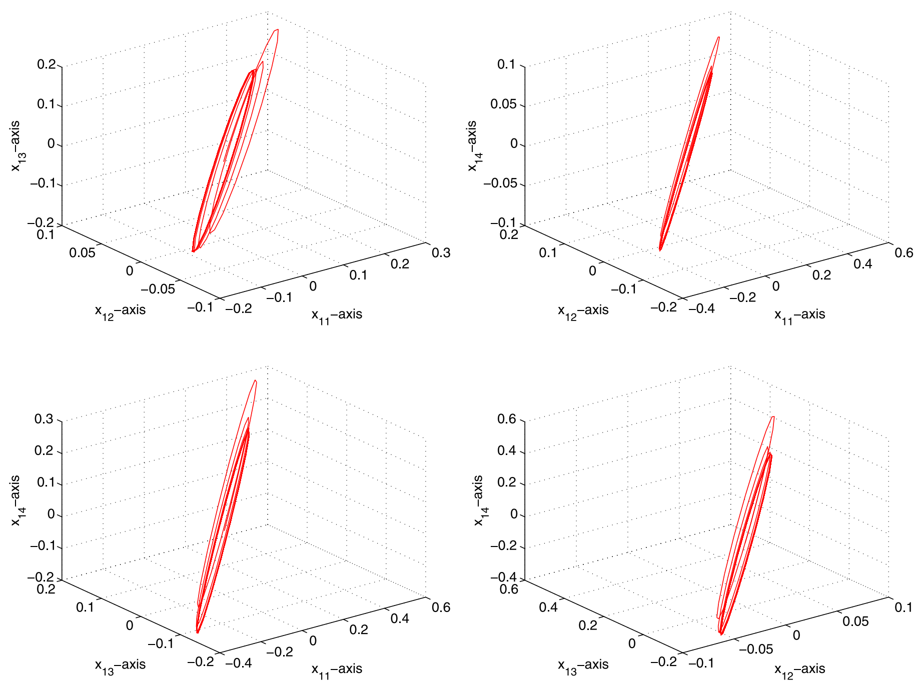

Figure 1 Phase responses of states $x_{11}, x_{12}, x_{13}, x_{14}$ in Example 5.1.

By calculating, we have

$$(\underline{a}_{ij})_{3\times 3} = \begin{pmatrix} 0.38 & 0.49 & 0.28 \\ 0.48 & 0.59 & 0.38 \\ 0.28 & 0.37 & 0.17 \end{pmatrix}, \qquad (\overline{C}_{ij})_{3\times 3} = \begin{pmatrix} 0.01 & 0.02 & 0.01 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{pmatrix},$$

$$(\overline{B}_{ij})_{3\times 3} = (\overline{D}_{ij})_{3\times 3} = \begin{pmatrix} 0.01 & 0.01 & 0.02 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{pmatrix}, \qquad (\overline{L}_{ij})_{3\times 3} = \begin{pmatrix} 0.2 & 0.3 & 0.1 \\ 0.6 & 0.3 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix},$$

$$(l_{ij})_{3\times 3} = \begin{pmatrix} 0.1 & 0.2 & 0.4 \\ 0.6 & 0.3 & 0.6 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}, \qquad L_f = L_g = M_f = M_g = 0.2,$$

$$\bar{k}_{ij} = 0.25, \quad i, j = 1, 2, 3.$$

Taking $p = 3$, $r = q = 1$, we get $\epsilon \approx 0.0947$, $\epsilon_1 \approx 0.0683$, $\epsilon_2 \approx 0.0341$, $\theta = 0.17$, that is, all conditions in Theorem 3.1 and Theorem 4.1 are satisfied. Therefore, we see that (5.1) has a 4-periodic solution, which is 3-exponentially stable (simulations in Figure 1 show that our result is feasible).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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