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Some new results for the (p,q)-Fibonacci and Lucas polynomials

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Abstract

In this paper, we investigate some arithmetic properties for the (p,q)-Fibonacci and Lucas polynomials associated with the classical Fibonacci and Lucas numbers. By applying some elementary methods and techniques, we establish some combinatorial identities for the (p,q)-Fibonacci and Lucas polynomials. It turns out that some known results in the references are obtained as special cases.

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1 Introduction

The polynomials defined recursively over the integers, such as the Dickson polynomials, Chebychev polynomials, Fibonacci polynomials and Lucas polynomials, have been extensively studied. Most of these polynomials share numerous interesting properties. They have been also found to be topics of interest in many different areas of pure and applied science; see, for example, [1–9].

Like the definitions of Dickson polynomials and Chebychev polynomials, the classical Fibonacci polynomials $F_n(x)$ studied by E.C. Catalan in 1883 are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \quad (n \geq 3), \tag{1.1}$$

with the initial condition $F_1(x) = 1$ and $F_2(x) = x$, and the classical Lucas polynomials $L_n(x)$ studied by M. Bicknell in 1970 are defined by

$$L_n(x) = xL_{n-1}(x) + L_{n-2}(x) \quad (n \geq 2), \tag{1.2}$$

with the initial condition $L_0(x) = 2$ and $L_1(x) = x$.

Since the above classical Fibonacci and Lucas polynomials appeared, some authors have explored their different extensions. For example, let $h(x)$ be a sequence of polynomials with real coefficients, Nalli and Haukkanen [10] introduced the $h(x)$-Fibonacci and Lucas polynomials with the $h(x)$-Fibonacci polynomials $F_{h,n}(x)$ defined by $F_{h,0}(x) = 0$, $F_{h,1}(x) = 1$ and

$$F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x) \quad (n \geq 1), \tag{1.3}$$

and the $h(x)$-Lucas polynomials $L_{h,n}(x)$ defined by

$$L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x) \quad (n \geq 1), \tag{1.4}$$

with the initial condition $L_{h,0}(x) = 2$ and $L_{h,1}(x) = h(x)$. More recently, Lee and Asci [11] considered the generalization of the $h(x)$-Fibonacci and Lucas polynomials, *i.e.*, the (p,q)-Fibonacci and Lucas polynomials, and derived some interesting combinatorial identities for these polynomials. And they also obtained the factorization of Pascal matrix involving the (p,q)-Fibonacci polynomials by applying the famous Riordan method.

The idea of the present paper stems from the work of He and Zhang [12] and Lee and Asci [11]. We perform a further investigation for the (p,q)-Fibonacci and Lucas polynomials and establish some more general combinatorial identities for these polynomials by applying some elementary methods and techniques. It turns out that some corresponding results including the ones described in [11, 12] are derived as special cases.

2 Preliminaries and known results

We begin here by recalling the (p,q)-Fibonacci and Lucas polynomials with some properties introduced by Lee and Asci [11] as follows.

Definition 2.1 Let $p(x)$ and $q(x)$ be two sequences of polynomials with real coefficients. The (p,q)-Fibonacci polynomials $F_{p,q,n}(x)$ are defined by

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x) \quad (n \geq 1), \tag{2.1}$$

with the initial conditions $F_{p,q,0}(x) = 0$ and $F_{p,q,1}(x) = 1$.

Definition 2.2 Let $p(x)$ and $q(x)$ be two sequences of polynomials with real coefficients. The (p,q)-Lucas polynomials $L_{p,q,n}(x)$ are defined by

$$L_{p,q,n+1}(x) = p(x)L_{p,q,n}(x) + q(x)L_{p,q,n-1}(x) \quad (n \geq 1), \tag{2.2}$$

with the initial conditions $L_{p,q,0}(x) = 2$ and $L_{p,q,1}(x) = p(x)$.

Obviously, the case $q(x) = 1$ in Definitions 2.1 and 2.2 respectively leads to the $h(x)$-Fibonacci polynomials and the $h(x)$-Lucas polynomials due to Nalli and Haukkanen [10]. In fact, the (p,q)-Fibonacci and Lucas polynomials can also be defined by means of the following generating functions (see, *e.g.*, [11]):

$$\frac{t}{1 - p(x)t - q(x)t^2} = \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n \tag{2.3}$$

and

$$\frac{2 - p(x)t}{1 - p(x)t - q(x)t^2} = \sum_{n=0}^{\infty} L_{p,q,n}(x)t^n. \tag{2.4}$$

Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation $t^2 - p(x)t - q(x) = 0$, then

$$\alpha(x) = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2}, \qquad \beta(x) = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}, \tag{2.5}$$

and

$$\alpha(x) + \beta(x) = p(x), \qquad \alpha(x)\beta(x) = -q(x), \qquad \alpha(x) - \beta(x) = \sqrt{D(x)}, \tag{2.6}$$

where $D(x) = p^2(x) + 4q(x)$ is called the discriminant of the equation. Based on the above, Lee and Asci [11] discovered the following results for the (p,q)-Fibonacci and Lucas polynomials.

Property 2.3 Let n be any non-negative integer. Then

$$F_{p,q,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \qquad L_{p,q,n}(x) = \alpha^n(x) + \beta^n(x). \tag{2.7}$$

Property 2.4 Let n be any positive integer. Then

$$t^n = F_{p,q,n}(x)t + q(x)F_{p,q,n-1}(x). \tag{2.8}$$

Property 2.5 Let m, n be any non-negative integers. Then

$$F_{p,q,m+n+1}(x) = F_{p,q,m+1}(x)F_{p,q,n+1}(x) + q(x)F_{p,q,m}(x)F_{p,q,n}(x). \tag{2.9}$$

It is worthy of mentioning that Property 2.3 is called the Binet formula for the (p,q)-Fibonacci and Lucas polynomials, and one can discover from Property 2.3 that, like the classical Fibonacci and Lucas polynomials, the (p,q)-Fibonacci and Lucas polynomials can be defined for negative indices by

$$F_{p,q,-n}(x) = -\bigl(-q(x)\bigr)^{-n}F_{p,q,n}(x), \qquad L_{p,q,-n}(x) = \bigl(-q(x)\bigr)^{-n}L_{p,q,n}(x). \tag{2.10}$$

The above properties will play important roles in establishing some combinatorial identities involving the (p,q)-Fibonacci and Lucas polynomials in the next two sections.

3 Several expressions of the (p,q)-Fibonacci and Lucas polynomials

In this section, we shall make use of the results stated in the second section to obtain several interesting expressions for the (p,q)-Fibonacci and Lucas polynomials in two- and three-variable cases, some of which can be regarded as the generalization of the properties showed in the second section. We firstly give the following formulae for the (p,q)-Fibonacci and Lucas polynomials in a two-variable case.

Theorem 3.1 *Let m, n be any non-negative integers. Then*

$$\begin{aligned} &F_{p,q,m+n}(x) \\ &\quad = D^{\frac{1}{2}}(x)F_{p,q,m}(x)F_{p,q,n}(x) + \beta^m(x)F_{p,q,n}(x) + \beta^n(x)F_{p,q,m}(x) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &L_{p,q,m+n}(x) \\ &\quad = L_{p,q,m}(x)L_{p,q,n}(x) - \beta^m(x)L_{p,q,n}(x) - \beta^n(x)L_{p,q,m}(x) + 2\beta^{m+n}(x). \end{aligned} \tag{3.2}$$

Proof It is easy to see that for any non-negative integers m, n,

$$\begin{aligned} &\{\alpha^m(x) - \beta^m(x)\}\{\alpha^n(x) - \beta^n(x)\} \\ &\quad = \alpha^{m+n}(x) - \alpha^m(x)\beta^n(x) - \alpha^n(x)\beta^m(x) + \beta^{m+n}(x) \\ &\quad = \alpha^{m+n}(x) - \beta^{m+n}(x) - \alpha^m(x)\beta^n(x) + \beta^{m+n}(x) - \alpha^n(x)\beta^m(x) + \beta^{m+n}(x) \\ &\quad = \alpha^{m+n}(x) - \beta^{m+n}(x) - \beta^n(x)\{\alpha^m(x) - \beta^m(x)\} - \beta^m(x)\{\alpha^n(x) - \beta^n(x)\}. \end{aligned} \tag{3.3}$$

Hence, applying the first formula of Property 2.3 to (3.3), we obtain

$$F_{p,q,m}(x)F_{p,q,n}(x) = \frac{F_{p,q,m+n}(x) - \beta^n(x)F_{p,q,m}(x) - \beta^m(x)F_{p,q,n}(x)}{\alpha(x) - \beta(x)}. \tag{3.4}$$

It follows from (2.6) and (3.4) that the formula (3.1) is complete. Similarly, we get

$$\begin{aligned} &L_{p,q,m}(x)L_{p,q,n}(x) \\ &\quad = \{\alpha^m(x) + \beta^m(x)\}\{\alpha^n(x) + \beta^n(x)\} \\ &\quad = \alpha^{m+n}(x) + \alpha^m(x)\beta^n(x) + \alpha^n(x)\beta^m(x) + \beta^{m+n}(x) \\ &\quad = \alpha^{m+n}(x) + \beta^{m+n}(x) + \alpha^m(x)\beta^n(x) + \beta^{m+n}(x) + \alpha^n(x)\beta^m(x) + \beta^{m+n}(x) - 2\beta^{m+n}(x) \\ &\quad = L_{p,q,m+n}(x) + \beta^n(x)L_{p,q,m}(x) + \beta^m(x)L_{p,q,n}(x) - 2\beta^{m+n}(x). \end{aligned} \tag{3.5}$$

This completes the proof of Theorem 3.1. □

From the above proof, one can also get that for any non-negative integers m, n,

$$\begin{aligned} &F_{p,q,m+n}(x) \\ &\quad = -D^{\frac{1}{2}}(x)F_{p,q,m}(x)F_{p,q,n}(x) + \alpha^m(x)F_{p,q,n}(x) + \alpha^n(x)F_{p,q,m}(x) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &L_{p,q,m+n}(x) \\ &\quad = L_{p,q,m}(x)L_{p,q,n}(x) - \alpha^m(x)L_{p,q,n}(x) - \alpha^n(x)L_{p,q,m}(x) + 2\alpha^{m+n}(x). \end{aligned} \tag{3.7}$$

Since $t^{m+n} = t^m \cdot t^n$, so by Property 2.4 one can easily derive the following result.

Theorem 3.2 *Let m, n be any positive integers. Then*

$$\begin{aligned} t^{n+m} &= F_{p,q,n}(x)F_{p,q,m}(x)t^2 + q(x)\{F_{p,q,n}(x)F_{p,q,m-1}(x) + F_{p,q,n-1}(x)F_{p,q,m}(x)\}t \\ &\quad + q^2(x)F_{p,q,n-1}(x)F_{p,q,m-1}(x). \end{aligned} \tag{3.8}$$

It is easy to see that the case $m = 0$ in Theorem 3.2 immediately gives Property 2.4. Based on the interest for Property 2.5, we have the following formula for the (p,q)-Lucas polynomials in the two-variable case.

Theorem 3.3 *Let $n \geq 2$ be any positive integer. Then, for any non-negative integer m,*

$$L_{p,q,n+m}(x) = F_{p,q,m+2}(x)L_{p,q,n-1}(x) + q(x)F_{p,q,m+1}(x)L_{p,q,n-2}(x). \tag{3.9}$$

Proof We shall prove the theorem by induction on m. It is obvious that (2.2) implies the case $m = 0$ in Theorem 3.3. Now, assume that Theorem 3.3 holds for any non-negative integer m. Then

$$L_{p,q,n+m}(x) = F_{p,q,m+2}(x)L_{p,q,n-1}(x) + q(x)F_{p,q,m+1}(x)L_{p,q,n-2}(x). \tag{3.10}$$

So, from (2.1) and (3.10), we get

$$\begin{aligned} L_{p,q,n+m+1}(x) &= p(x)L_{p,q,n+m}(x) + q(x)L_{p,q,n+m-1}(x) \\ &= p(x)\{F_{p,q,m+2}(x)L_{p,q,n-1}(x) + q(x)F_{p,q,m+1}(x)L_{p,q,n-2}(x)\} \\ &\quad + q(x)\{F_{p,q,m+1}(x)L_{p,q,n-1}(x) + q(x)F_{p,q,m}(x)L_{p,q,n-2}(x)\} \\ &= p(x)F_{p,q,m+2}(x)L_{p,q,n-1}(x) + p(x)q(x)F_{p,q,m+1}(x)L_{p,q,n-2}(x) \\ &\quad + q(x)F_{p,q,m+1}L_{p,q,n-1}(x) + q^2(x)F_{p,q,m}(x)L_{p,q,n-2}(x) \\ &= \{p(x)F_{p,q,m+2} + q(x)F_{p,q,m+1}(x)\}L_{p,q,n-1}(x) \\ &\quad + q(x)\{p(x)F_{p,q,m+1}(x) + q(x)F_{p,q,m}\}L_{p,q,n-2}(x) \\ &= F_{p,q,m+3}(x)L_{p,q,n-1}(x) + q(x)F_{p,q,m+2}(x)L_{p,q,n-2}(x). \end{aligned} \tag{3.11}$$

Thus, we conclude the induction step and give the proof of Theorem 3.3. □

We next use Property 2.5 and Theorem 3.3 to give the formulae for the (p,q)-Fibonacci and Lucas polynomials in a three-variable case. We have the following.

Theorem 3.4 *Let k, m, n be any non-negative integers with $k \geq 2$. Then*

$$\begin{aligned} F_{p,q,n+m+k}(x) &= F_{p,q,k}(x)F_{p,q,m+1}(x)F_{p,q,n+1}(x) + q(x)F_{p,q,k-1}(x) \\ &\quad \times \{F_{p,q,m}(x)F_{p,q,n+1}(x) + F_{p,q,m+1}(x)F_{p,q,n}(x)\} \\ &\quad + q^2(x)F_{p,q,k-2}(x)F_{p,q,m}(x)F_{p,q,n}(x) \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} L_{p,q,n+m+k}(x) &= L_{p,q,k}(x)F_{p,q,m+1}(x)F_{p,q,n+1}(x) + q(x)L_{p,q,k-1}(x) \\ &\quad \times \{F_{p,q,m}(x)F_{p,q,n+1}(x) + F_{p,q,m+1}(x)F_{p,q,n}(x)\} \\ &\quad + q^2(x)L_{p,q,k-2}(x)F_{p,q,m}(x)F_{p,q,n}(x). \end{aligned} \tag{3.13}$$

Proof The detailed proof is left as an exercise for the interested readers. □

4 Sums of powers for the (p,q)-Fibonacci and Lucas polynomials

In [11], Lee and Asci applied Property 2.3 to extend a result of Koshy [3] to the (p,q)-Fibonacci and Lucas polynomials and give two sum relations for the (p,q)-Fibonacci and Lucas polynomials. In this section, we shall extend the two sum relations due to Lee and Asci and give the close formulae of sums of any positive powers for the (p,q)-Fibonacci and Lucas polynomials. In order to do that, we first present the following formulae for the (p,q)-Fibonacci and Lucas polynomials which can be also regarded as the generalization of the results of He and Zhang [12] on the Lucas sequences of the first kind and second kind.

Theorem 4.1 *Let n be a positive integer and k be a non-negative integer. Suppose that the discriminant $D(x) \neq 0$. Then*

$$\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{ik} L_{p,q,(n-2i)k}(x) = 2L_{p,q,k}^{n}(x), \tag{4.1}$$

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{ik} F_{p,q,(n-2i)k}(x) = 2D^{\frac{n-1}{2}}(x) F_{p,q,k}^{n}(x) \quad (2 \nmid n) \tag{4.2}$$

and

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{ik} L_{p,q,(n-2i)k}(x) = 2D^{\frac{n}{2}}(x) F_{p,q,k}^{n}(x) \quad (2 \mid n). \tag{4.3}$$

Proof We firstly prove (4.1). It is easy to see from Property 2.3 that

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{ik} L_{p,q,(n-2i)k}(x) \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} \bigl(\alpha(x)\beta(x)\bigr)^{ik} \bigl\{\alpha^{(n-2i)k}(x) + \beta^{(n-2i)k}(x)\bigr\} \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} \alpha^{(n-i)k}(x)\beta^{ik}(x) + \sum_{i=0}^{n} \binom{n}{i} \alpha^{ik}(x)\beta^{(n-i)k}(x) \\
&\quad = \bigl\{\alpha^k(x) + \beta^k(x)\bigr\}^n + \bigl\{\alpha^k(x) + \beta^k(x)\bigr\}^n \\
&\quad = 2L_{p,q,k}^{n}(x).
\end{aligned} \tag{4.4}$$

Hence, the formula (4.1) is complete. On the other hand, we discover

$$\begin{aligned}
&\bigl\{\alpha(x) - \beta(x)\bigr\} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{ik} F_{p,q,(n-2i)k}(x) \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(\alpha(x)\beta(x)\bigr)^{ik} \bigl\{\alpha^{(n-2i)k}(x) - \beta^{(n-2i)k}(x)\bigr\} \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{(n-i)k}(x)\beta^{ik}(x) - \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{ik}(x)\beta^{(n-i)k}(x) \\
&\quad = \bigl\{\alpha^k(x) - \beta^k(x)\bigr\}^n - \bigl\{\beta^k(x) - \alpha^k(x)\bigr\}^n.
\end{aligned} \tag{4.5}$$

So if $2 \nmid n$ then by (4.5) we have

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{ik}F_{p,q,(n-2i)k}(x) &= \frac{2\{\alpha^k(x)-\beta^k(x)\}^n}{\alpha(x)-\beta(x)}\\ &= \frac{2\{\alpha(x)-\beta(x)\}^n}{\alpha(x)-\beta(x)}\left(\frac{\alpha^k(x)-\beta^k(x)}{\alpha(x)-\beta(x)}\right)^n\\ &= 2D^{\frac{n-1}{2}}(x)F^n_{p,q,k}(x),\end{aligned}\tag{4.6}$$

which gives the formula (4.2). In the same way,

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{ik}L_{p,q,(n-2i)k}(x)\\ &\quad=\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(\alpha(x)\beta(x)\bigr)^{ik}\bigl\{\alpha^{(n-2i)k}(x)+\beta^{(n-2i)k}(x)\bigr\}\\ &\quad=\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{(n-i)k}(x)\beta^{ik}(x)+\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{ik}(x)\beta^{(n-i)k}(x)\\ &\quad=\bigl\{\alpha^k(x)-\beta^k(x)\bigr\}^n+\bigl\{\beta^k(x)-\alpha^k(x)\bigr\}^n.\end{aligned}\tag{4.7}$$

It follows from (4.7) that if $2 \mid n$ then

$$\begin{aligned}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{ik}L_{p,q,(n-2i)k}(x) &= 2\bigl\{\alpha^k(x)-\beta^k(x)\bigr\}^n\\ &= 2\bigl\{\alpha(x)-\beta(x)\bigr\}^n\left(\frac{\alpha^k(x)-\beta^k(x)}{\alpha(x)-\beta(x)}\right)^n\\ &= 2D^{\frac{n}{2}}(x)F^n_{p,q,k}(x).\end{aligned}\tag{4.8}$$

Thus, we complete the proof of Theorem 4.1. □

Remark 4.2 Using the above methods, one can also derive that for any positive integer n,

$$\sum_{i=0}^{n}\binom{n}{i}\bigl(-q(x)\bigr)^{ik}F_{p,q,(n-2i)k}(x)=0,\tag{4.9}$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{ik}F_{p,q,(n-2i)k}(x)=0\quad(2\mid n)\tag{4.10}$$

and

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{ik}L_{p,q,(n-2i)k}(x)=0\quad(2\nmid n).\tag{4.11}$$

In fact, there exist some similar formulae to Theorem 4.1. We have the following theorem.

Theorem 4.3 *Let n be a positive integer and k be a non-negative integer. Suppose that the discriminant $D(x) \neq 0$. Then*

$$\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}(x) L_{p,q,ik}(x) = F_{p,q,nk}(x) L_{p,q,k}^{n}(x), \tag{4.12}$$

$$\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} L_{p,q,ik}^{2}(x) = L_{p,q,nk}(x) L_{p,q,k}^{n}(x) + 2^{n+1} \bigl(-q(x)\bigr)^{nk} \tag{4.13}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}^{2}(x) \\
&\quad = D^{-1}(x) \bigl\{ L_{p,q,nk}(x) L_{p,q,k}^{n}(x) - 2^{n+1} \bigl(-q(x)\bigr)^{nk} \bigr\}.
\end{aligned} \tag{4.14}$$

Proof By Property 2.3, we obtain

$$\begin{aligned}
&\bigl\{\alpha(x) - \beta(x)\bigr\} \sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}(x) L_{p,q,ik}(x) \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} \bigl(\alpha(x)\beta(x)\bigr)^{(n-i)k} \bigl\{\alpha^{2ik}(x) - \beta^{2ik}(x)\bigr\} \\
&\quad = \alpha^{nk}(x) \sum_{i=0}^{n} \binom{n}{i} \alpha^{ik}(x) \beta^{(n-i)k}(x) - \beta^{nk}(x) \sum_{i=0}^{n} \binom{n}{i} \alpha(x)^{(n-i)k} \beta^{ik}(x) \\
&\quad = \bigl\{\alpha^{nk}(x) - \beta^{nk}(x)\bigr\} \sum_{i=0}^{n} \binom{n}{i} \alpha^{ik}(x) \beta^{(n-i)k}(x) \\
&\quad = \bigl\{\alpha^{nk}(x) - \beta^{nk}(x)\bigr\} \bigl\{\alpha^{k}(x) + \beta^{k}(x)\bigr\}^{n}.
\end{aligned} \tag{4.15}$$

It follows from (4.15) that

$$\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}(x) L_{p,q,ik}(x) = F_{p,q,nk}(x) L_{p,q,k}^{n}(x), \tag{4.16}$$

as desired. In the same way, from Property 2.3, we get

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} \bigl(-q(x)\bigr)^{(n-i)k} L_{p,q,ik}^{2}(x) \\
&\quad = \sum_{i=0}^{n} \binom{n}{i} \bigl(\alpha(x)\beta(x)\bigr)^{(n-i)k} \bigl\{\alpha^{ik}(x) + \beta^{ik}(x)\bigr\}^{2} \\
&\quad = \bigl\{\alpha^{nk}(x) + \beta^{nk}(x)\bigr\} \sum_{i=0}^{n} \binom{n}{i} \alpha^{ik}(x) \beta^{(n-i)k}(x) + 2^{n+1} \bigl(-q(x)\bigr)^{n} \\
&\quad = L_{p,q,nk}(x) L_{p,q,k}^{n}(x) + 2^{n+1} \bigl(-q(x)\bigr)^{n}
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\{\alpha(x)-\beta(x)\}^2\sum_{i=0}^{n}\binom{n}{i}\bigl(-q(x)\bigr)^{(n-i)k}F_{p,q,ik}^2(x)\\
&=\sum_{i=0}^{n}\binom{n}{i}\bigl(\alpha(x)\beta(x)\bigr)^{(n-i)k}\{\alpha^{ik}(x)-\beta^{ik}(x)\}^2\\
&=\alpha^{nk}(x)\{\alpha^k(x)+\beta^k(x)\}^n-2^{n+1}\bigl(-q(x)\bigr)^{nk}+\beta^{nk}(x)\{\alpha^k(x)+\beta^k(x)\}^n\\
&=\{\alpha^{nk}(x)+\beta^{nk}(x)\}\{\alpha^k(x)+\beta^k(x)\}^n-2^{n+1}\bigl(-q(x)\bigr)^{nk}\\
&=L_{p,q,nk}(x)L_{p,q,k}^n(x)-2^{n+1}\bigl(-q(x)\bigr)^{nk}.
\end{aligned}\tag{4.18}$$

This completes the proof of Theorem 4.3. □

Theorem 4.4 *Let n be a positive integer and k be a non-negative integer. Suppose that the discriminant* $D(x)\neq 0$. *Then*

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{(n-i)k}F_{p,q,ik}(x)L_{p,q,ik}(x)\\
&=\begin{cases}D^{\frac{n}{2}}(x)F_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\mid n,\\ -D^{\frac{n-1}{2}}(x)L_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\nmid n,\end{cases}
\end{aligned}\tag{4.19}$$

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{(n-i)k}L_{p,q,ik}^2(x)\\
&=\begin{cases}D^{\frac{n}{2}}(x)L_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\mid n,\\ -D^{\frac{n+1}{2}}(x)F_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\nmid n,\end{cases}
\end{aligned}\tag{4.20}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{(n-i)k}F_{p,q,ik}^2(x)\\
&=\begin{cases}D^{\frac{n-2}{2}}(x)L_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\mid n,\\ -D^{\frac{n-1}{2}}(x)F_{p,q,nk}(x)F_{p,q,k}^n(x), & 2\nmid n.\end{cases}
\end{aligned}\tag{4.21}$$

Proof By Property 2.3 we have

$$\begin{aligned}
&\{\alpha(x)-\beta(x)\}\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{(n-i)k}F_{p,q,ik}(x)L_{p,q,ik}(x)\\
&=\sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(\alpha(x)\beta(x)\bigr)^{(n-i)k}\{\alpha^{2ik}(x)-\beta^{2ik}(x)\}\\
&=\alpha^{nk}(x)\sum_{i=0}^{n}\binom{n}{i}(-1)^i\alpha^{ik}(x)\beta^{(n-i)k}(x)
\end{aligned}$$

$$
\begin{aligned}
&- \beta^{nk}(x) \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{(n-i)k}(x) \beta^{ik}(x) \\
&= \alpha^{nk}(x) \{\beta^k(x) - \alpha^k(x)\}^n - \beta^{nk}(x) \{\alpha^k(x) - \beta^k(x)\}^n. \qquad (4.22)
\end{aligned}
$$

It follows from (4.22) that if $2 \mid n$ then

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(-q(x)\big)^{(n-i)k} F_{p,q,ik}(x) L_{p,q,ik}(x) \\
&= \frac{1}{\alpha(x) - \beta(x)} \{\alpha^k(x) - \beta^k(x)\}^n \{\alpha^{nk}(x) - \beta^{nk}(x)\} \\
&= \frac{\{\alpha(x) - \beta(x)\}^{n+1}}{\alpha(x) - \beta(x)} \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)} \right)^n \frac{\alpha^{nk}(x) - \beta^{nk}(x)}{\alpha(x) - \beta(x)} \\
&= D^{\frac{n}{2}}(x) F_{p,q,nk}(x) F_{p,q,k}^n(x), \qquad (4.23)
\end{aligned}
$$

and if $2 \nmid n$ then

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(-q(x)\big)^{(n-i)k} F_{p,q,ik}(x) L_{p,q,ik}(x) \\
&= \frac{-1}{\alpha(x) - \beta(x)} \{\alpha^k(x) - \beta^k(x)\}^n \{\alpha^{nk}(x) + \beta^{nk}(x)\} \\
&= -\frac{\{\alpha(x) - \beta(x)\}^{n}}{\alpha(x) - \beta(x)} \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)} \right)^n \{\alpha^{nk}(x) + \beta^{nk}(x)\} \\
&= -D^{\frac{n-1}{2}}(x) L_{p,q,nk}(x) F_{p,q,k}^n(x). \qquad (4.24)
\end{aligned}
$$

In the same way, we get

$$
\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(-q(x)\big)^{(n-i)k} L_{p,q,ik}^2(x) \\
&= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(\alpha(x)\beta(x)\big)^{(n-i)k} \{\alpha^{ik}(x) + \beta^{ik}(x)\}^2 \\
&= \alpha^{nk}(x) \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{ik}(x) \beta^{(n-i)k}(x) + \beta^{nk}(x) \sum_{i=0}^{n} \binom{n}{i} (-1)^i \alpha^{(n-i)k}(x) \beta^{ik}(x) \\
&= \alpha^{nk} \{\beta^k(x) - \alpha^k(x)\}^n + \beta^{nk} \{\alpha^k(x) - \beta^k(x)\}^n \qquad (4.25)
\end{aligned}
$$

and

$$
\begin{aligned}
&\{\alpha(x) - \beta(x)\}^2 \sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(-q(x)\big)^{(n-i)k} F_{p,q,ik}^2(x) \\
&= \sum_{i=0}^{n} \binom{n}{i} (-1)^i \big(\alpha(x)\beta(x)\big)^{(n-i)k} \{\alpha^{ik}(x) - \beta^{ik}(x)\}^2 \\
&= \alpha^{nk}(x) \{\beta^k(x) - \alpha^k(x)\}^n + \beta^{nk}(x) \{\alpha^k(x) - \beta^k(x)\}^n. \qquad (4.26)
\end{aligned}
$$

It follows from (4.25) and (4.26) that if $2 \mid n$ then

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{(n-i)k} L_{p,q,ik}^2(x) \\
&\quad = \bigl\{\alpha(x) + \beta(x)\bigr\}^n \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)}\right)^n \bigl\{\alpha^{nk}(x) + \beta^{nk}(x)\bigr\} \\
&\quad = D^{\frac{n}{2}}(x) L_{p,q,nk}(x) F_{p,q,k}^n(x)
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}^2(x) \\
&\quad = \bigl\{\alpha(x) - \beta(x)\bigr\}^{n-2} \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)}\right)^n \bigl\{\alpha^{nk}(x) + \beta^{nk}(x)\bigr\} \\
&\quad = D^{\frac{n-2}{2}}(x) L_{p,q,nk}(x) F_{p,q,k}^n(x),
\end{aligned} \tag{4.28}$$

and if $2 \nmid n$ then

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{(n-i)k} L_{p,q,ik}^2(x) \\
&\quad = -\bigl\{\alpha(x) - \beta(x)\bigr\}^{n+1} \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)}\right)^n \frac{\alpha^{nk}(x) - \beta^{nk}(x)}{\alpha(x) - \beta(x)} \\
&\quad = -D^{\frac{n+1}{2}}(x) F_{p,q,nk}(x) F_{p,q,k}^n(x)
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
&\sum_{i=0}^{n} \binom{n}{i} (-1)^i \bigl(-q(x)\bigr)^{(n-i)k} F_{p,q,ik}^2(x) \\
&\quad = -\bigl\{\alpha(x) - \beta(x)\bigr\}^{n-1} \left(\frac{\alpha^k(x) - \beta^k(x)}{\alpha(x) - \beta(x)}\right)^n \frac{\alpha^{nk}(x) - \beta^{nk}(x)}{\alpha(x) - \beta(x)} \\
&\quad = -D^{\frac{n-1}{2}}(x) F_{p,q,nk}(x) F_{p,q,k}^n(x).
\end{aligned} \tag{4.30}$$

Thus, we conclude the proof of Theorem 4.4. □

It becomes obvious that Theorem 4.1, Remark 4.2, Theorems 4.3 and 4.4 give the results of He and Zhang [12] on the Lucas sequences of the first kind and second kind when $p(x)$ and $q(x)$ in Definitions 2.1 and 2.2 satisfy $p(x) = 1$, $q(x) = -1$ and $p(x) = 2$, $q(x) = -1$, respectively. It is worthy of noticing that the case k being an even number in (4.12) gives a sophisticated identity for the classical Fibonacci and Lucas numbers which was asked by Hoggatt as an advanced problem, and the case $k = 1$ in (4.19) and (4.21) gives the familiar combinatorial identities for the classical Fibonacci and Lucas numbers stated in [8].

We now use Theorem 4.1 to obtain the following close formulae of sums of any positive powers for the (p, q)-Fibonacci and Lucas polynomials. We have the following theorem.

Theorem 4.5 *Let m, n be positive integers and a, b be non-negative integers. Suppose that the discriminant $D(x) \neq 0$. Then*

$$\sum_{k=0}^{m} L^n_{p,q,ak+b}(x) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\frac{A_i}{(-q(x))^{an} - (-q(x))^{ia}L_{p,q,(n-2i)a}(x) + 1}, \tag{4.31}$$

where A_i is denoted by

$$\begin{aligned} A_i = {} & \bigl(-q(x)\bigr)^{i(am+b)+an} L_{p,q,(n-2i)(am+b)}(x) \\ & - \bigl(-q(x)\bigr)^{i(a(m+1)+b)} L_{p,q,(n-2i)(a(m+1)+b)}(x) \\ & - \bigl(-q(x)\bigr)^{i(a-b)+nb} L_{p,q,(n-2i)(a-b)}(x) + \bigl(-q(x)\bigr)^{ib} L_{p,q,(n-2i)b}(x). \end{aligned} \tag{4.32}$$

Proof In view of (4.1), we have

$$L^n_{p,q,ak+b}(x) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\bigl(-q(x)\bigr)^{i(ak+b)} L_{p,q,(n-2i)(ak+b)}(x). \tag{4.33}$$

It follows from (4.33), Property 2.3 and $\alpha(x)\beta(x) = -q(x)$ that

$$\begin{aligned} & \sum_{k=0}^{m} L^n_{p,q,ak+b}(x) \\ & = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\sum_{k=0}^{m}\bigl(-q(x)\bigr)^{i(ak+b)}\bigl\{\alpha^{(n-2i)(ak+b)}(x) + \beta^{(n-2i)(ak+b)}(x)\bigr\} \\ & = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\left\{\frac{(-q(x))^{ib}\alpha^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\alpha^{(n-2i)a(m+1)}(x) - 1]}{(-q(x))^{ia}\alpha^{(n-2i)a}(x) - 1}\right. \\ & \quad \left. + \frac{(-q(x))^{ib}\beta^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\beta^{(n-2i)a(m+1)}(x) - 1]}{(-q(x))^{ia}\beta^{(n-2i)a}(x) - 1}\right\} \\ & = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\frac{1}{(-q(x))^{an} - (-q(x))^{ia}L_{p,q,(n-2i)a}(x) + 1} \\ & \quad \times \bigl\{\bigl(-q(x)\bigr)^{i(am+b)+an} L_{p,q,(n-2i)(am+b)}(x) + \bigl(-q(x)\bigr)^{ib} L_{p,q,(n-2i)b}(x) \\ & \quad - \bigl(-q(x)\bigr)^{i(a(m+1)+b)} L_{p,q,(n-2i)(a(m+1)+b)}(x) \\ & \quad - \bigl(-q(x)\bigr)^{i(a-b)+nb} L_{p,q,(n-2i)(a-b)}(x)\bigr\}. \end{aligned} \tag{4.34}$$

Thus, we complete the proof of Theorem 4.5. □

Theorem 4.6 *Let m, n be positive integers and a, b be non-negative integers. Suppose that the discriminant $D(x) \neq 0$. If $2 \nmid n$ then*

$$\sum_{k=0}^{m} F^n_{p,q,ak+b}(x) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\frac{D^{-\frac{n-1}{2}}(x)(-1)^i B_i}{(-q(x))^{an} - (-q(x))^{ia}L_{p,q,(n-2i)a}(x) + 1}, \tag{4.35}$$

and if $2 \mid n$ *then*

$$\sum_{k=0}^{m} F^{n}_{p,q,ak+b}(x) = \frac{1}{2}\sum_{i=0}^{n}\binom{n}{i}\frac{D^{-\frac{n}{2}}(x)(-1)^i C_i}{(-q(x))^{an} - (-q(x))^{ia}L_{p,q,(n-2i)a}(x) + 1}, \tag{4.36}$$

where B_i *and* C_i *are respectively denoted by*

$$\begin{aligned} B_i &= \bigl(-q(x)\bigr)^{i(am+b)+an}F_{p,q,(n-2i)(am+b)}(x) \\ &\quad - \bigl(-q(x)\bigr)^{i(a(m+1)+b)}F_{p,q,(n-2i)(a(m+1)+b)}(x) \\ &\quad + \bigl(-q(x)\bigr)^{i(a-b)+nb}F_{p,q,(n-2i)(a-b)}(x) + \bigl(-q(x)\bigr)^{ib}F_{p,q,(n-2i)b}(x) \end{aligned} \tag{4.37}$$

and

$$\begin{aligned} C_i &= \bigl(-q(x)\bigr)^{i(am+b)+an}L_{p,q,(n-2i)(am+b)}(x) \\ &\quad - \bigl(-q(x)\bigr)^{i(a(m+1)+b)}L_{p,q,(n-2i)(a(m+1)+b)}(x) \\ &\quad - \bigl(-q(x)\bigr)^{i(a-b)+nb}L_{p,q,(n-2i)(a-b)}(x) + \bigl(-q(x)\bigr)^{ib}L_{p,q,(n-2i)b}(x). \end{aligned} \tag{4.38}$$

Proof According to (4.2), we have

$$2D^{\frac{n-1}{2}}(x)F^{n}_{p,q,ak+b}(x) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{i(ak+b)}F_{p,q,(n-2i)(ak+b)}(x). \tag{4.39}$$

It follows from (4.39), Property 2.3 and $\alpha(x)\beta(x) = -q(x)$ that

$$\begin{aligned} &2D^{\frac{n-1}{2}}(x)\sum_{k=0}^{m}F^{n}_{p,q,ak+b}(x) \\ &\quad = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\sum_{k=0}^{m}\bigl(-q(x)\bigr)^{i(ak+b)}\left\{\frac{\alpha^{(n-2i)(ak+b)}(x) - \beta^{(n-2i)(ak+b)}(x)}{\alpha(x)-\beta(x)}\right\} \\ &\quad = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\left\{\frac{(-q(x))^{ib}\alpha^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\alpha^{(n-2i)a(m+1)}(x)-1]}{\{(-q(x))^{ia}\alpha^{(n-2i)a}(x)-1\}\{\alpha(x)-\beta(x)\}}\right. \\ &\qquad \left. - \frac{(-q(x))^{ib}\beta^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\beta^{(n-2i)a(m+1)}(x)-1]}{\{(-q(x))^{ia}\beta^{(n-2i)a}(x)-1\}\{\alpha(x)-\beta(x)\}}\right\} \\ &\quad = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\frac{1}{(-q(x))^{an} - (-q(x))^{ia}L_{p,q,(n-2i)a}(x)+1} \\ &\qquad \times \Bigl\{\bigl(-q(x)\bigr)^{i(am+b)+an}F_{p,q,(n-2i)(am+b)}(x) + \bigl(-q(x)\bigr)^{ib}F_{p,q,(n-2i)b}(x) \\ &\qquad - \bigl(-q(x)\bigr)^{i(a(m+1)+b)}F_{p,q,(n-2i)(a(m+1)+b)}(x) \\ &\qquad + \bigl(-q(x)\bigr)^{i(a-b)+nb}F_{p,q,(n-2i)(a-b)}(x)\Bigr\}. \end{aligned} \tag{4.40}$$

Similarly, from (4.3), we have

$$2D^{\frac{n}{2}}(x)F^{n}_{p,q,ak+b}(x) = \sum_{i=0}^{n}\binom{n}{i}(-1)^i\bigl(-q(x)\bigr)^{i(ak+b)}L_{p,q,(n-2i)(ak+b)}(x). \tag{4.41}$$

Hence

$$
\begin{aligned}
&2D^{\frac{n}{2}}(x)\sum_{k=0}^{m}F_{p,q,ak+b}^{n}(x)\\
&=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\sum_{k=0}^{m}\bigl(-q(x)\bigr)^{i(ak+b)}\bigl\{\alpha^{(n-2i)(ak+b)}(x)+\beta^{(n-2i)(ak+b)}(x)\bigr\}\\
&=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\biggl\{\frac{(-q(x))^{ib}\alpha^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\alpha^{(n-2i)a(m+1)}(x)-1]}{(-q(x))^{ia}\alpha^{(n-2i)a}(x)-1}\\
&\quad+\frac{(-q(x))^{ib}\beta^{(n-2i)b}(x)[(-q(x))^{ia(m+1)}\beta^{(n-2i)a(m+1)}(x)-1]}{(-q(x))^{ia}\beta^{(n-2i)a}(x)-1}\biggr\}\\
&=\sum_{i=0}^{n}\binom{n}{i}(-1)^{i}\frac{1}{(-q(x))^{an}-(-q(x))^{ia}L_{p,q,(n-2i)a}(x)+1}\\
&\quad\times\bigl\{\bigl(-q(x)\bigr)^{i(am+b)+an}L_{p,q,(n-2i)(am+b)}(x)+\bigl(-q(x)\bigr)^{ib}L_{p,q,(n-2i)b}(x)\\
&\quad-\bigl(-q(x)\bigr)^{i(a(m+1)+b)}L_{p,q,(n-2i)(a(m+1)+b)}(x)\\
&\quad-\bigl(-q(x)\bigr)^{i(a-b)+nb}L_{p,q,(n-2i)(a-b)}(x)\bigr\}.
\end{aligned}
\tag{4.42}
$$

This concludes the proof of Theorem 4.6. □

Remark 4.7 From (2.10) and Theorem 4.6, one can easily obtain that for any non-negative integers i, j, k, n,

$$
\sum_{i=0}^{n}F_{p,q,ki+j}(x)=\frac{F_{p,q,nk+j}(x)(-q(x))^{k}+F_{p,q,j}(x)-F_{p,q,(n+1)k+j}(x)+F_{p,q,k-j}(-q(x))^{j}}{(-q(x))^{k}-L_{p,q,k}(x)+1}
\tag{4.43}
$$

when $j<k$, and if $j\geq k$ then

$$
\begin{aligned}
&\sum_{i=0}^{n}F_{p,q,ki+j}(x)\\
&\quad=\frac{F_{p,q,nk+j}(x)(-q(x))^{k}+F_{p,q,j}(x)-F_{p,q,(n+1)k+j}(x)+F_{p,q,j-k}(x)(-q(x))^{k}}{(-q(x))^{k}-L_{p,q,k}(x)+1},
\end{aligned}
\tag{4.44}
$$

where an equivalent version appeared in [11, Theorem 2.14] (but note a misprint: $p(x)$ should be $-q(x)$). For another close formulae of sums of powers of the classical Fibonacci and Lucas numbers, one is referred to [13, 14].

Competing interests
The author declares that they have no competing interests.

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