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Periodic and subharmonic solutions for a 2nth-order difference equation containing both advance and retardation with ϕ-Laplacian

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Abstract
In this paper, by using critical point theory, we obtain some new sufficient conditions on the existence and multiplicity of periodic and subharmonic solutions to a 2nth-order nonlinear difference equation containing both advance and retardation with ϕ-Laplacian. Some previous results have been generalized.

Keywords: periodic and subharmonic solutions; 2nth-order; critical point theory; difference equations; ϕ-Laplacian

1 Introduction

Let N, Z, and R denote the sets of all natural numbers, integers and real numbers, respectively. For $a, b \in Z$, define $Z(a) = \{a, a+1, \ldots\}$, $Z(a,b) = \{a, a+1, \ldots, b\}$ when $a \le b$. $\cdot^{\text{tr}}$ denotes the transpose of a vector.

Consider the following 2nth-order difference equation containing both advance and retardation with ϕ-Laplacian of the type:

$$\triangle^n\big(r_{k-n}\phi\big(\triangle^n u_{k-1}\big)\big) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad k \in Z, \tag{1.1}$$

where $n \in Z$, $\triangle$ is forward difference operator defined by $\triangle u_k = u_{k+1} - u_k$, $\triangle^n u_k = \triangle(\triangle^{n-1}u_k)$, $\phi \in C(R,R)$ satisfied $\phi(0) = 0, f \in C(Z \times R^3, R)$, $r_k > 0$ for each $k \in Z$, $\{r_k\}$ and $\{f(k, v_1, v_2, v_3)\}$ are T-periodic in k and T is a given positive integer.

In this paper, given positive integer m, we will study the existence of mT-periodic solutions for (1.1). As usual, such a mT-periodic solution will be called a subharmonic solution.

We may think of (1.1) as a discrete analog of the following 2nth-order functional differential equation:

$$\frac{d^n}{dt^n}\left[r(t)\phi\left(\frac{d^n u(t)}{dt^n}\right)\right] = (-1)^n f\big(t, u(t+1), u(t), u(t-1)\big), \quad t \in R. \tag{1.2}$$

Equations similar in structure to (1.2) have been studied by many authors. For example, for the case where $\phi(x) = x$, $n = 1$, Smets and Willem [1] have considered solitary waves with prescribed speed on infinite lattices of particles with nearest neighbor interaction for the following forward and backward differential difference equation:

$$c^2u''(t) = V'\big(u(t+1) - u(t)\big) - V'\big(u(t) - u(t-1)\big), \quad t \in R.$$

For the case where $\phi(x) = |x|^{p-2}x$, $p > 1$, $n = 1$, Wang [2] has studied the existence of positive solutions of the equation

$$\big(|u'|^{p-2}u'\big)' + a(t)f(u) = 0, \quad t \in R.$$

For the case where $\phi(x) = |x|^{p-2}x$, $p > 1$, $n = 2$, Agarwal, Lu, and O'Regan [3] have studied the existence of positive solutions of the equation

$$\big(|u''|^{p-2}u''\big)'' = \lambda q(t)f(u), \quad t \in R.$$

For the case where $\phi(x) = \frac{x}{\sqrt{1+x^2}}$, $n = 1$, Bonheure and Habets [4] have studied classical and non-classical solutions of a prescribed curvature equation

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \lambda f(t,u), \quad t \in R.$$

In recent years, many authors have studied the existence of periodic solutions of difference equations. To mention a few, see [5–8] for second-order difference equations and [9, 10] for higher-order equations. Since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [11–13] and Zhou *et al.* [14] established sufficient conditions on the existence of periodic solutions of second-order nonlinear difference equations. In 2007, by using the Linking Theorem, Cai and Yu [15] obtained some criteria for the existence of periodic solutions of the following equation:

$$\triangle^n\big(r_{k-n}\triangle^n u_{k-n}\big) + f(k,u_k) = 0, \quad k \in Z, \tag{1.3}$$

for the case where f grows superlinearly at both 0 and ∞, where $n \in Z(3)$. In 2010, by using the Linking Theorem and the Saddle Point Theorem, Zhou [16] extended f in (1.3) into sublinear or asymptotically linear and improved the results of [15] when f is superlinear. In particular, a necessary and sufficient condition for the existence of the unique periodic solution of (1.3) is also established in [16]. In 2013, by using the Linking Theorem, Deng [17] provided some sufficient conditions of the existence and multiplicity of periodic solutions and subharmonic solutions of the following equation:

$$\triangle^n\big(r_{k-n}\varphi_p\big(\triangle^n u_{k-1}\big)\big) = (-1)^n f(k,u_{k+1},u_k,u_{k-1}), \quad k \in Z, \tag{1.4}$$

where $n \in N$, φ_p is the p-Laplacian operator given by $\varphi_p(u) = |u|^{p-2}u$ $(1 < p < \infty)$ and where f satisfies some growth conditions near both 0 and ∞. In 2012, Mawhin [18] considered T-periodic solutions of systems of difference equations of the form

$$\triangle\phi\big[\triangle u(k-1)\big] = \nabla_u F\big[k,u(k)\big] + h(k), \quad k \in Z, \tag{1.5}$$

under various conditions upon $F : Z \times R^n \to R$ and $h : Z \to R^n$, where $n \in Z$, $\phi = \nabla\Phi$, in which $\Phi : R^n \to [0, \infty)$ is continuously differentiable and strictly convex, satisfies $\phi(0) = 0$ and is a homeomorphism of R^n onto the ball $B_a \subseteq R^n$ or of B_a onto R^n. By using direct variational method, he gave sufficient conditions for the existence of a minimizing sequence for the case of coercive potential, or some averaged coercivity conditions of the Ahmad-Lazer-Paul type adding the nonlinearity satisfies some growth conditions, or the convex potential. Using the Saddle Point Theorem, previously obtained results are extended to the case of an averaged anticoercivity condition in [18]. However, the results on periodic solutions of higher-order nonlinear difference equations involving ϕ-Laplacian are very scarce in the literature. Furthermore, since (1.1) contains both advance and retardation, there are very few works dealing with this subject; see [10, 19]. The main purpose of this paper is to give some sufficient conditions for the existence and multiplicity of periodic and subharmonic solutions of (1.1). Particularly, our results generalize the results in the literature [17, 20]; see Remark 3.4 and Remark 3.5 for details.

2 Preliminaries

Throughout this paper, we assume that,

(F_1) there exists a functional $F(k, v_1, v_2) \in C^1(Z \times R^2, R)$ with $F(k, v_1, v_2) \geq 0$ and satisfies

$$F(k + T, v_1, v_2) = F(k, v_1, v_2),$$
$$\frac{\partial F(k-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(k, v_1, v_2)}{\partial v_2} = f(k, v_1, v_2, v_3).$$

In this section, we first establish the variational setting associated with (1.1).

Let S be the set of all two-sided sequences, that is,

$$S = \{u = \{u_k\} | u_k \in R, k \in Z\}.$$

Then S is a vector space with $au + bv = \{au_k + bv_k\}$ for $u, v \in S$, $a, b \in R$. For any fixed positive integer m and T, we define the subspace E_m of S as

$$E_m = \{u = \{u_k\} \in S | u_{k+mT} = u_k, k \in Z\}.$$

Obviously, E_m is isomorphic to R^{mT} and hence E_m can be equipped with the inner product $(\cdot, \cdot)$ and norm $\|\cdot\|$ as

$$(u, v) = \sum_{j=1}^{mT} u_j v_j, \quad u, v \in E_m,$$

and

$$\|u\| = \left(\sum_{j=1}^{mT} u_j^2\right)^{\frac{1}{2}}, \quad u \in E_m.$$

On the other hand, we define the norm $\|\cdot\|_q$ on E_m as follows:

$$\|u\| = \left(\sum_{j=1}^{mT} |u_j|^q\right)^{\frac{1}{q}},$$

for all $u \in E_m$ and $q \geq 1$. By Hölder' inequality and Jensen' inequality, we have

$$\|u\|_2 \leq \|u\|_q \leq (mT)^{\frac{2-q}{2q}} \|u\|_2, \quad 1 \leq q < 2,$$
$$(mT)^{\frac{2-q}{2q}} \|u\|_2 \leq \|u\|_q \leq \|u\|_2, \quad 2 \leq q.$$

Let

$$d_{1,q} = \begin{cases} 1, & 1 \leq q < 2, \\ (mT)^{\frac{2-q}{2q}}, & 2 \leq q, \end{cases} \qquad d_{2,q} = \begin{cases} (mT)^{\frac{2-q}{2q}}, & 1 \leq q < 2, \\ 1, & 2 \leq q. \end{cases}$$

Therefore,

$$d_{1,q}\|u\|_2 \leq \|u\|_q \leq d_{2,q}\|u\|_2, \quad u \in E_m. \tag{2.1}$$

Clearly, $\|u\| = \|u\|_2$. For all $u \in E_m$, define the functional J on E_m as follows:

$$J(u) = \sum_{k=1}^{mT} r_{k-1}\Phi\left(\triangle^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k), \tag{2.2}$$

where

$$\Phi(u) = \int_0^u \phi(s)\, ds$$

is the primitive function of $\phi(u)$.

Clearly, $J \in C^1(E_m, R)$ and for any $u = \{u_k\}_{k\in Z} \in E_m$, by using $u_j = u_{mT+j}$ for $j \in Z(0, mT - 1)$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_k} = (-1)^n \triangle^n\left(r_{k-n}\phi\left(\triangle^n u_{k-1}\right)\right) - f(k, u_{k+1}, u_k, u_{k-1}).$$

Thus, u is a critical point of J on E_m if and only if

$$\triangle^n\left(r_{k-n}\phi\left(\triangle^n u_{k-1}\right)\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad k \in Z(1, mT).$$

Due to the periodicity of $u = \{u_k\}_{k\in Z} \in E_m$ and $f(k, v_1, v_2, v_3)$ in the first variable k, we reduce the existence of periodic solutions of (1.1) to the existence of critical points of J on E_m.

Let M be the $mT \times mT$ matrix defined by

$$M = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

By matrix theory, we see that the eigenvalues of M are

$$\lambda_j = 2\left(1 - \cos\frac{2j}{mT}\right), \quad j = 0, 1, 2, \ldots, mT - 1.$$

Thus, $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 > 0, \ldots, \lambda_{mT-1} > 0$. Therefore,

$$\lambda_{\min} = \min\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} = 2\left(1 - \cos\frac{2}{mT}\right),$$

$$\lambda_{\max} = \max\{\lambda_1, \lambda_2, \ldots, \lambda_{mT-1}\} = \begin{cases} 4, & \text{when } mT \text{ is even,} \\ 2(1 + \cos\frac{1}{mT}), & \text{when } mT \text{ is odd.} \end{cases}$$

For convenience, we identify $u \in E_m$ with $u = (u_1, u_2, \ldots, u_{mT})^{\mathrm{tr}}$. Let

$$\overline{E}_m = \left\{u = (u_1, u_2, \ldots, u_{mT})^{\mathrm{tr}} \in E_m | \triangle^{n-1} u_j = 0, j \in Z(1, mT)\right\}.$$

Then

$$\overline{E}_m = \left\{u \in E_m | u = \{a\}, a \in R\right\}.$$

Let $\widetilde{E}_m$ be the direct orthogonal complement of E_m to $\overline{E}_m$, *i.e.*, $E_m = \overline{E}_m \oplus \widetilde{E}_m$.

For $u = (u_1, u_2, \ldots, u_{mT})^{\mathrm{tr}} \in E_m$ and $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \ldots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$, we have

$$\|x\|_2^q = \left[\sum_{k=1}^{mT} \left(\triangle^{n-2}u_{k+1} - \triangle^{n-2}u_k\right)^2\right]^{\frac{q}{2}} \le \left[\lambda_{\max} \sum_{k=1}^{mT} \left(\triangle^{n-2}u_k\right)^2\right]^{\frac{q}{2}} \le \lambda_{\max}^{\frac{(n-1)q}{2}} \|u\|_2^q. \tag{2.3}$$

For $u = (u_1, u_2, \ldots, u_{mT})^{\mathrm{tr}} \in \widetilde{E}_m$ and $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \ldots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$, we have

$$\|x\|_2^q = \left[\sum_{k=1}^{mT} \left(\triangle^{n-2}u_{k+1} - \triangle^{n-2}u_k\right)^2\right]^{\frac{q}{2}} \ge \left[\lambda_{\min} \sum_{k=1}^{mT} \left(\triangle^{n-2}u_k\right)^2\right]^{\frac{q}{2}} \ge \lambda_{\min}^{\frac{(n-1)q}{2}} \|u\|_2^q. \tag{2.4}$$

Let H be a Hilbert space and $C^1(H, R)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H. Let $J \in C^1(H, R)$. A sequence $\{x_j\} \subset H$ is called a Palais-Smale sequence (P. S. sequence for short) for J if $\{J(x_j)\}$ is bounded and $J'(x_j) \to 0$ as $j \to \infty$. We say J satisfies the Palais-Smale condition (P. S. condition for short) if any P. S. sequence for J possesses a convergent subsequence.

Let B_r be the open ball in H with radius r and center 0, and let ∂B_r denote its boundary. Lemma 2.1 is taken from [21].

Lemma 2.1 (Linking Theorem) *Let H be a real Hilbert space and $H = H_1 \oplus H_2$, where H_1 is a finite-dimensional subspace of H. Assume that $J \in C^1(H,R)$ satisfies the P. S. condition and the following conditions.*

(J_1) *There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap H_2} \geq a$;*
(J_2) *There exist an $e \in \partial B_1 \cap H_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$ where $Q = (\overline{B}_{R_0} \cap H_1) \oplus \{re|0 < r < R_0\}$.*

Then J possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$c = \inf_{h\in\Gamma} \sup_{x\in Q} J\bigl(h(x)\bigr),$$

where $\Gamma = \{h \in C(\overline{Q},H) : h|_{\partial Q} = \mathrm{id}|_{\partial Q}\}$ and $\mathrm{id}|_{\partial Q}$ is the identity operator on ∂Q.

3 Main results

Let

$$\underline{r} = \min_{k\in Z(1,T)} \{r_k\}, \qquad \overline{r} = \max_{k\in Z(1,T)} \{r_k\}.$$

Here we give some conditions.

(Φ_1) There exist constants $\epsilon_1 > 0$, $a_1 > 0$ and $\mu \geq 1$ such that

$$\Phi(u) \geq a_1|u|^\mu \quad \text{for } |u| \leq \epsilon_1.$$

(Φ_2) There exist constants $\delta_1 > 0$, $b_1 > 0$, $c_1 \geq 0$ and $\nu \geq 1$ such that

$$\Phi(u) \leq b_1|u|^\nu + c_1 \quad \text{for } |u| \geq \delta_1.$$

(F_2) There exist constants $\epsilon_2 > 0$, $a_2 > 0$ and $\theta \geq 1$ such that

$$F(k,v_1,v_2) \leq a_2\Bigl(\sqrt{v_1^2+v_2^2}\Bigr)^\theta \quad \text{for } \sqrt{v_1^2+v_2^2} \leq \epsilon_2.$$

(F_3) There exist constants $\delta_2 > 0$, $b_2 > 0$, $c_2 > 0$ and $\vartheta \geq 1$ such that

$$F(k,v_1,v_2) \geq b_2\Bigl(\sqrt{v_1^2+v_2^2}\Bigr)^\vartheta - c_2 \quad \text{for } \sqrt{v_1^2+v_2^2} \geq \delta_2.$$

($H_{1,s}$) $\mu = \theta = s$ and $\frac{a_1}{a_2}(\frac{d_{1,s}}{d_{2,s}})^s \frac{\underline{r}\lambda_{\min}^{\frac{ns}{2}}}{2^{\frac{s}{2}}} > 1$.
($H_{1,p}$) $\nu = \vartheta = p$ and $\frac{b_1}{b_2}(\frac{d_{2,p}}{d_{1,p}})^p \frac{\overline{r}\lambda_{\max}^{\frac{np}{2}}}{2^{\frac{p}{2}}} < 1$.
($H_{2,s}$) $\mu < \theta$.
($H_{2,p}$) $\nu < \vartheta$.

Remark 3.1 By (Φ_2) it is easy to see that there exists a constant $c_1' > 0$ such that

$$\Phi(u) \le b_1|u|^{\nu} + c_1', \quad u \in R. \tag{3.1}$$

Remark 3.2 By (F_3) it is easy to see that there exists a constant $c_2' > 0$ such that

$$F(k, \nu_1, \nu_2) \ge b_2\Big(\sqrt{\nu_1^2 + \nu_2^2}\Big)^{\vartheta} - c_2', \quad (k, \nu_1, \nu_2) \in Z \times R^2. \tag{3.2}$$

Remark 3.3 The p-Laplacian operator given by $\varphi_p(u) = |u|^{p-2}u$ $(1 < p < \infty)$, the curvature-type operator given by $\phi_q(u) = \frac{|u|^{q-2}u}{\sqrt{1+|u|^q}}$ $(2 \le q < \infty)$ and the identity operator given by $\phi_I(u) = u$ satisfy (Φ_1) and (Φ_2).

Our main results are as follows.

Theorem 3.1 *Assume that* (Φ_1), (Φ_2), (F_1), (F_2), (F_3) *are satisfied. If one of the following four cases is satisfied*:

(1) *Assume that* $(H_{2,s})$ *and* $(H_{2,p})$ *are satisfied.*
(2) *Assume that* $(H_{1,s})$ *and* $(H_{1,p})$ *are satisfied.*
(3) *Assume that* $(H_{1,s})$ *and* $(H_{2,p})$ *are satisfied.*
(4) *Assume that* $(H_{2,s})$ *and* $(H_{1,p})$ *are satisfied.*

Then for any given positive integer m, (1.1) *has at least three* mT*-periodic solutions.*

Remark 3.4 If $\phi(u) = |u|^{p-2}u$ $(1 < p < \infty)$, $r_k = 1$ and $n = 1$, Theorem 3.1 reduces to Theorem 3.1 in [20].

Remark 3.5 If $\phi(u) = |u|^{p-2}u$ $(1 < p < \infty)$, Theorem 3.1 reduces to Theorem 1.1 in [17].

Corollary 3.1 *Assume that* (F_1) *and the following conditions are satisfied.*

(Φ_1') *There exists constant* $\mu \ge 1$ *such that* $\lim_{|u|\to 0} \frac{\Phi(u)}{|u|^{\mu}} = d > 0$.
(Φ_2') *There exist constants* $\delta_1 > 0$ *and* $\nu \ge 1$ *such that*

$$0 < \phi(u)u \le \nu\Phi(u), \quad |u| \ge \delta_1.$$

(F_2') *There exists constant* $\theta \ge \mu$ *such that*

$$\lim_{(\nu_1,\nu_2)\to(0,0)} \frac{F(k, \nu_1, \nu_2)}{(\nu_1^2 + \nu_2^2)^{\frac{\theta}{2}}} = 0, \quad (k, \nu_1, \nu_2) \in Z \times R^2.$$

(F_3') *There exist constants* $\delta_2 > 0$ *and* $\vartheta > \nu$ *such that*

$$0 < \vartheta F(k, \nu_1, \nu_2) \le \frac{\partial F(k, \nu_1, \nu_2)}{\partial \nu_1}\nu_1 + \frac{\partial F(k, \nu_1, \nu_2)}{\partial \nu_2}\nu_2, \quad \sqrt{\nu_1^2 + \nu_2^2} \ge \delta_2.$$

Then for any given positive integer m, (1.1) *has at least three* mT*-periodic solutions.*

4 Proof of the main results

Lemma 4.1 *Assume that* (Φ_2), (F_1), (F_3), *and* $(H_{2,p})$ *are satisfied. Then the functional* J *is bounded from above in* E_m.

Proof By (2.1), (2.3), (3.1), and (3.2), for any $u \in E_m$, we have

$$\begin{aligned} J(u) &= \sum_{k=1}^{mT} r_{k-1}\Phi(\triangle^n u_{k-1}) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\ &\le \sum_{k=1}^{mT} r_{k-1} b_1 |\triangle^n u_{k-1}|^{\nu} + mTc_1' - \sum_{k=1}^{mT} b_2 \left(\sqrt{u_{k+1}^2 + u_k^2}\right)^{\vartheta} + mTc_2' \\ &\le b_1 \bar{r} d_{2,\nu}^{\nu} \left(\sum_{k=1}^{mT} |\triangle^n u_{k-1}|^2\right)^{\frac{\nu}{2}} - b_2 \left[\left(\sum_{k=1}^{mT} \left(\sqrt{u_{k+1}^2 + u_k^2}\right)^{\vartheta}\right)^{\frac{1}{\vartheta}}\right]^{\vartheta} + mT(c_1' + c_2') \\ &\le b_1 \bar{r} d_{2,\nu}^{\nu} (x^{\mathrm{tr}} M x)^{\frac{\nu}{2}} - b_2 d_{1,\vartheta}^{\vartheta} (2\|u\|^2)^{\frac{\vartheta}{2}} + mT(c_1' + c_2') \\ &\le b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{\nu}{2}} \|x\|^{\nu} - 2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta} \|u\|^{\vartheta} + mT(c_1' + c_2') \\ &\le b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{n\nu}{2}} \|u\|^{\nu} - 2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta} \|u\|^{\vartheta} + mT(c_1' + c_2') \\ &\le \max_{\|u\| \le \rho_0} \left\{ b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{n\nu}{2}} \|u\|^{\nu} - 2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta} \|u\|^{\vartheta} \right\} + mT(c_1' + c_2'), \end{aligned} \tag{4.1}$$

where $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \dots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$ and $\rho_0 = (\frac{b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{n\nu}{2}}}{2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta}})^{\frac{1}{\vartheta-\nu}}$.

The proof of Lemma 4.1 is complete. □

Remark 4.1 The case $mT = 1$ is trivial. For the case $mT = 2$, M has a different form, namely,

$$M = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

However, in this special case, the argument need not be changed and we omit it.

Lemma 4.2 *Assume that* (Φ_2), (F_1), (F_3), *and* $(H_{2,p})$ *are satisfied. Then the functional* J *satisfies the P. S. condition in* E_m.

Proof Let $\{u^{(j)}\}$ be a P. S. sequence, then there exists a positive constant M_1 such that

$$-M_1 \le J(u^{(j)}), \quad j \in N.$$

By (4.1), it is easy to see that

$$-M_1 \le J(u^{(j)}) \le b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{n\nu}{2}} \|u^{(j)}\|^{\nu} - 2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta} \|u^{(j)}\|^{\vartheta} + mT(c_1' + c_2'), \quad j \in N.$$

Therefore,

$$2^{\frac{\vartheta}{2}} b_2 d_{1,\vartheta}^{\vartheta} \|u^{(j)}\|^{\vartheta} - b_1 \bar{r} d_{2,\nu}^{\nu} \lambda_{\max}^{\frac{n\nu}{2}} \|u^{(j)}\|^{\nu} \le M_1 + mT(c_1' + c_2'), \quad j \in N.$$

Since $\vartheta > \nu$, it is not difficult to see that $\{u^{(j)}\}$ is a bounded sequence in E_m. As a consequence, $\{u^{(j)}\}$ possesses a convergence subsequence in E_m. Thus the P. S. condition is verified. □

Lemma 4.3 *Assume that* (Φ_2), (F_1), (F_3), *and* $(H_{1,p})$ *are satisfied. Then the functional* J *is bounded from above in* E_m.

Proof Similar to the proof of Lemma 4.1, we have

$$J(u) \le b_1\bar{r}d_{2,p}^p\lambda_{\max}^{\frac{np}{2}}\|u\|^p - 2^{\frac{p}{2}}b_2d_{1,p}^p\|u\|^p + mT\left(c_1' + c_2'\right), \tag{4.2}$$

where $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \dots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$. Since $\frac{b_1}{b_2}(\frac{d_{2,p}}{d_{1,p}})^p\frac{\bar{r}\lambda_{\max}^{\frac{np}{2}}}{2^{\frac{p}{2}}} < 1$, we have

$$J(u) \le mT\left(c_1' + c_2'\right).$$

The proof of Lemma 4.3 is complete. □

Lemma 4.4 *Assume that* (Φ_2), (F_1), (F_3), *and* $(H_{1,p})$ *are satisfied. Then the functional* J *satisfies the P. S. condition in* E_m.

Proof Let $\{u^{(j)}\}$ be a P. S. sequence, then there exists a positive constant M_2 such that

$$-M_2 \le J\left(u^{(j)}\right), \quad j \in N.$$

By (4.2), it is easy to see that

$$-M_2 \le J\left(u^{(j)}\right) \le b_1\bar{r}d_{2,p}^p\lambda_{\max}^{\frac{np}{2}}\left\|u^{(j)}\right\|^p - 2^{\frac{p}{2}}b_2d_{1,p}^p\left\|u^{(j)}\right\|^p + mT\left(c_1' + c_2'\right), \quad j \in N.$$

Therefore,

$$2^{\frac{p}{2}}b_2d_{1,p}^p\left\|u^{(j)}\right\|^p - b_1\bar{r}d_{2,p}^p\lambda_{\max}^{\frac{np}{2}}\left\|u^{(j)}\right\|^p \le M_2 + mT\left(c_1' + c_2'\right), \quad j \in N.$$

Since $\frac{b_1}{b_2}(\frac{d_{2,p}}{d_{1,p}})^p\frac{\bar{r}\lambda_{\max}^{\frac{np}{2}}}{2^{\frac{p}{2}}} < 1$, we know that $\{u^{(j)}\}$ is a bounded sequence in E_m. As a consequence, $\{u^{(j)}\}$ possesses a convergence subsequence in E_m. Thus the P. S. condition is verified. □

Proof of Theorem 3.1 Assumptions (F_1) and (F_2) imply that $F(k,0) = 0$ and $f(k,0) = 0$ for $k \in Z$. Adding $\phi(0) = 0$, then $u = 0$ is a trivial mT-periodic solution of (1.1).

By Lemma 4.1 or Lemma 4.3, J is bounded from above on E_m. We define $\alpha_0 = \sup_{u\in E_m} J(u)$. Equation (4.1) implies $\lim_{\|u\|\to\infty} J(u) = -\infty$. This means that $-J$ is coercive. By the continuity of J, there exists $\overline{u} \in E_m$ such that $J(\overline{u}) = \alpha_0$. Clearly, $\overline{u}$ is a critical point of J.

Case 1. Assume that $(H_{2,s})$ and $(H_{2,p})$ are satisfied. We claim that $\alpha_0 > 0$.

Let

$$\rho = \min\left\{\epsilon_1\lambda_{\max}^{-\frac{n}{2}}, \epsilon_2, \frac{1}{2}\left(\frac{a_1\underline{r}d_{1,\mu}^{\mu}\lambda_{\min}^{\frac{n\mu}{2}}}{2^{\frac{\theta}{2}}a_2d_{2,\theta}^{\theta}}\right)^{\frac{1}{\theta-\mu}}\right\}.$$

By (Φ_1), (F_2), and $(H_{2,s})$, for any $u \in \widetilde{E}_m$, $\|u\| \le \rho$, we have

$$\begin{aligned}
J(u) &= \sum_{k=1}^{mT} r_{k-1}\Phi\bigl(\triangle^n u_{k-1}\bigr) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\ge \sum_{k=1}^{mT} r_{k-1}a_1\bigl|\triangle^n u_{k-1}\bigr|^{\mu} - \sum_{k=1}^{mT} a_2\Bigl(\sqrt{u_{k+1}^2 + u_k^2}\Bigr)^{\theta} \\
&\ge a_1\underline{r}d_{1,\mu}^{\mu}\Biggl(\sum_{k=1}^{mT}\bigl|\triangle^n u_{k-1}\bigr|^2\Biggr)^{\frac{\mu}{2}} - a_2\Biggl[\Biggl(\sum_{k=1}^{mT}\Bigl(\sqrt{u_{k+1}^2 + u_k^2}\Bigr)^{\theta}\Biggr)^{\frac{1}{\theta}}\Biggr]^{\theta} \\
&\ge a_1\underline{r}d_{1,\mu}^{\mu}\bigl(x^{\mathrm{tr}}Mx\bigr)^{\frac{\mu}{2}} - a_2 d_{2,\theta}^{\theta}\bigl(2\|u\|^2\bigr)^{\frac{\theta}{2}} \ge a_1\underline{r}d_{1,\mu}^{\mu}\lambda_{\min}^{\frac{\mu}{2}}\|x\|^{\mu} - 2^{\frac{\theta}{2}}a_2 d_{2,\theta}^{\theta}\|u\|^{\theta} \\
&\ge a_1\underline{r}d_{1,\mu}^{\mu}\lambda_{\min}^{\frac{n\mu}{2}}\|u\|^{\mu} - 2^{\frac{\theta}{2}}a_2 d_{2,\theta}^{\theta}\|u\|^{\theta},
\end{aligned} \tag{4.3}$$

where $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \ldots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$.

Take $\sigma = a_1\underline{r}d_{1,\mu}^{\mu}\lambda_{\min}^{\frac{n\mu}{2}}\rho^{\mu} - 2^{\frac{\theta}{2}}a_2 d_{2,\theta}^{\theta}\rho^{\theta}$. Then $\sigma \ge \frac{1}{2}a_1\underline{r}d_{1,\mu}^{\mu}\lambda_{\min}^{\frac{n\mu}{2}}\rho^{\mu} > 0$ and

$$J(u) \ge \sigma, \quad u \in \widetilde{E}_m \cap \partial B_{\rho}. \tag{4.4}$$

Therefore, $\alpha_0 = \sup_{u\in E_m} J(u) \ge \sigma > 0$. From (4.4), we have also proved that J satisfies the condition (J_1) of the Linking Theorem.

For all $u \in \overline{E}_m$, we have

$$J(u) = -\sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \le 0.$$

Thus, the critical point $\overline{u}$ of J corresponding to the critical value α_0 is a nontrivial mT-periodic solution of (1.1). In the following, we will verify the condition (J_2).

Take $e \in \partial B_1 \cap \widetilde{E}_m$, for any $z \in \overline{E}_m$ and $r > 0$, let $u = re + z$. Then

$$\begin{aligned}
J(u) &= \sum_{k=1}^{mT} r_{k-1}\Phi\bigl(\triangle^n u_{k-1}\bigr) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&= \sum_{k=1}^{mT} r_{k-1}\Phi\bigl(r\triangle^n e_{k-1}\bigr) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\le \sum_{k=1}^{mT} r_{k-1}b_1\bigl|r\triangle^n e_{k-1}\bigr|^{\nu} + mTc_1' - \sum_{k=1}^{mT} b_2\Bigl(\sqrt{u_{k+1}^2 + u_k^2}\Bigr)^{\vartheta} + mTc_2' \\
&\le b_1\overline{r}r^{\nu}d_{2,\nu}^{\nu}\Biggl(\sum_{k=1}^{mT}\bigl|\triangle^n e_{k-1}\bigr|^2\Biggr)^{\frac{\nu}{2}} - b_2\Biggl[\Biggl(\sum_{k=1}^{mT}\Bigl(\sqrt{u_{k+1}^2 + u_k^2}\Bigr)^{\vartheta}\Biggr)^{\frac{1}{\vartheta}}\Biggr]^{\vartheta} + mT\bigl(c_1' + c_2'\bigr) \\
&\le b_1\overline{r}r^{\nu}d_{2,\nu}^{\nu}\bigl(y^{\mathrm{tr}}My\bigr)^{\frac{\nu}{2}} - b_2 d_{1,\vartheta}^{\vartheta}\bigl(2\|u\|^2\bigr)^{\frac{\vartheta}{2}} + mT\bigl(c_1' + c_2'\bigr) \\
&\le b_1\overline{r}r^{\nu}d_{2,\nu}^{\nu}\lambda_{\max}^{\frac{\nu}{2}}\|y\|^{\nu} - 2^{\frac{\vartheta}{2}}b_2 d_{1,\vartheta}^{\vartheta}\bigl(r^{\vartheta} + \|z\|^{\vartheta}\bigr) + mT\bigl(c_1' + c_2'\bigr) \\
&\le b_1\overline{r}d_{2,\nu}^{\nu}\lambda_{\max}^{\frac{n\nu}{2}}r^{\nu} - 2^{\frac{\vartheta}{2}}b_2 d_{1,\vartheta}^{\vartheta}r^{\vartheta} - 2^{\frac{\vartheta}{2}}b_2 d_{1,\vartheta}^{\vartheta}\|z\|^{\vartheta} + mT\bigl(c_1' + c_2'\bigr),
\end{aligned} \tag{4.5}$$

where $y = (\triangle^{n-1}e_1, \triangle^{n-1}e_2, \ldots, \triangle^{n-1}e_{mT})^{\mathrm{tr}}$.

Let $g_1(t) = b_1\overline{r}d_{2,\nu}^{\nu}\lambda_{\max}^{\frac{n\nu}{2}}t^{\nu} - 2^{\frac{\vartheta}{2}}b_2d_{1,\vartheta}^{\vartheta}t^{\vartheta}$, $g_2(t) = -2^{\frac{\vartheta}{2}}b_2d_{1,\vartheta}^{\vartheta}t^{\vartheta} + mT(c_1' + c_2')$. We have $\lim_{t\to+\infty} g_1(t) = -\infty$ and $\lim_{t\to+\infty} g_2(t) = -\infty$, and $g_1(t)$, $g_2(t)$ are bounded from above, and $J(z) \leq 0$ for $z \in \overline{E}_m$. Thus there exists a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$ where $Q = (\overline{B}_{R_0} \cap \widetilde{E}_m) \oplus \{re|0 < r < R_0\}$.

Case 2. Assume that $(H_{1,s})$ and $(H_{1,p})$ are satisfied. We claim that $\alpha_0 > 0$.

Let $\rho = \min\{\epsilon_1\lambda_{\max}^{-\frac{n}{2}}, \epsilon_2\}$. By $(H_{1,s})$, (Φ_1), and (F_2), for any $u \in \widetilde{E}_m$, $\|u\| \leq \rho$, we have

$$\begin{aligned}
J(u) &= \sum_{k=1}^{mT} r_{k-1}\Phi\left(\triangle^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\geq \sum_{k=1}^{mT} r_{k-1}a_1\left|\triangle^n u_{k-1}\right|^s - \sum_{k=1}^{mT} a_2\left(\sqrt{u_{k+1}^2 + u_k^2}\right)^s \\
&\geq a_1\underline{r}d_{1,s}^s\left(\sum_{k=1}^{mT}\left|\triangle^n u_{k-1}\right|^2\right)^{\frac{s}{2}} - a_2\left[\left(\sum_{k=1}^{mT}\left(\sqrt{u_{k+1}^2 + u_k^2}\right)^s\right)^{\frac{1}{s}}\right]^s \\
&\geq a_1\underline{r}d_{1,s}^s\left(x^{\mathrm{tr}}Mx\right)^{\frac{s}{2}} - a_2d_{2,s}^s\left(2\|u\|^2\right)^{\frac{s}{2}} \geq a_1\underline{r}d_{1,s}^s\lambda_{\min}^{\frac{s}{2}}\|x\|^s - 2^{\frac{s}{2}}a_2d_{2,s}^s\|u\|^s \\
&\geq a_1\underline{r}d_{1,s}^s\lambda_{\min}^{\frac{ns}{2}}\|u\|^s - 2^{\frac{s}{2}}a_2d_{2,s}^s\|u\|^s,
\end{aligned} \tag{4.6}$$

where $x = (\triangle^{n-1}u_1, \triangle^{n-1}u_2, \ldots, \triangle^{n-1}u_{mT})^{\mathrm{tr}}$.

Take $\sigma = a_1\underline{r}d_{1,s}^s\lambda_{\min}^{\frac{ns}{2}}\rho^s - 2^{\frac{s}{2}}a_2d_{2,s}^s\rho^s$. Then $\sigma \geq 0$ and

$$J(u) \geq \sigma, \quad u \in \widetilde{E}_m \cap \partial B_\rho. \tag{4.7}$$

Therefore, $\alpha_0 = \sup_{u\in E_m} J(u) \geq \sigma > 0$. From (4.7), we have also proved that J satisfies the condition (J_1) of the Linking Theorem.

For all $u \in \overline{E}_m$, we have

$$J(u) = -\sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \leq 0.$$

Thus, the critical point $\overline{u}$ of J corresponding to the critical value α_0 is a nontrivial mT-periodic solution of (1.1). In the following, we will verify the condition (J_2).

Take $e \in \partial B_1 \cap \widetilde{E}_m$, for any $z \in \overline{E}_m$ and $r > 0$, let $u = re + z$. Then

$$\begin{aligned}
J(u) &= \sum_{k=1}^{mT} r_{k-1}\Phi\left(\triangle^n u_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&= \sum_{k=1}^{mT} r_{k-1}\Phi\left(r\triangle^n e_{k-1}\right) - \sum_{k=1}^{mT} F(k, u_{k+1}, u_k) \\
&\leq \sum_{k=1}^{mT} r_{k-1}b_1\left|r\triangle^n e_{k-1}\right|^p + mTc_1' - \sum_{k=1}^{mT} b_2\left(\sqrt{u_{k+1}^2 + u_k^2}\right)^p + mTc_2' \\
&\leq b_1\overline{r}r^p d_{2,p}^p\left(\sum_{k=1}^{mT}\left|\triangle^n e_{k-1}\right|^2\right)^{\frac{p}{2}} - b_2\left[\left(\sum_{k=1}^{mT}\left(\sqrt{u_{k+1}^2 + u_k^2}\right)^p\right)^{\frac{1}{p}}\right]^p + mT\left(c_1' + c_2'\right)
\end{aligned}$$

$$\begin{aligned}
&\le b_1\bar{r}r^p d_{2,p}^p\left(y^{\mathrm{tr}}My\right)^{\frac{p}{2}} - b_2 d_{1,p}^p\left(2\|u\|^2\right)^{\frac{p}{2}} + mT\left(c_1' + c_2'\right)\\
&\le b_1\bar{r}r^p d_{2,p}^p\lambda_{\max}^{\frac{p}{2}}\|y\|^p - 2^{\frac{p}{2}} b_2 d_{1,p}^p\left(r^p + \|z\|^p\right) + mT\left(c_1' + c_2'\right)\\
&\le \left(b_1\bar{r}d_{2,p}^p\lambda_{\max}^{\frac{np}{2}} - 2^{\frac{p}{2}} b_2 d_{1,p}^p\right)r^p - 2^{\frac{p}{2}} b_2 d_{1,p}^p\|z\|^p + mT\left(c_1' + c_2'\right),
\end{aligned} \tag{4.8}$$

where $y = (\triangle^{n-1}e_1, \triangle^{n-1}e_2, \dots, \triangle^{n-1}e_{mT})^{\mathrm{tr}}$.

Since $\frac{b_1}{b_2}(\frac{d_{2,p}}{d_{1,p}})^p\frac{\bar{r}\lambda_{\max}^{\frac{np}{2}}}{2^{\frac{p}{2}}} < 1$, and $J(z) \le 0$ for $z \in \overline{E}_m$, thus there exists a constant $R_0 > \rho$ such that $J|_{\partial Q} \le 0$ where $Q = (\overline{B}_{R_0} \cap \widetilde{E}_m) \oplus \{re|0 < r < R_0\}$.

Case 3. Assume that $(H_{1,s})$ and $(H_{2,p})$ are satisfied. Similar to Case 1, by (4.6), we see that $\alpha_0 > 0$. Similar to Case 2, by (4.5), we see that there exists a constant $R_0 > \rho$ such that $J|_{\partial Q} \le 0$ where $Q = (\overline{B}_{R_0} \cap \widetilde{E}_m) \oplus \{re|0 < r < R_0\}$. We have also proved that J satisfies the condition (J_1) and (J_2) of the Linking Theorem.

Case 4. Assume that $(H_{2,s})$ and $(H_{1,p})$ are satisfied. Similar to Case 1, by (4.3), we see that $\alpha_0 > 0$. Similar to Case 2, by (4.8), we see that there exists a constant $R_0 > \rho$ such that $J|_{\partial Q} \le 0$ where $Q = (\overline{B}_{R_0} \cap \widetilde{E}_m) \oplus \{re|0 < r < R_0\}$. We have also proved that J satisfies the condition (J_1) and (J_2) of the Linking Theorem.

By one of the above four cases and the Linking Theorem, J possesses critical value $\alpha \ge \sigma > 0$. Moreover, α can be characterized as

$$\alpha = \inf_{h\in\Gamma}\sup_{x\in Q} J\left(h(x)\right),$$

where $\Gamma = \{h \in C(\overline{Q}, E_m) : h|_{\partial Q} = \mathrm{id}|_{\partial Q}\}$ and $\mathrm{id}|_{\partial Q}$ is the identity operator on ∂Q. Let $\widetilde{u} \in E_m$ be a critical point associated to the critical value α of J, *i.e.*, $J(\widetilde{u}) = \alpha$. If $\widetilde{u} \ne \overline{u}$, then The proof is complete. Otherwise, $\widetilde{u} = \overline{u}$. Then $\alpha_0 = J(\overline{u}) = J(\widetilde{u}) = \alpha$, *i.e.*, $\sup_{u\in E_m} J(u) = \inf_{h\in\Gamma}\sup_{x\in Q} J(h(x))$. Choosing $h = \mathrm{id}$, we have $\sup_{u\in Q} J(u) = \alpha_0$. Take $-e \in \partial B_1 \cap \widetilde{E}_m$. Similarly, there exists a positive number $R_1 > \rho$, $J|_{\partial Q_1} \le 0$, where $Q_1 = (\overline{B}_{R_1} \cap \widetilde{E}_m) \oplus \{-re|0 < r < R_1\}$. Again, by the Linking Theorem, J possesses a critical value $\alpha' \ge \sigma > 0$. Moreover, α' can be characterized as

$$\alpha' = \inf_{h\in\Gamma_1}\sup_{x\in Q_1} J\left(h(x)\right),$$

where $\Gamma_1 = \{h \in C(\overline{Q}_1, E_m) : h|_{\partial Q_1} = \mathrm{id}|_{\partial Q_1}\}$ and $\mathrm{id}|_{\partial Q_1}$ is the identity operator on ∂Q_1. If $\alpha' \ne \alpha_0$, then the proof is finished. If $\alpha' = \alpha_0$, then $\sup_{u\in Q_1} J(u) = \alpha_0$. Due to the fact that $J|_{\partial Q} \le 0$ and $J|_{\partial Q_1} \le 0$, J attains its maximum at some points in the interior of sets Q and Q_1. However, $Q \cap Q_1 \subset \overline{E}_m$ and $J|_{\overline{E}_m} \le 0$. Therefore, there must be a point $u' \in E_m$, $u' \ne \widetilde{u}$ and $J(u') = \alpha' = \alpha_0$.

The proof of Theorem 3.1 is complete. □

Proof of Corollary 3.1 By (Φ_1'), there exists constant $\epsilon_1 > 0$ such that $\Phi(u) \ge \frac{d}{2}|u|^\mu$, for $|u| \le \epsilon_1$. Hence (Φ_1') implies (Φ_1). By (Φ_2'), there exist constants $\delta_1 > 0$, $b_1 > 0$ and $c_1 \ge 0$ such that $\Phi(u) \le b_1|u|^\nu + c_1$, for $|u| \ge \delta_1$. So (Φ_2') implies (Φ_2).

By (F_2'), there exist constants $\epsilon_2 > 0$ and $a_2 > 0$ such that

$$F(k, v_1, v_2) \le a_2\left(\sqrt{v_1^2 + v_2^2}\right)^\theta, \quad \sqrt{v_1^2 + v_2^2} \le \epsilon_2.$$

So (F_2') implies (F_2).

By (F_3'), there exist constants $\delta_2 > 0$, $b_2 > 0$ and $c_2 > 0$ such that

$$F(k, v_1, v_2) \geq b_2\Big(\sqrt{v_1^2 + v_2^2}\Big)^{\vartheta} - c_2, \quad \sqrt{v_1^2 + v_2^2} \geq \delta_2.$$

So (F_3') implies (F_3). Since $\vartheta > \nu$, (F_3') implies $(H_{2,p})$.

If $\theta > \mu$, then (F_2') implies $(H_{2,s})$. If $\theta = \mu = s$, then by (F_2'), there exist constants $\epsilon_2' > 0$ and $a_2' = a_1(\frac{d_{1,s}}{d_{2,s}})^s \frac{r\lambda_{\min}^{\frac{ns}{2}}}{2^{\frac{s+2}{2}}}$ such that

$$F(k, v_1, v_2) \leq a_2'\Big(\sqrt{v_1^2 + v_2^2}\Big)^{s}, \quad \sqrt{v_1^2 + v_2^2} \leq \epsilon_2'.$$

we have $\frac{a_1}{a_2'}(\frac{d_{1,s}}{d_{2,s}})^s \frac{r\lambda_{\min}^{\frac{ns}{2}}}{2^{\frac{s}{2}}} = 2 > 1$. So, if $\theta = \mu = s$, then (F_2') implies $(H_{1,s})$.

So, by Theorem 3.1, Corollary 3.1 holds. □

5 Example

As an application of Theorem 3.1, we give an example to illustrate our result.

Example 5.1 For a given positive integer T, consider the following $2n$th-order difference equation:

$$\triangle^n\left(\frac{\triangle^n u_{k-1}}{\sqrt{1 + |\triangle^n u_{k-1}|^2}}\right) = (-1)^n f(k, u_{k+1}, u_k, u_{k-1}), \quad n \in Z(1), k \in Z, \tag{5.1}$$

where

$$f(k, v_1, v_2, v_3) = v_2\left(\left(2 + \cos\frac{2\pi k}{T}\right)\big(v_1^2 + v_2^2\big) + \left(2 + \cos\frac{2\pi(k-1)}{T}\right)\big(v_2^2 + v_3^2\big)\right).$$

Let

$$F(k, v_1, v_2) = \frac{2 + \cos\frac{2\pi k}{T}}{4}\big(v_1^2 + v_2^2\big)^2.$$

It is easy to verify that all the assumptions of Theorem 3.1 are satisfied. So, for any given positive integer m, (5.1) has at least three mT-periodic solutions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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