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The bistability of higher-order differences of discrete periodic signals

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Abstract

A theorem on higher-order absolute differences taken from successive terms of bounded sequences is proved. This theorem establishes the property of bistability of such difference series and suggests a method for converting periodic discrete-time signals into the binary digital form based only on the computation of absolute differences.

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1 Introduction

This paper advances an approach in dynamical systems [1–5] which is based on considering higher-order differences taken from the orbits of a given system. Such an approach is motivated by the observation that some natural systems process the information contained in the signals' difference structure. For example, it is claimed in brain theory that the visual cortex 'responds to contrast rather than the uniform luminosity' and the higher differences up to fourth order in these problems are considered [6].

The difference method reveals some new aspects in analyzing arbitrary time series and discrete dynamical systems. For instance, the difference characteristic γ introduced in [2, 3] demonstrates a strong correlation with the Lyapunov exponent [7] used in deterministic chaos (the definition of γ can be found in [2, 3]; see [8, 9] for its modification and discussions). Another possible generalization could relate to fractional analysis: it would be interesting to define the fractional analog of γ and apply it to chaos discrimination in fractional dynamical systems (such systems are studied, *e.g.*, in [10–12]).

In the present paper another new aspect of the difference analysis, the bistability of higher-order differences taken from a bounded time series (discrete-time signals) is proved. Some possible applications to digital communication and signal processing in Section 2 are considered.

The content of this paper is the following. In Section 2 our main Theorem 1 is formulated. This theorem establishes the property of bistability for higher-order absolute differences taken from discrete-time signals. In the sense of applications it asserts that some systems operating on the basis of their inputs' higher-order differences should rather be classified over digital ones. The theorem suggests a method for converting some discrete (and sampled analog) signals into a binary digital form based only on computation of absolute differences; if applicable (*e.g.*, as for the case of arbitrary periodic signals), this method

is more effective than the traditional ones used in communication theory. The proof of
Theorem 1, based on a version of notion of peak sets originated in Banach algebras and
approximation theory, is presented in Section 3.

2 Main results

In the difference analysis a given time series (or an orbit of a dynamical system) X is de-
composed into two constituents-the sign and magnitude components. The *sign compo-
nent* S reflects the alternation in monotony (increase/decrease) of higher-order absolute
differences taken from successive terms of X and does not depend on their exact values.
The *magnitude* (or *height*) *component* H consists of these absolute differences and does
not depend on the sign distribution.

Let us proceed to formal definitions. Let

$$X = (f_1, f_2, \ldots, f_n, \ldots) \tag{1}$$

be an infinite bounded numerical sequence-this can be some time series, discrete-time
signal, experimental data (long enough), or an orbit of iterates of some map. For natural
m we consider mth-order *absolute differences*

$$H_n^{(0)} = f_n, \qquad H_n^{(m+1)} = \left| H_{n+1}^{(m)} - H_n^{(m)} \right| \quad (m \geq 0, n \geq 1) \tag{2}$$

and define the mth *absolute difference sequence* as $H_m = (H_n^{(m)})_{n=1}^{\infty}$. Furthermore, we define

$$S_m = \left(\delta_1^{(m)}, \delta_2^{(m)}, \ldots, \delta_n^{(m)}, \ldots \right), \quad \text{where } \delta_n^{(m)} = \begin{cases} +1, & H_{n+1}^{(m)} \geq H_n^{(m)}, \\ -1, & H_{n+1}^{(m)} < H_n^{(m)}. \end{cases} \tag{3}$$

The matrices $S = \{\delta_n^{(m)} : m \geq 0, n \geq 1\}$ and $H = \{H_n^{(m)} : m \geq 0, n \geq 1\}$ are called S- and H-
components of X, the sequences S_m and H_m are called mth S- and H-*components* of X and
denoted $S_m = S_m(X)$ and $H_m = H_m(X)$. In what follows instead of $H_n^{(m)}$ we use the notation
H_n^m (omitted brackets).

The next proposition (see also [5]) states that H_n^m can be written as some linear forms of
terms of X. This is a consequence of Eq. (2) and the proof, which can be conducted by the
induction method, is straightforward (omitted). The $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the collections
of natural, integer, and real numbers, respectively; $\binom{m}{i}$ $(= \frac{m!}{i!(m-i)!})$ denotes the binomial
coefficients.

Proposition 1 *Let $X = (f_n)_{n=1}^{\infty}$ and $m \geq 1$ be arbitrary. Then the mth-order absolute differ-
ences H_n^m can be represented as*

$$H_n^m = k_{m,n}^{(0)} f_n + k_{m,n}^{(1)} f_{n+1} + \cdots + k_{m,n}^{(m)} f_{n+m}, \tag{4}$$

where the coefficients $k_{m,n}^{(i)} \in \mathbb{Z}$ are constructed recurrently: for $p \geq 1$

$$k_{1,n}^{(0)} = -k_{1,n}^{(1)} = -\delta_n^{(0)}, \qquad k_{p+1,n}^{(i)} = \begin{cases} -\delta_n^{(p)} k_{p,n}^{(i)}, & i = 0, \\ \delta_n^{(p)} (k_{p,n+1}^{(i-1)} - k_{p,n}^{(i)}), & 1 \leq i \leq p, \\ \delta_n^{(p)} k_{p,n+1}^{(i-1)}, & i = p + 1. \end{cases} \tag{5}$$

One can see that $k_{m,n}^{(i)}$ in Eq. (4) depend only on S-components $S_1, \ldots, S_m$ of X. The rule (5) is a generalization of the additive scheme for constructing the Pascal triangle of binomial coefficients. It can be proved (we omit this) that for $m, n \geq 1$

$$\sum_{i=0}^{m} k_{m,n}^{(i)} = 0 \quad \text{and} \quad \left| k_{m,n}^{(i)} \right| \leq \binom{m}{i}. \tag{6}$$

The equality sign here is assumed when $\delta_n^{(m)} = (-1)^m$ for all $n \geq 1$; for this case, (4) gains the form

$$H_n^m = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f_{n+i}$$

which is the so-called δ-transform (of the sequence X) studied in the Hausdorff transforms of divergent series [13]. This particular sign distribution appears also in definition of *completely monotonic functions* (see, *e.g.*, the Bernstein theorem in [14]).

In this paper we are interested in the H-components of bounded sequences (discrete-time signals). To formulate our main theorem, Theorem 1, we present Definition 1 - namely in the context of this definition the (physics-related) term "bistability" in Section 1 is understood. In what follows, for bounded $X = (x_i)_{i=1}^{\infty}$ we consider the *sup*-norm:

$$\|X\| = \sup\{|x_i| : 1 \leq i < \infty\}. \tag{7}$$

A finite subsequence $h = (x_{i+1}, \ldots, x_{i+k})$ of $X = (x_i)_{i=1}^{\infty}$ is called a segment of X of length k, $|h| = k$; for two functions of natural argument $a(m)$ and $b(m)$, differed from 0 we denote $a(m) \sim b(m)$ if $a(m)/b(m) \to 1$ as $m \to \infty$. For a number $x \in \mathbb{R}$ we denote $[x]$ and $\{x\}$ its entire and fractional parts, respectively.

Definition 1 Let the numbers $0 \leq \mu < 1$ and $0 \leq \varepsilon < 1$ be given. A sequence $X = (x_i)_{i=1}^{\infty}$ is called μ-*binary* if $x_i \in \{0, \mu\}$ for every $i \geq 1$; X is called (μ, ε)-*binary* if there exists μ-binary $Z = (z_i)_{i=1}^{\infty}$ such that $\|X - Z\| \leq \varepsilon$. A collection of sequences $(X_m)_{m \geq 1}$ is called μ-binary $((\mu, \varepsilon)$-binary) if X_m is μ-binary $((\mu, \varepsilon)$-binary) for all large enough m; $(X_m)_{m \geq 1}$ is called *asymptotically μ-binary* if for arbitrary $\varepsilon > 0$ it is (μ, ε)-binary.

The $(\mu, 0)$-binary sequences are μ-binary sequences, $(0, \varepsilon)$-binary sequences are the sequences upper bounded by ε. For the case of finite sequences X of length n the above-given definitions are analogous, replacing in Eq. (7) and in Definition 1 the symbol ∞ by $n + 1$ and n, respectively.

The proof of next proposition is straightforward (and is omitted).

Proposition 2 *A collection of sequences $(X_m)_{m \geq 1}$ is asymptotically μ-binary if and only if there exists a μ-binary collection $(Z_m)_{m \geq 1}$ such that $\|X_m - Z_m\| \to 0$ as $m \to \infty$.*

Theorem 1 is the main result of this paper. We define some notions involved in its formulation. For $X = (f_n)_{n=1}^{\infty}$ and $H_m = H_m(X)$ we consider

$$\mu_m = \|H_m\| \quad \text{and} \quad \mu = \lim_{m \to \infty} \mu_m \quad (\mu \geq 0). \tag{8}$$

For periodic X the *supremum* in Eq. (7) can be replaced by *maximum*. We note that Eq. (2) yields the result that for every X we have $\mu_{m+1} \leq \mu_m$ ($m \geq 1$). Due to this monotony of μ_m, for arbitrary X one and only one of next two options (a) and (b) is possible:

$$\text{(a)} \quad \mu = \mu_m \quad \text{for all large enough } m, \qquad \text{(b)} \quad \mu < \mu_m \quad \text{for } m = 1, 2, \ldots. \qquad (9)$$

We define the *peak sets P_m*,

$$P_m = \left\{ n \geq 1 : H_n^m \geq \mu \right\} \qquad (10)$$

(it follows from Eq. (8) that $P_m \neq \emptyset$ for $m \geq 1$), modifying the corresponding notion from theory of Banach algebras (*e.g.*, [14]). Let P_m be numbered in increasing order of its entries,

$$P_m = \left\{ n_m^{(1)}, n_m^{(2)}, \ldots \right\} \quad \text{and denote} \quad d_m = \max\left\{ \left| n_m^{(i+1)} - n_m^{(i)} \right| : i \geq 1 \right\}. \qquad (11)$$

In item (b) of Theorem 1 the following restrictions on P_m are imposed:

$$1 \leq d_m \leq m, \qquad \lim_{m \to \infty} (m - d_m) = \infty. \qquad (12)$$

The d_m characterize the density of P_m in natural series $\mathbb{N}$. The first condition in (12) means that P_m is dense enough in $\mathbb{N}$ (every segment of $\mathbb{N}$ of length m contains a point of P_m); the second relation implies that the set

$$E = \{ m - d_m : m \in \mathbb{N} \}, \quad E \subseteq \mathbb{N} \qquad (13)$$

is infinite. If $n \in E$, then there is some $m \geq 1$ such $n = m - d_m$; we denote such an m as $m = m_n$; hence, one can consider a function of the natural argument,

$$\varphi : \mathbb{N} \to \mathbb{N}, \quad \text{defined as} \quad \varphi(n) = m_n - d_{m_n}. \qquad (14)$$

For example, if $d_m = [m/2]$ (entire part of half of m), then (12) is satisfied and one can assign $\varphi(n) = [n/2]$; another example is

$$d_m = T \ (= \text{const.}), \quad \text{and then} \quad \varphi(n) = n - T. \qquad (15)$$

In these examples, E coincides with $\mathbb{N}$; Eq. (15) holds, *e.g.*, for T-periodic X.

The next theorem establishes the bistability of higher-order absolute differences taken from discrete signals. In addition, it asserts (descriptively) that the denser the peak set P_m is, the denser is the asymptotically binary H_m.

Theorem 1 *Let $X = (f_n)_{n=1}^{\infty}$, $H_m = H_m(X)$, and μ_m and μ be defined by Eq. (8). Then the following statements, provided that a finite collection (whose cardinality is smaller than d_m) of entries of H_m can be excluded, are true:*

(a) For every $m \in \mathbb{N}$ there exists a finite segment $h_m \subset H_m$ such that $|h_m| \sim m$ as $m \to \infty$ and the collection $(h_m)_{m=1}^{\infty}$ is asymptotically μ-binary.

(b) *If the peak sets* (10) *satisfy Eq.* (12) *then the collection* $(H_n)_{n \in E}$, *where E is defined by Eq.* (13), *is asymptotically μ-binary; more precisely, for every $n \in E$ the H_n is (μ, ε_n)-binary with $\varepsilon_n = \mu_{\varphi(n)} - \mu$ where φ is defined by Eq.* (14).

(c) *Let X be periodic with a period $T \geq 1$. Then the collection $(H_m)_{m=1}^{\infty}$ is either μ-binary or asymptotically μ-binary. More precisely, if Eq.* (9a) *holds, $(H_m)_{m=1}^{\infty}$ is μ-binary, and if Eq.* (9b) *holds, $(H_m)_{m=1}^{\infty}$ is asymptotically μ-binary: for every $m > T$ the H_m is (μ, ε_m)-binary with $\varepsilon_m = \mu_{m-T} - \mu$.*

The condition Eq. (12) on peak sets in this theorem is imposed on the difference series H_m, but not immediately on X, which leads to certain difficulties to decide whether for a given X its higher order difference series possess the asymptotic bistability. Theorem 1(c) asserts that this property always holds for arbitrary periodic X.

In the next corollary both options from Theorem 1(c) are presented (below, the real numbers $f_1, \ldots, f_T$ are called H-*independent* if for arbitrary $\lambda_1, \ldots, \lambda_T \in \mathbb{Z}$ for which $\lambda_1 + \cdots + \lambda_T = 0$, the relation $\sum_{i=1}^{T} \lambda_i f_i = 0$ implies that all the λ_i are zero):

Corollary 1 *Let $X = (f_n)_{n=1}^{\infty}$ be periodic with a period $T \geq 1$, $H_m = H_m(X)$, and let μ be defined by Eq.* (8). *Then if $f_1, \ldots, f_T$ are rational numbers then the collection $(H_m)_{m=1}^{\infty}$ is μ-binary, and if $f_1, \ldots, f_T$ are H-independent then $(H_m)_{m=1}^{\infty}$ is asymptotically μ-binary.*

Let us discuss the reason why a bistability, considered in Theorem 1, is emerged. Despite the X (the line H_0 in the matrix H) can be arbitrary, Eq. (2) sets a strong interrelation between the entries of every triplet $\{H_n^{m+1}, H_n^m, H_{n+1}^m\}$ of the matrix H. Namely such intrinsic local restrictions are usually the reason of some special features (in our case this is the bistability) of a given object (in our case this is the matrix H); *e.g.*, the maximum principle for harmonic functions (both classical and discrete [15] versions) is due to their local mean value property. For proving Theorem 1 we exploit the following situation: if the absolute difference ($= H_n^{m+1}$) of two positive quantities (H_n^m and H_{n+1}^m) is close to their upper bound, then one of them should also be close to this bound while another one should be close to zero. Similar situations arise also in Banach algebras and approximation theory, see, *e.g.*, the Bishop-de Leeuw theorem ([14], Ch. 8), Bishop's "$\frac{1}{4} - \frac{3}{4}$"-criterion ([16], Ch. II, Comments) and Mergelyan's earlier work [17] on rational approximations where a binary-valued function $\mathfrak{m}(\zeta)$ is of the main interest.

Before discussing on applications of Theorem 1 we note that a given signal $X = (f_n)_{n=1}^{\infty}$ can be completely restored by its first entries and components S and H. Namely, the next proposition (see also [5]; the proof is by the induction method and is omitted) provides us with analytical expressions for the computation of the original X by given $S_1, \ldots, S_m$, H_m and the first terms $f_1, \ldots, f_m$ of X (or, by first terms of H_1^m denoted now as $H_1^k = [f_1, \ldots, f_k]$, $2 \leq k \leq m$).

Proposition 3 *Let $m \geq 1$ be given, let $H = (h_n)_{n=1}^{\infty}$ be some infinite sequence, $h_n \geq 0$ and $S_k = (s_{k,n})_{n=1}^{\infty}$, $s_{k,n} = \pm 1$, $0 \leq k \leq m$ be some infinite binary sequences. Let $X = (f_n)_{n=1}^{\infty}$ be defined as follows: $f_1, \ldots, f_m$ are arbitrary and for every natural $n \geq 1$*

$$f_n = f_1 + \sum_{k=2}^{m} [f_1, \ldots, f_k] \sum_{i=1}^{n-k+1} B_{n,i,k-2} + \sum_{i=1}^{n-m} h_i B_{n,i,m-1}, \tag{16}$$

where $\sum_{i}^{j} = 0$ if $j < i$ and $B_{n,i,p} \in \mathbb{Z}$ are constructed recurrently: for $p \geq 1$

$$B_{n,i,0} = s_{0,i}, \qquad B_{n,i,p} = s_{p,i} \sum_{j=i+1}^{n-p} B_{n,j,p-1}.$$

Then for all $1 \leq k \leq m$ the relations

$$S_k(X) = S_k \quad and \quad H_m(X) = H \tag{17}$$

hold.

Let us outline some possible applications of the above-presented theory to signal processing. Theorem 1 suggests a method for converting the discrete signals into the binary ones based only on the computation of differences. For processing the signals $X = (f_n)_{n=1}^{\infty}$ for which μ_m converges to μ fast enough, this method can be far more effective than the ones which deal with replacing f_n by their binary codes. Namely, we claim that, when applicable (*e.g.*, if X is periodic), the difference converting method can reduce the data to be transmitted and can increase significantly the transmission speed. One can suggest the following scheme for processing (not only the transmission) the discrete-time signals by the digital systems: by Theorem 1 (and using Proposition 1) a given X is converted into a binary form, which is then processed as a digital signal, and then the obtained (binary) signal is deconverted (provided that a number of S-components and some finite set of the resulting signal entries are given) by Proposition 3.

To illustrate this, we consider the periodic case. Let a signal $X = (f_n)_{n=1}^{\infty}$ be periodic; by using the binary symbols it should be transmitted (it suffices to transmit the period $f_1, \ldots, f_T$). The traditional method, where each of the f_i is replaced by a binary code of some length q, requires qT binary digits (bits) to be transmitted. By the difference method, this transmission task is solved as follows. Let m be a minimal number for which the sequence H_m can be treated as a binary one (the ε_m from Theorem 1 is small enough). According to Proposition 3, to transmit X one should transmit $m(q + T + 2)$ bits ($(m+1)q$ of them to transmit $f_1, \ldots, f_m$ and μ by usual method, mT to transmit $S_1, \ldots, S_m$, and m bits to transmit H_m). If $m < T$, the ratio

$$\frac{m(q + T + 2)}{qT}$$

can be smaller than 1, *i.e.*, the difference method indeed is more effective; *e.g.*, if $f_i = i$ $(1 \leq i \leq T)$, then $m = 2$ and the above-mentioned ratio is indeed small if q and T are large.

3 Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma, where the notion of peak sets defined by Eq. (10) is essentially used.

Lemma 1 *Let $X = (f_n)_{n=1}^{\infty}$, $H_m = H_m(X)$, μ_m and μ be defined by Eq. (8), and for some $m, n \geq$ 1, the inequality*

$$H_n^m \geq \mu \tag{18}$$

holds. Then for every $0 \le k \le m$ and every $0 \le i \le k$ one of the two relations (19) *or* (20),

$$\mu \le H_{n+i}^{m-k} \le \mu_{m-k} \quad and \quad 0 \le H_{n+i+1}^{m-k} \le \mu_{m-k} - \mu, \tag{19}$$

$$\mu \le H_{n+i+1}^{m-k} \le \mu_{m-k} \quad and \quad 0 \le H_{n+i}^{m-k} \le \mu_{m-k} - \mu, \tag{20}$$

holds; that means that for every $1 \le k \le m$ the finite sequence

$$H_n^{m-k}, H_{n+1}^{m-k}, \ldots, H_{n+k}^{m-k} \tag{21}$$

is (μ, ε)-binary with $\varepsilon = \mu_{m-k} - \mu$.

Proof The proof is conducted by the induction method with respect to the variable k, $1 \le k \le m - 1$. Let us first prove the lemma for the case $k = 1$. From definition of μ_m and μ, we have

$$\mu \le H_n^m \le \mu_m, \quad \text{that is, by Eq. (2),} \quad \mu \le \left| H_n^{m-1} - H_{n+1}^{m-1} \right| \le \mu_m. \tag{22}$$

Let us assume that $H_n^{m-1} \ge H_{n+1}^{m-1}$ (for the contrary case $H_n^{m-1} \le H_{n+1}^{m-1}$ the proof is analogous); that is, the second relation in Eq. (22) can be written as

$$\mu \le H_n^{m-1} - H_{n+1}^{m-1} \le \mu_m. \tag{23}$$

From the left-hand side of (23) we obtain $\mu \le H_n^{m-1}$, and from the definition in Eq. (8) it follows that $H_n^{m-1} \le \mu_{m-1}$, that is, $\mu \le H_n^{m-1} \le \mu_{m-1}$. Furthermore, from Eq. (23) one obtains

$$H_{n+1}^{m-1} \le H_n^{m-1} - \mu \le \mu_{m-1} - \mu$$

and, hence, Lemma 1 for the case $k = 1$ is proved.

To prove the lemma for arbitrary $2 \le k \le m$, we apply the induction method: we assume that one of the relations (19) or (20) for some $1 \le k \le m - 2$ holds, and we prove that it holds also when k is substituted by $k + 1$. Let us assume that Eq. (19) holds (for the case of Eq. (20) the proof is analogous). By Eq. (2)

$$H_{n+i}^{m-k} = \left| H_{n+i}^{m-(k+1)} - H_{n+i+1}^{m-(k+1)} \right|$$

and let us assume that the difference here is positive (if it is negative, the proof is the same),

$$H_{n+i}^{m-k} = H_{n+i}^{m-(k+1)} - H_{n+i+1}^{m-(k+1)}. \tag{24}$$

From (19) we have $H_{n+i}^{m-(k+1)} \ge \mu$. We also have

$$H_{n+i}^{m-(k+1)} \le \mu_{m-(k+1)} = \mu + (\mu_{m-(k+1)} - \mu)$$

and, hence, the first relation in Eq. (19), where k is substituted by $k + 1$, is proved. Let us now consider the second relation in (19). Eqs. (24) and (19) give

$$H_{n+i+1}^{m-(k+1)} = H_{n+i}^{m-(k+1)} - H_{n+i}^{m-k} \leq \mu_{m-(k+1)} - \mu = \varepsilon_{m-(k+1)}$$

and Eqs. (19) and (20), where k is substituted by $k + 1$, are proved. Lemma 1 is proved.

$\square$

Proof of Theorem 1 (a) Let $X = (f_n)_{n=1}^{\infty}$ be given, $H_m = H_m(X)$, and μ_m and μ be defined by Eq. (8). It follows from definition of μ that for every $m \geq 1$ there is some $n = n_m$ such that $H_n^m \geq \mu$ (*cf.* Eq. (18)). Then by Lemma 1 for every $1 \leq k \leq m$ the sequence

$$H_n^{m-k}, H_{n+1}^{m-k}, \ldots, H_{n+k}^{m-k}$$

is $(\mu, \mu_{m-k} - \mu)$-binary. Let $k(m)$ be such that $k(m) < m$, $(m - k(m)) \to \infty$, and $k(m) \sim m$ as $m \to \infty$. Then by Lemma 1 the finite sequence

$$h_m = \left(H_{n_m}^{m-k(m)}, H_{n_m+1}^{m-k(m)}, \ldots, H_{n_m+k(m)}^{m-k(m)}\right) \tag{25}$$

with length $|h_m| = k(m) + 1$ is (μ, ε)-binary with $\varepsilon = \mu_{m-k(m)} - \mu$. Since $\varepsilon \to 0$ and $|h_m| \to \infty$ as $m \to \infty$, item (a) of the theorem is proved.

(b) As in Eq. (25), with every $n_m^{(i)} \in P_m$ (see Eq. (11)) we associate a sequence $h_m^{(i)}$. Since $d_m < m$ and $(m - d_m) \to \infty$, one can assign in Eq. (25) $k(m) = d_m$ what means that

$$\bigcup_{i=1}^{\infty} h_m^{(i)} = H_{m-d_m}.$$

Since every ith $h_m^{(i)}$ is (μ, ε)-binary with $\varepsilon = \mu_{m-d_m} - \mu$, their union H_{m-d_m} is also (μ, ε)-binary with the same ε. Since $\varepsilon \to 0$ as $m \to \infty$, item (b) of the theorem is proved.

(c) The proof can be deduced from the previous item (b) and the example from Eq. (15). We present a more detailed proof. If Eq. (9a) holds, the proof of the theorem follows *ab absurdo*, if one takes into account Eq. (2) and the periodicity of every H_m. Let us consider the case when Eq. (9b) holds. Since every H_m is T-periodic, to prove that H_m is (μ, ε)-binary it suffices to prove that for some n the finite sequence

$$H_n^m, H_{n+1}^m, \ldots, H_{n+T}^m \tag{26}$$

is (μ, ε)-binary. This statement follows from Lemma 1 if one considers the sequence H_{m+T} and chooses some $n \geq 1$ such that $H_n^{m+T} \geq \mu$ (*cf.* Eq. (18)): indeed, then Lemma 1 yields the result that the finite sequence (26) is (μ, ε)-binary with $\varepsilon = \mu_{m-T} - \mu$. Then due to the T-periodicity of H_m we see that H_m is also (μ, ε)-binary. Item (c) of the theorem is proved. Theorem 1 is proved.

$\square$

Proof of Corollary 1 To prove the first point of Corollary 1, we note that since X is periodic and the f_i are rational, by considering the common denominator for $f_1, \ldots, f_T$ one can suppose that X is a (periodic) sequence of natural numbers; then it follows that Eq. (9a)

for such X holds, and hence $(H_m)_{m \geq 1}$ is μ-binary. The second point of Corollary 1 follows from Eq. (6). Indeed, due to the assumption on H-independence none of the H_n^m can take the value 0, and hence by Theorem 1(c) $(H_m)_{m \geq 1}$ cannot be μ-binary and then Eq. (9a) is impossible; then the alternative option Eq. (9b) mentioned in Theorem 1(c) asserts that $(H_m)_{m \geq 1}$ is asymptotically μ-binary. Corollary 1 is proved. □

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