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Multiplicity of solutions for second-order impulsive differential equations with Sturm-Liouville boundary conditions

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Abstract
In this paper, we use variational methods to investigate the solutions of impulsive differential equations with Sturm-Liouville boundary conditions. The conditions for the existence and multiplicity of solutions are established. The main results are also demonstrated with examples.

Keywords: variational methods; impulsive differential equations; boundary value problem

1 Introduction

Impulsive differential equations arising from the real world describe the dynamics of a process in which sudden, discontinuous jumps occur. Such processes are naturally seen in biology, medicine, mechanics, engineering, chaos theory, and so on. Due to its significance, a great deal of work has been done in the theory of impulsive differential equations [1–8].

In this paper, we consider the following second-order impulsive differential equations with Sturm-Liouville boundary conditions:

$$\begin{cases} -(p(t)u'(t))' + q(t)u(t) = f(t,u(t)), & t \neq t_j, t \in [0,1], \\ -\Delta(p(t_j)u'(t_j)) = I_j(u(t_j)), & j = 1,2,\ldots,n, \\ u'(0) + au(0) = 0, \qquad u'(1) - bu(1) = 0, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = 1$, $p \in C'[0,1]$, $q \in C[0,1]$, $p(t) > 0$, $q(t) > 0$, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ for $u'(t_j^{\pm}) = \lim_{t\to t_j^{\pm}} u'(t)$, $j = 1,2,\ldots,n$.

In recent years, boundary value problems for impulsive and Sturm-Liouville equations have been studied extensively in the literature. There have been many approaches to the study of positive solutions of differential equations, such as fixed point theory, topological degree theory and the comparison method [9–14]. On the other hand, many researchers have used variational methods to study the existence of solutions for boundary value problems [15–21]. However, to our knowledge, the study of solutions for impulsive differential equations as (1.1) using variational methods has received considerably less of attention.

More precisely, Tian and Ge [22] studied a linear impulsive problem with Sturm-Liouville boundary conditions:

$$\begin{cases} -u''(t) - \lambda u(t) = g(t), & t \neq t_j, t \in [0,T], \\ -\Delta u'(t_j) = d_j, & j = 1,2,\ldots,l, \\ \alpha u'(0) - \beta u(0) = 0, & ru'(T) + \sigma u(T) = 0 \end{cases} \tag{1.2}$$

and a nonlinear impulsive problem:

$$\begin{cases} -u''(t) - \lambda u(t) = f(t,u(t)), & t \neq t_j, t \in [0,T], \\ -\Delta u'(t_j) = I_j(u(t_j)), & j = 1,2,\ldots,l, \\ \alpha u'(0) - \beta u(0) = 0, & ru'(T) + \sigma u(T) = 0. \end{cases} \tag{1.3}$$

They obtained the existence of positive solutions for problems (1.2) and (1.3) by using the variational method.

Inspired by the work [22], in this paper we use critical point theory and variational methods to investigate the multiple solutions of (1.1). Our main results extend the study made in [22], in the sense that we deal with a class of problems that is not considered in those papers.

We need the following conditions.

(H1) There exist $M > m > 0$, and $\beta > \frac{2M}{m}$, such that for all $(t,u) \in [0,1] \times R$,

$$0 < \beta F(t,u) \leq uf(t,u),$$
$$0 < \beta \int_0^u I_j(s)\,ds \leq uI_j(u), \quad j = 1,2,\ldots,n,$$

where $F(t,u) = \int_0^u f(t,\xi)\,d\xi$.

(H2) $\overline{\lim}_{u\to 0} \frac{F(t,u)}{|u|^\beta} = 0$ uniformly for $t \in [0,1]$, $\overline{\lim}_{u\to 0} \frac{\int_0^u I_j(t)\,dt}{|u|^\beta} = 0$, $j = 1,2,\ldots,n$.

(H3) $f(t,u)$ and $I_j(u)$ are odd with respect to u.

2 Preliminaries and statements

Firstly, we introduce some notations and some necessary definitions.

In the Sobolev space $X = H_0^1(0,1)$, consider the inner product

$$(u,v) = \int_0^1 u'(t)v'(t)\,dt + \int_0^1 u(t)v(t)\,dt,$$

inducing the norm

$$\|u\| = \left(\int_0^1 \left|u'(t)\right|^2 dt + \int_0^1 \left|u(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

We also consider the inner product

$$(u,v)_X = \int_0^1 p(t)u'(t)v'(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt$$

and the norm

$$\|u\|_X = \left(\int_0^1 p(t)\left|u'(t)\right|^2 dt + \int_0^1 q(t)\left|u(t)\right|^2 dt \right)^{\frac{1}{2}}.$$

Then the norm $\|\cdot\|_X$ is equivalent to the usual norm $\|\cdot\|$ in $H_0^1(0,1)$. Hence, X is reflexive. Denote $\|u\|_\infty = \max_{t\in(0,1)} |u(t)|$.

For $u \in H^2(0,1)$, we find that u and u' are both absolutely continuous, and $u'' \in L^2(0,1)$, hence $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = 0$ for any $t \in [0,1]$. If $u \in H_0^1(0,1)$, then u is absolutely continuous and $u' \in L^2[0,1]$. In this case, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-) = 0$ is not necessarily valid for every $t \in (0,1)$ and the derivative u' may present some discontinuities. This leads to the impulsive effects. As a consequence, we need to introduce a different concept of solution. We say that $u \in C[0,1]$ is a classical solution of IBVP (1.1) if it satisfies the following conditions: u satisfies the first equation of (1.1) a.e. on $[0,1]$; the limits $u'(t_j^+)$, $u'(t_j^-)$, $j = 1,2,\ldots,n$ exist and the impulsive condition of (1.1) holds; u satisfies the boundary condition of (1.1); for every $j = 0,1,2,\ldots,n$, $u_j = u|_{(t_j,t_{j+1})} \in H^2(0,1)$.

We multiply the two sides of the first equation of (1.1) by $v \in X$ and integrate from 0 to 1, and we have

$$-\int_0^1 \bigl(p(t)u'(t)\bigr)' v(t)\,dt + \int_0^1 q(t)u(t)v(t)\,dt = \int_0^1 f\bigl(t,u(t)\bigr)v(t)\,dt.$$

Moreover,

$$\begin{aligned}
&-\int_0^1 \bigl(p(t)u'(t)\bigr)' v(t)\,dt \\
&\quad = -\sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} \bigl(p(t)u'(t)\bigr)' v(t)\,dt \\
&\quad = -\sum_{j=0}^{n} \left(p\bigl(t_{j+1}^-\bigr)u'\bigl(t_{j+1}^-\bigr)v\bigl(t_{j+1}^-\bigr) - p\bigl(t_j^+\bigr)u'\bigl(t_j^+\bigr)v\bigl(t_j^+\bigr) - \int_{t_j}^{t_{j+1}} p(t)u'(t)v'(t)\,dt \right) \\
&\quad = \sum_{j=1}^{n} \Delta\bigl(p(t_j)u'(t_j)\bigr)v(t_j) + p(0)u'(0)v(0) - p(1)u'(1)v(1) + \int_0^1 p(t)u'(t)v'(t)\,dt \\
&\quad = -\sum_{j=1}^{n} I_j\bigl(u(t_j)\bigr)v(t_j) - ap(0)u(0)v(0) - bp(1)u(1)v(1) + \int_0^1 p(t)u'(t)v'(t)\,dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_0^1 \bigl(p(t)u'(t)v'(t) + q(t)u(t)v(t)\bigr)\,dt - \int_0^1 f\bigl(t,u(t)\bigr)v(t)\,dt \\
&\quad = \sum_{j=1}^{n} I_j\bigl(u(t_j)\bigr)v(t_j) + ap(0)u(0)v(0) + bp(1)u(1)v(1).
\end{aligned}$$

Considering the above, we need to introduce a different concept of solution for problem (1.1).

Definition 2.1 We say that a function $u \in H_0^1(0,1)$ is a weak solution of problem (1.1) if the identity

$$\begin{aligned}
&\int_0^1 \bigl(p(t)u'(t)v'(t) + q(t)u(t)v(t)\bigr)\,dt - \int_0^1 f\bigl(t,u(t)\bigr)v(t)\,dt \\
&\quad = \sum_{j=1}^{n} I_j\bigl(u(t_j)\bigr)v(t_j) + ap(0)u(0)v(0) + bp(1)u(1)v(1)
\end{aligned}$$

holds for any $v \in H_0^1(0,1)$.

We consider the functional $\varphi : X \to R$, defined by

$$\begin{aligned}
\varphi(u) &= \int_0^1 \left(\frac{1}{2}\bigl(p(t)\bigl|u'(t)\bigr|^2 + q(t)\bigl|u(t)\bigr|^2\bigr)\right)dt - \int_0^1 F\bigl(t,u(t)\bigr)\,dt - \sum_{j=1}^{n}\int_0^{u(t_j)} I_j(t)\,dt \\
&\quad - \frac{a}{2}p(0)u^2(0) - \frac{b}{2}p(1)u^2(1) \\
&= \frac{1}{2}\|u\|_X^2 - \int_0^1 F\bigl(t,u(t)\bigr)\,dt - \sum_{j=1}^{n}\int_0^{u(t_j)} I_j(t)\,dt - \frac{a}{2}p(0)u^2(0) - \frac{b}{2}p(1)u^2(1),
\end{aligned} \tag{2.1}$$

where $F(t,u) = \int_0^u f(t,\xi)\,d\xi$. Using the continuity of f and I_j, $j = 1,2,\ldots,n$, one has $\varphi \in C^1(X,R)$. For any $v \in X$, we have

$$\begin{aligned}
\varphi'(u)v &= \int_0^1 \bigl(p(t)u'(t)v'(t) + q(t)u(t)v(t)\bigr)\,dt - \int_0^1 f\bigl(t,u(t)\bigr)v(t)\,dt \\
&\quad - \sum_{j=1}^{n} I_j\bigl(u(t_j)\bigr)v(t_j) - ap(0)u(0)v(0) - bp(1)u(1)v(1).
\end{aligned} \tag{2.2}$$

Thus, the solutions of problem (1.1) are the corresponding critical points of φ.

Lemma 2.1 *If $u \in X$ is a weak solution of* (1.1), *then u is a classical solution of* (1.1).

Proof The proof is similar to [15]. For any $j \in \{1,2,\ldots,n\}$ and $v \in X$ with $v(t) = 0$, for every $t \in [0,t_j] \cup [t_{j+1},1]$. Then

$$\int_{t_j}^{t_{j+1}} \bigl[p(t)u'(t)v'(t) + q(t)u(t)v(t) - f\bigl(t,u(t)\bigr)v(t)\bigr]\,dt = 0. \tag{2.3}$$

By the definition of weak derivative, the above equality implies

$$-\bigl(p(t)u'(t)\bigr)' + q(t)u(t) = f\bigl(t,u(t)\bigr), \quad \text{a.e. } t \in (t_j,t_{j+1}). \tag{2.4}$$

Hence $u_j \in H^2(t_j,t_{j+1})$ and u satisfies the first equation of (1.1) a.e. on $[0,1]$.

Now, multiplying by $v \in H_0^1(0,1)$ and integrating between 0 and 1, we get

$$\begin{aligned}
&\sum_{j=1}^{n}\bigl[\Delta\bigl(p(t_j)u'(t_j)\bigr) + I_j\bigl(u(t_j)\bigr)\bigr]v(t_j) + p(0)\bigl[au(0) + u'(0)\bigr]v(0) \\
&\quad + p(1)\bigl[bu(1) - u'(1)\bigr]v(1) = 0.
\end{aligned} \tag{2.5}$$

Next we will show that u satisfies the impulsive conditions in (1.1). If not, without loss of generality, we assume that there exists $j \in \{1, 2, \ldots, n\}$ such that

$$I_j\big(u(t_j)\big) + \Delta\big(p(t_j)u'(t_j)\big) \neq 0. \tag{2.6}$$

Let $v(t) = \prod_{i=0, i\neq j}^{n+1}(t - t_i)$, then by (2.5), we get

$$\big[\Delta\big(p(t_j)u'(t_j)\big) + I_j\big(u(t_j)\big)\big]v(t_j) = 0,$$

which contradicts (2.6), so u satisfies the impulsive conditions of (1.1). Similarly, u satisfies the boundary conditions. Therefore, u is a classical solution of problem (1.1). □

Lemma 2.2 *Let $u \in X$, then*

$$\|u\|_\infty \leq C_1\|u\|_X, \tag{2.7}$$

where $C_1 = \sqrt{2}\max\{\frac{1}{(\min_{t\in[0,1]} p(t))^{\frac{1}{2}}}, \frac{1}{(\min_{t\in[0,1]} q(t))^{\frac{1}{2}}}\}$.

Proof By using the same methods as [22], we can obtain the result, here we omit it. □

Defining

$$a(u,v) = \int_0^1 \big[p(t)u'(t)v'(t) + q(t)u(t)v(t)\big]\,dt - ap(0)u(0)v(0) - bp(1)u(1)v(1),$$

then we have the following.

Lemma 2.3 *If $a < \frac{1}{p(0)C_1^2}$ and $b \leq 0$, or $a \leq 0$ and $b < \frac{1}{p(1)C_1^2}$, there exist constants $0 < m < M$ such that*

$$m\|u\|_X \leq a(u,v) \leq M\|u\|_X. \tag{2.8}$$

Proof Firstly we prove the left part of (2.8),

$$a(u,u) = \int_0^1 \big(p(t)\big|u'(t)\big|^2 + q(t)\big|u(t)\big|^2\big)\,dt - ap(0)u^2(0) - bp(1)u^2(1).$$

(i) If $a \leq 0$ and $b \leq 0$, then $a(u,u) \geq \|u\|_X^2$.

(ii) If $0 \leq a < \frac{1}{p(0)C_1^2}$ and $b \leq 0$, then

$$\begin{aligned} a(u,u) &\geq \|u\|_X^2 - ap(0)u^2(0) \geq \|u\|_X^2 - ap(0)\|u\|_\infty^2 \\ &\geq \|u\|_X^2 - ap(0)C_1^2\|u\|_X^2 = \big(1 - ap(0)C_1^2\big)\|u\|_X^2. \end{aligned}$$

(iii) If $a \leq 0$ and $0 \leq b < \frac{1}{p(1)C_1^2}$, then

$$\begin{aligned} a(u,u) &\geq \|u\|_X^2 - bp(1)u^2(0) \geq \|u\|_X^2 - bp(1)\|u\|_\infty^2 \\ &\geq \|u\|_X^2 - bp(1)C_1^2\|u\|_X^2 = \big(1 - bp(1)C_1^2\big)\|u\|_X^2. \end{aligned}$$

From (i), (ii), and (iii), set $m = \max\{1, 1 - ap(0)C_1^2, 1 - bp(1)C_1^2\}$, and we have

$$a(u,u) \geq m\|u\|_X^2.$$

On the other hand,

$$\begin{aligned} a(u,u) &= \|u\|_X^2 - ap(0)u^2(0) - bp(1)u^2(1) \\ &\leq \|u\|_X^2 + |a|p(0)\|u\|_\infty^2 + |b|p(1)\|u\|_\infty^2 \\ &\leq \bigl[1 + |a|p(0)C_1^2 + |b|p(1)C_1^2\bigr]\|u\|_X^2. \end{aligned}$$

Set $M = 1 + |a|p(0)C_1^2 + |b|p(1)C_1^2$, then

$$a(u,u) \leq M\|u\|_X^2.$$

This is the end of the proof. □

We state some basic notions and celebrated results from critical points theory.

Definition 2.2 Let X be a real Banach space (in particular a Hilbert space) and $\varphi \in C^1(X,R)$. φ is said to be satisfying the P.S. condition on X if any sequence $\{x_n\} \in X$ for which $\varphi(x_n)$ is bounded and $\varphi'(x_n) \to 0$ as $n \to \infty$, possesses a convergent subsequence in X.

Lemma 2.4 (see [23]) *Let $\varphi \in C^1(X,R)$, and let φ satisfy the P.S. condition. Assume that there exist $u_0, u_1 \in X$ and a bounded neighborhood Ω of u_0 such that u_1 is not in Ω and*

$$\inf_{v\in\partial\Omega} \varphi(v) > \max\bigl\{\varphi(u_0), \varphi(u_1)\bigr\}.$$

Then there exists a critical point u of φ, i.e., $\varphi'(u) = 0$, with

$$\varphi(u) > \max\bigl\{\varphi(u_0), \varphi(u_1)\bigr\}.$$

Note that if either u_0 or u_1 is a critical point of φ, then we obtain the existence of at least two critical points for φ.

Lemma 2.5 (see [24]) *Let E be an infinite dimensional real Banach space. Let $\varphi \in C^1(E,R)$ be an even functional which satisfies the P.S. condition, and $\varphi(0) = 0$. Suppose that $E = V \oplus X$, where V is finite dimensional, and φ satisfies*:

(i) *there exist $\alpha > 0$ and $\rho > 0$ such that $\varphi|_{\partial B_\rho \cap X} \geq \alpha$*;
(ii) *for any finite dimensional subspace $W \subset E$, there is an $R = R(W)$ such that $\varphi(u) \leq 0$ for every $u \in W$ with $\|u\| > R$.*

Then φ possesses an unbounded sequence of critical values.

Lemma 2.6 (see [25]) *For the functional $F : M \subseteq X \to R$ with M not empty,* $\min_{u\in M} F(u) = a$ *has a solution in case the following hold*:

(i) *X is a real reflexive Banach space*;
(ii) *M is bounded and weak sequentially closed*;

(iii) *F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_n\}$ in M such that $u_n \rightharpoonup u$ as $n \to \infty$, we have $F(u) \le \underline{\lim}_{n\to\infty} F(u_n)$.*

3 Main results

To prove our main results, we need the following lemmas.

Lemma 3.1 *The function $\varphi : X \to R$ defined by* (2.1) *is continuous, continuously differentiable and weakly lower semi-continuous. Moreover, if $a < \frac{1}{p(0)C_1^2}$, $b \le 0$, or $a \le 0$, $b < \frac{1}{p(1)C_1^2}$, and* (H1) *holds, then φ satisfies the P.S. condition.*

Proof From the continuity of f and I_j, $j = 1, 2, \dots, n$, we obtain the continuity and differentiability of φ and φ'.

To show that φ is weakly lower semi-continuous, let $\{u_k\}$ be a weakly convergent sequence to u in X. Then $\|u\| \le \underline{\lim}_{k\to\infty} \|u_k\|$, and $\{u_k\}$ converges uniformly to u in $C[0, 1]$, and

$$\begin{aligned}
&\lim_{k\to\infty} \frac{1}{2}\int_0^1 q(t)\bigl|u_k(t)\bigr|^2\,dt - \int_0^1 F\bigl(t, u_k(t)\bigr)\,dt \\
&\quad - \sum_{j=1}^n \int_0^{u_k(t_j)} I_j(t)\,dt - \frac{a}{2}p(0)u_k^2(0) - \frac{b}{2}p(1)u_k^2(1) \\
&= \frac{1}{2}\int_0^1 q(t)\bigl|u(t)\bigr|^2\,dt - \int_0^1 F\bigl(t, u(t)\bigr)\,dt \\
&\quad - \sum_{j=1}^n \int_0^{u(t_j)} I_j(t)\,dt - \frac{a}{2}p(0)u_k^2(0) - \frac{b}{2}p(1)u_k^2(1).
\end{aligned}$$

We conclude that $\varphi(u) \le \underline{\lim}_{k\to\infty} \varphi(u_k)$. Then φ is weakly lower semi-continuous.

Next we show that φ satisfies the P.S. condition. Let $\{\varphi(u_k)\}$ be a bounded sequence such that $\lim_{k\to\infty} \varphi'(u_k) = 0$, then there exists a constant $C_2 > 0$ such that

$$\bigl\|\varphi(u_k)\bigr\|_X \le C_2, \qquad \bigl\|\varphi'(u_k)\bigr\|_X \le C_2.$$

By (2.2) and (2.8), we get

$$\begin{aligned}
&\int_0^1 f\bigl(t, u_k(t)\bigr)u_k(t)\,dt + \sum_{j=1}^n I_j\bigl(u_k(t_j)\bigr)u_k(t_j) \\
&= \int_0^1 p(t)\bigl|u_k'(t)\bigr|^2 + q(t)\bigl|u_k(t)\bigr|^2\,dt - ap(0)u_k^2(0) - bp(1)u_k^2(1) - \varphi'(u_k)u_k \\
&\le M\|u_k\|_X^2 - \varphi'(u_k)u_k.
\end{aligned} \tag{3.1}$$

From (2.8) and (3.1), we have

$$\begin{aligned}
\varphi(u_k) &\ge \frac{m}{2}\|u_k\|_X^2 - \int_0^1 F\bigl(t, u_k(t)\bigr)\,dt - \sum_{j=1}^n \int_0^{u_k(t_j)} I_j(t)\,dt \\
&\ge \frac{m}{2}\|u_k\|_X^2 - \frac{1}{\beta}\int_0^1 f\bigl(t, u_k(t)\bigr)u_k(t)\,dt - \frac{1}{\beta}\sum_{j=1}^n I_j\bigl(u_k(t_j)\bigr)u_k(t_j)
\end{aligned}$$

$$\begin{aligned}&\geq \frac{m}{2}\|u_k\|_X^2 - \frac{M}{\beta}\|u_k\|_X^2 - \frac{1}{\beta}\varphi'(u_k)u_k \\ &\geq \left(\frac{m}{2} - \frac{M}{\beta}\right)\|u_k\|_X^2 - \frac{C_2}{\beta}\|u_k\|_X. \end{aligned} \tag{3.2}$$

Since $\{\varphi(u_k)\}$ is bounded, from (3.2) we see that $\|u_k\|_X$ is bounded.

From the reflexivity of X, we may extract a weakly convergent subsequence, which, for simplicity, we call $\{u_k\}$, $u_k \rightharpoonup u$ in X. In the following we will verify that $\{u_k\}$ strongly converges to u. We have

$$\begin{aligned}&\big(\varphi'(u_k) - \varphi'(u)\big)(u_k - u) \\ &\quad = \|u_k - u\|_X^2 - \sum_{j=1}^n \big[I_j\big(u_k(t_j)\big) - I_j\big(u(t_j)\big)\big]\big(u_k(t_j) - u(t_j)\big) \\ &\qquad - \int_0^1 \big[f\big(t, u_k(t)\big) - f\big(t, u(t)\big)\big]\big(u_k(t) - u(t)\big)\,dt \\ &\qquad - \big[ap(0)u_k^2(0) - ap(0)u^2(0)\big] \\ &\qquad - \big[bp(1)u_k^2(1) - bp(1)u^2(1)\big].\end{aligned}$$

By $u_k \rightharpoonup u$ in X, we see that $\{u_k\}$ uniformly converges to u in $C[0,1]$. So

$$\begin{aligned}&\sum_{j=1}^n \big[I_j\big(u_k(t_j)\big) - I_j\big(u(t_j)\big)\big]\big(u_k(t_j) - u(t_j)\big) \to 0, \\ &\int_0^1 \big[f\big(t, u_k(t)\big) - f\big(t, u(t)\big)\big]\big(u_k(t) - u(t)\big)\,dt \to 0, \\ &\big[ap(0)u_k^2(0) - ap(0)u^2(0)\big] \to 0, \\ &\big[bp(1)u_k^2(1) - bp(1)u^2(1)\big] \to 0, \\ &\big(\varphi'(u_k) - \varphi'(u)\big)(u_k - u) \to 0, \quad \text{as } k \to +\infty.\end{aligned}$$

So we obtain $\|u_k - u\|_X \to 0$, as $k \to +\infty$. That is, $\{u_k\}$ strongly converges to u in X, which means φ satisfies the P.S. condition. □

Lemma 3.2 *Assume that* (H1) *holds, then there exist* $l, d, d_j, k_j > 0$, $j = 1, 2, \ldots, n$, *such that*

$$F(t,u) \geq l|u|^\beta + d, \quad \forall u \in R, \tag{3.3}$$

$$\int_0^u I_j(t)\,dt \geq k_j|u|^\beta + d_j, \quad \forall u \in R. \tag{3.4}$$

Proof From (H1), we get

$$\begin{aligned}&\frac{\beta}{u} \leq \frac{f(t,u)}{F(t,u)}, \quad \forall u > 0, \\ &\frac{\beta}{u} \geq \frac{f(t,u)}{F(t,u)}, \quad \forall u < 0.\end{aligned}$$

Integrating the above two inequalities from 1 to u and u to -1, respectively, we have

$$\beta \ln u \le \ln \frac{F(t,u)}{F(t,1)}, \quad \forall u > 1,$$
$$\beta \ln u \ge \ln \frac{F(t,-1)}{F(t,u)}, \quad \forall u < -1.$$

That is,

$$F(t,u) \ge F(t,1)u^{\beta}, \quad \forall u > 1,$$
$$F(t,u) \ge F(t,-1)(-u)^{\beta}, \quad \forall u < -1.$$

So there exists a constant $l > 0$ such that

$$F(t,u) \ge l|u|^{\beta}, \quad \forall |u| > 1. \tag{3.5}$$

From the continuity of $F(t,u)$, there exists a constant $d > 0$, such that

$$F(t,u) \ge d, \quad \forall |u| \le 1. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$F(t,u) \ge l|u|^{\beta} + d, \quad \forall u \in R.$$

Using the same methods, we know that there exist two constants $k_j > 0$ and $d_j > 0$ such that

$$\int_0^u I_j(t)\,dt \ge k_j|u|^{\beta} + d_j, \quad \forall u \in R.$$

This is the end of the proof. □

Now we get the main results of this paper.

Theorem 3.1 *Suppose* $a < \frac{1}{p(0)C_1^2}$, $b \le 0$, *or* $a \le 0$, $b < \frac{1}{p(1)C_1^2}$, *and* (H1) *and* (H2) *hold, then* (1.1) *has at least two solutions.*

Proof In our case it is clear that $\varphi(0) = 0$, Lemma 3.1 has shown that φ satisfies the P.S. condition.

Firstly, we will show that there exists $k > 0$ such that the functional φ has a local minimum $u_0 \in B_k = \{u \in H_0^1(0,1) : \|u\|_X < k\}$.

Let $k > 0$, which will be determined later. Since $X = H_0^1(0,1)$ is a Hilbert space, it is easy to deduce that $\bar{B}_k$ is bounded and weak sequentially closed. Lemma 3.1 has shown that φ is weak sequentially lower semi-continuous on $\bar{B}_k$. So by Lemma 2.6, we know that φ has a local minimum $u_0 \in \bar{B}_k$.

Without loss of generality, we assume that $\varphi(u_0) = \min_{u\in\bar{B}_k} \varphi(u)$. Now we will show that $\varphi(u_0) < \inf_{u\in\partial B_k} \varphi(u)$.

In fact, by (H2), we can choose $k > 0$, then there exist $h, h_j > 0$ satisfying

$$F(t,u) \le h|u|^\beta, \qquad \int_0^u I_j(t)\,dt \le h_j|u|^\beta, \quad \text{for } \|u\|_X \le k,$$
$$\frac{m}{2}k^2 - \left(h + \sum_{j=1}^n h_j\right)(C_1k)^\beta > 0.$$

For any $u \in \partial B_k$, $\|u\|_X = k$, we have

$$\begin{aligned}
\varphi(u) &= \int_0^1 \left(\frac{1}{2}\left(p(t)\left|u'(t)\right|^2 + q(t)\left|u(t)\right|^2\right)\right) dt - \int_0^1 F\big(t,u(t)\big)\,dt \\
&\quad - \sum_{j=1}^n \int_0^{u(t_j)} I_j(t)\,dt - \frac{a}{2}p(0)u^2(0) - \frac{b}{2}p(1)u^2(1) \\
&\ge \frac{m}{2}\|u\|_X^2 - \int_0^1 F\big(t,u(t)\big)\,dt - \sum_{j=1}^n \int_0^{u(t_j)} I_j(t)\,dt \\
&\ge \frac{m}{2}\|u\|_X^2 - h\int_0^1 |u|^\beta\,dt - \sum_{j=1}^n h_j\left|u(t_j)\right|^\beta \\
&\ge \frac{m}{2}\|u\|_X^2 - h\int_0^1 |u|^\beta\,dt - \sum_{j=1}^n h_j\left|u(t_j)\right|^\beta \\
&\ge \frac{m}{2}k^2 - \left(h + \sum_{j=1}^n h_j\right)(C_1k)^\beta > 0.
\end{aligned} \tag{3.7}$$

So $\varphi(u) > 0$ for any $u \in \partial B_k$. Besides, $\varphi(u_0) \le \varphi(0) = 0$. Then $\varphi(u) > 0 = \varphi(0) \ge \varphi(u_0)$ for any $u \in \partial B_k$. So $\varphi(u_0) \le \inf\{\varphi(u) : u \in \partial B_k\}$. Hence, φ has a local minimum $u_0 \in B_k = \{u \in X : \|u\|_X < k\}$.

Next we will verify that there exists a u_1 with $\|u_1\| > k$ such that $\varphi(u_1) < \inf_{\partial B_k} \varphi(u)$.

Let $r \in R \setminus \{0\}$, $e(t) = 1$. From (3.3) and (3.4), we have

$$\begin{aligned}
\varphi(re) &= \int_0^1 \left(\frac{r^2}{2}\left(p(t)\left|e'(t)\right|^2 + q(t)\left|e(t)\right|^2\right)\right) dt - \int_0^1 F\big(t,re(t)\big)\,dt \\
&\quad - \sum_{j=1}^n \int_0^{re(t_j)} I_j(t)\,dt - \frac{ar^2}{2}p(0)e^2(0) - \frac{br^2}{2}p(1)e^2(1) \\
&\le \frac{Mr^2}{2} - \int_0^1 \left(l\left|re(t)\right|^\beta + d\right) dt - \sum_{j=1}^n \left(k_j\left|re(t_j)\right|^\beta + d_j\right) \\
&\le \frac{Mr^2}{2} - r^\beta\left(l\int_0^1 \left|e(t)\right|^\beta dt + \sum_{j=1}^n k_j\left|e(t_j)\right|^\beta\right) - d - \sum_{j=1}^n d_j.
\end{aligned}$$

Since $\int_0^1 |e(t)|^\beta\,dt > 0$, $\sum_{j=1}^n k_j|e(t_j)|^\beta > 0$, $\beta > 2$, then we get $\lim_{|r|\to+\infty} \varphi(re) = -\infty$. Hence, there exists a sufficiently large $r_0 > 0$ with $\|r_0e\|_X > k$ such that $\varphi(r_0e) < \inf_{u\in\partial B_k} \varphi(u)$. Set $u_1 = r_0e$, then $\varphi(u_1) < \inf_{u\in\partial B_k} \varphi(u)$. Hence, by Lemma 2.4, there exists $u_2 \in X$ such that $\varphi'(u_2) = 0$. Therefore, u_0 and u_2 are two critical points of φ, and they are classical solutions of (1.1). □

Theorem 3.2 *Suppose* $a < \frac{1}{p(0)C_1^2}$, $b \leq 0$, *or* $a \leq 0$, $b < \frac{1}{p(1)C_1^2}$, *and* (H1), (H2), *and* (H3) *hold, then* (1.1) *has infinitely many classical solutions.*

Proof By (H3), we know that $f(t,u)$ and $I_j(u)$ are odd about u, then φ is even. Moreover, by Lemma 3.1, we know that $\varphi \in C^1(X,R)$, $\varphi(0) = 0$, and φ satisfies the P.S. condition.

Next, we will verify the conditions (i) and (ii) of Lemma 2.5.

Let $V \subset H_0^1(0,1)$ is a finite dimensional subspace, for any $u \in V^\perp$, by (3.7), we can easily verify (i) in the same way as in Theorem 3.1.

For each finite dimensional subspace $V_1 \subset H_0^1(0,1)$, for any $r \in R \setminus \{0\}$ and $u \in V_1 \setminus \{0\}$, the inequality

$$\begin{aligned}
\varphi(ru) &= \int_0^1 \left(\frac{r^2}{2} \bigl(p(t)\bigl|u'(t)\bigr|^2 + q(t)\bigl|u(t)\bigr|^2\bigr) \right) dt - \int_0^1 F\bigl(t, ru(t)\bigr)\, dt \\
&\quad - \sum_{j=1}^n \int_0^{ru(t_j)} I_j(t)\, dt - \frac{ar^2}{2} p(0)u^2(0) - \frac{br^2}{2} p(1)u^2(1) \\
&\leq \frac{Mr^2}{2} - \int_0^1 \bigl(l\bigl|ru(t)\bigr|^\beta + d\bigr)\, dt - \sum_{j=1}^n \bigl(k_j\bigl|ru(t_j)\bigr|^\beta + d_j\bigr) \\
&\leq \frac{Mr^2}{2} - r^\beta \left(l \int_0^1 \bigl|u(t)\bigr|^\beta\, dt + \sum_{j=1}^n k_j\bigl|u(t_j)\bigr|^\beta \right) - d - \sum_{j=1}^n d_j
\end{aligned} \tag{3.8}$$

holds. Take $w \in V_1$ such that $\|w\| = 1$, since $\int_0^1 |u(t)|^\beta\, dt > 0$, $\sum_{j=1}^n k_j|u(t_j)|^\beta > 0$, $\beta > 2$, (3.8) implies that there exists $r_w > 0$ such that $\|rw\| > R$ and $\varphi(rw) < 0$ for every $r \geq r_w > 0$. Since V_1 is a finite dimensional subspace, we can choose an $R = R(V_1) > 0$ such that $\varphi(u) < 0$, $\forall u \in V_1 \setminus B_R$.

According to Lemma 2.5, φ possesses infinitely many critical points, *i.e.*, the impulsive problem (1.1) has infinitely many solutions. □

4 Example

Example 4.1 Let $p(t) = 3t^2 + 2$, $q(t) = t^4 + 4$, $a = \frac{1}{3}$, $b = -\frac{1}{3}$, $\beta = 3$, we consider the Sturm-Liouville boundary value problem with impulse

$$\begin{cases}
-[(3t^2 + 2)u'(t)]' + (t^4 + 4)u(t) = 4u^3(t) + 3u^2(t), & t \neq t_j, t \in [0,1], \\
-\Delta[(3t_j^2 + 2)u'(t_j)] = \frac{9}{2}u^{\frac{7}{2}}(t_j), \quad j = 1,2,\ldots,n, \\
u'(0) + \frac{1}{3}u(0) = 0, \qquad u'(1) + \frac{1}{3}u(1) = 0.
\end{cases} \tag{4.1}$$

Compared with (1.1), $f(t,u) = 4u^3(t) + 3u^2(t)$, $I_j(u) = \frac{9}{2}u^{\frac{7}{2}}$.

The conditions (H1), (H2) are satisfied. Applying Theorem 3.1, problem (4.1) has at least two solutions.

Example 4.2 Let $p(t) = e^t$, $q(t) = \sin^2 t + 9$, $a = \frac{1}{4}$, $b = -\frac{1}{4}$, $\beta = 4$, consider the Sturm-Liouville boundary value problem with impulse

$$\begin{cases}
-[e^t u'(t)]' + (\sin^2 t + 9)u(t) = 6u^5(t) + 4u^3(t), & t \neq t_j, t \in [0,1], \\
-\Delta[e^{t_j} u'(t_j)] = u^5(t_j), \quad j = 1,2,\ldots,n, \\
u'(0) + \frac{1}{4}u(0) = 0, \qquad u'(1) + \frac{1}{4}u(1) = 0.
\end{cases} \tag{4.2}$$

Compared with (1.1), $f(t,u) = 6u^5(t) + 4u^3(t)$, $I_j(u) = u^5$.

The conditions (H1), (H2), (H3) are satisfied. Applying Theorem 3.2, problem (4.2) has infinitely many solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LY and JL carried out the proof of the main part of this article, ZL corrected the manuscript and participated in its design and coordination. All authors have read and approved the final manuscript.

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