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Existence of three symmetric positive solutions for a second-order multi-point boundary value problem on time scales

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Abstract

In this article, we investigate the existence of at least three symmetric positive solutions of a second-order multi-point boundary value problem on time scales. The ideas involve Bai and Ge's fixed point theorem. As an application, we give an example to demonstrate our main result.

MSC: 34B10; 34B18; 39A10

Keywords: boundary value problem; fixed point theorem; symmetric positive solutions; time scales

1 Introduction

Calculus on time scales was introduced by Hilger [1] as a theory which includes both differential and difference calculus as a special cases. In the past few years, it has found a considerable amount of interest and attracted the attention of many researchers. Time scale calculus would allow the exploration of a variety of situations in economic, biological, heat transfer, stock market, and epidemic models; see the monographs of Aulbach and Hilger [2], Bohner and Peterson [3, 4], and Lakshmikantham *et al.* [5] and the references therein.

The study of multi-point linear boundary value problems was initiated by II'in and Moiseev [6, 7]. Since then, the more general nonlinear multi-point boundary value problems have been widely studied by many authors. The multi-point boundary value problems arise frequently in applied mathematics and physics, see for instance [8–16] and the references therein. At the same time, interest in obtaining the solutions on time scales has been on-going for several years.

On the other hand, the existence of symmetric positive solutions of second-order boundary value problems have been studied by some authors, see [17, 18]. Most of the study of the symmetric positive solution is limited to the Dirichlet boundary value problem, the Sturm-Liouville boundary value problem and the Neumann boundary value problem. However, there is not so much work on symmetric positive solutions for second-order m-point boundary value problems; see [19–21].

Yao [22] studied the following boundary value problem (BVP):

$$\begin{cases} x''(t) + \omega(t)f(x(t)) = 0, & t \in (0,1), \\ \alpha x(0) - \beta x'(0) = 0, & \alpha x(1) + \beta x'(1) = 0. \end{cases}$$

This author obtained the existence of n symmetric positive solutions and established a corresponding iterative scheme by using a monotone iterative technique.

Kosmatov [23, 24] studied the following BVP:

$$\begin{cases} -u''(t) = \alpha(t)f(t,u(t),|u'(t)|), & t \in (0,1), \\ u(0) = \sum_{i=1}^{n} \mu_i u(\xi_i), \qquad u(t) = u(t-1), & t \in [0,1]. \end{cases}$$

By using the Leggett-Wiliams fixed point theorem and the coincidence degree theorem of Mawhin, he studied the existence of three positive solutions for a multi-point boundary value problem.

Motivated by the results mentioned above, in this paper, we consider the following multi-point BVP:

$$\begin{cases} -(p(t)y^{\Delta}(t))^{\nabla} = f(t,y(t),y^{\Delta}(t)), \quad t \in (a,b), \\ -\alpha y(a) + \beta y^{[\Delta]}(a) = \sum_{i=1}^{m-2} \alpha_i y^{[\Delta]}(\xi_i), \\ \alpha y(b) + \beta y^{[\Delta]}(b) = \sum_{i=1}^{m-2} \alpha_i y^{[\Delta]}(\eta_i), \end{cases} \tag{1.1}$$

where $\mathbb{T} \subset \mathbb{R}$ be a symmetric bounded time scale, with $a = \min \mathbb{T}$, $b = \max \mathbb{T}$, $[a,b] \subset \mathbb{T}$ such that $[a,b] = \{t \in \mathbb{T} : a \leq t \leq b\}$ and $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$ is called the quasi-Δ-derivative of $y(t)$.

In the rest of this paper, we make the following assumptions:

(C1) $p \in C^{\nabla}([a,b],(0,\infty)), f \in C([a,b] \times (0,\infty) \times (-\infty,\infty),(0,\infty))$, $f(t,u,v) = f(b+a-t,u,v), f(t,u,v) = f(t,u,-v), p(t) = p(b+a-t)$;

(C2) $\alpha, \beta \in (0,\infty)$, $\alpha_i \in [0,\infty)$, with $\beta > \sum_{i=1}^{m-2} \alpha_i$, $\xi_i, \eta_i \in (a,b)$ such that $\xi_i = b+a-\eta_i$, for $i \in \{1,2,3,\ldots,m-2\}$.

By using Bai and Ge's fixed point theorem [25], we get the existence of at least three symmetric positive solutions for the BVP (1.1). In fact, our results are new when $\mathbb{T} = \mathbb{R}$ (the differential case) and $\mathbb{T} = \mathbb{Z}$. Hence, our new results naturally complement recent advances in the literature.

This paper is organized as follows. In Section 2, we provide some preliminary lemmas which are key tools for our main results. We give and prove our main results in Section 3. Finally, in Section 4, we give an example to demonstrate our results.

2 Preliminaries

In this section, we present auxiliary lemmas which will be used later.

Define by $\theta(t)$ and $\varphi(t)$ the solutions of the corresponding homogeneous equation

$$\big(p(t)y^{\Delta}(t)\big)^{\nabla} = 0, \quad t \in (a,b), \tag{2.1}$$

under the initial conditions

$$\begin{cases} \theta(a) = \beta, & p(a)\theta^{\Delta}(a) = \alpha, \\ \varphi(b) = \beta, & p(b)\varphi^{\Delta}(b) = -\alpha. \end{cases} \tag{2.2}$$

Using the initial conditions (2.2), we can deduce from (2.1) for $\theta(t)$ and $\varphi(t)$, the following equations:

$$\theta(t) = \beta + \alpha \int_a^t \frac{\Delta\tau}{p(\tau)}, \tag{2.3}$$

$$\varphi(t) = \beta + \alpha \int_t^b \frac{\triangle\tau}{p(\tau)}. \tag{2.4}$$

Denote $D := 2\alpha\beta + \alpha^2 \int_a^b \frac{\triangle\tau}{p(\tau)}$.

Lemma 2.1 *Suppose that the condition $D \neq 0$ holds. Then, for $f \in C([a,b] \times (0,\infty) \times (-\infty,\infty),(0,\infty))$, the BVP* (1.1) *has a unique solution*

$$y(t) = \int_a^b G(t,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s,$$

where $G(t,s)$ is the Green function for (1.1) *and is given by*

$$G(t,s) = \frac{1}{D}\begin{cases}\theta(t)\varphi(s), & a \le t \le s \le b,\\ \theta(s)\varphi(t), & a \le s \le t \le b,\end{cases} \tag{2.5}$$

where $\theta(t)$, $\varphi(t)$ are given in (2.3) *and* (2.4), *respectively, and $G_t{}^{[\triangle]}(t,s) := p(t)G_t{}^{\triangle}(t,s)$.*

Proof Let

$$\begin{aligned} y(t) = {} & \int_a^b G(t,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ & + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \end{aligned}$$

be a solution of (1.1), then we have

$$\begin{aligned} y(t) = {} & \frac{1}{D}\int_a^t \theta(s)\varphi(t) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ & + \frac{1}{D}\int_t^b \theta(t)\varphi(s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ & + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s, \\ p(t)y^{\triangle}(t) = {} & \frac{1}{D}p(t)\varphi^{\triangle}(t)\int_a^t \theta(s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ & + \frac{1}{D}p(t)\theta^{\triangle}(t)\int_t^b \varphi(s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ = {} & \int_a^b G_t{}^{[\triangle]}(t,s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s, \end{aligned}$$

and

$$\begin{aligned} \bigl(p(t)y^{\triangle}(t)\bigr)^{\nabla} = {} & \frac{1}{D}\bigl(p(t)\varphi^{\triangle}(t)\bigr)^{\nabla}\int_a^t \theta(s) f\bigl(s,y(s),y^{\triangle}(s)\bigr)\nabla s \\ & + \frac{1}{D}p\bigl(\rho(t)\bigr)\varphi^{\triangle}\bigl(\rho(t)\bigr)\theta(t) f\bigl(t,y(t),y^{\triangle}(t)\bigr) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{D}\big(p(t)\theta^{\triangle}(t)\big)^{\nabla}\int_t^b \varphi(s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s \\
&-\frac{1}{D}p\big(\rho(t)\big)\theta^{\triangle}\big(\rho(t)\big)\varphi(t)f\big(t,y(t),y^{\triangle}(t)\big) \\
=&-\frac{p(\rho(t))}{D}\big[-\varphi^{\triangle}\big(\rho(t)\big)\theta(t)+\theta^{\triangle}\big(\rho(t)\big)\varphi(t)\big]f\big(t,y(t),y^{\triangle}(t)\big) \\
=&-f\big(t,y(t),y^{\triangle}(t)\big).
\end{aligned}
$$

Since

$$
\begin{aligned}
y(a)&=\frac{1}{D}\int_a^b \theta(a)\varphi(s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s \\
&\quad+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t{}^{[\triangle]}(\eta_i,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s, \\
p(a)y^{\triangle}(a)&=p(a)\theta^{\triangle}(a)\int_a^b \frac{1}{D}\varphi(s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s,
\end{aligned}
$$

we have

$$
-\alpha y(a)+\beta p(a)y(a)=-\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t{}^{[\triangle]}(\eta_i,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s.
$$

For $t,s\in[a,b]$, $G_t^{[\triangle]}(t,s)=G_t^{[\triangle]}(b+a-t,b+a-s)$, we get

$$
-\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t{}^{[\triangle]}(\eta_i,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s=\sum_{i=1}^{m-2}\alpha_i y^{[\triangle]}(\xi_i).
$$

Similarly, we can see that the other boundary condition is satisfied. □

Lemma 2.2 *For $t,s\in[a,b]$, we have $G(b+a-t,b+a-s)=G(t,s)$.*

Proof In fact, if $t\le s$, then $1-t\ge 1-s$. In view of (2.5) and the assumption (C1), we get

$$
\begin{aligned}
G(b+a-t,b+a-s)&=\frac{1}{D}\left(\beta+\alpha\int_a^{b+a-s}\frac{1}{p(\tau)}\triangle\tau\right)\left(\beta+\alpha\int_{b+a-t}^b\frac{1}{p(\tau)}\triangle\tau\right) \\
&=\frac{1}{D}\left(\beta+\alpha\int_s^b\frac{1}{p(\tau)}\triangle\tau\right)\left(\beta+\alpha\int_a^t\frac{1}{p(\tau)}\triangle\tau\right) \\
&=G(t,s),\quad a\le t\le s\le b.
\end{aligned}
$$

Similarly, we can prove that $G(b+a-t,b+a-s)=G(t,s)$, $a\le s\le t\le b$.

So, we have $G(b+a-t,b+a-s)=G(t,s)$ for all $(t,s)\in[a,b]\times[a,b]$, *i.e.*, $G(t,s)$ is a symmetric function on $[a,b]\times[a,b]$. □

Lemma 2.3 *Let* (C1) *and* (C2) *hold. Then the unique solution y of the BVP* (1.1) *satisfies*

$$
y(t)\ge 0\quad \textit{for } t\in[a,b].
$$

Proof By using conditions (C1), (C2) and definition of $y(t)$, we get

$$\begin{aligned} y(t) &= \frac{1}{D}\int_a^t \theta(s)\left[\varphi(t) - \sum_{i=1}^{m-2}\alpha_i\right] f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\quad + \frac{1}{D}\int_t^b \varphi(s)\left[\theta(t) + \sum_{i=1}^{m-2}\alpha_i\right] f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\geq 0. \end{aligned}$$

The proof is complete. □

Lemma 2.4 *Suppose that* (C1), (C2) *hold, then* $\min_{t\in[a,b]} y(t) \geq \Gamma \max_{t\in[a,b]} y(t)$ *where* $\Gamma = \frac{\beta}{\beta+\alpha\int_a^b \frac{\triangle\tau}{p(\tau)}} < 1$.

Proof We have from (2.5) $0 \leq G(t,s) \leq G(s,s)$ for $t,s \in [a,b]$.

For $a \leq t \leq s \leq b$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\theta(t)}{\theta(s)} \geq \frac{\theta(a)}{\theta(b)} = \frac{\beta}{\theta(b)}.$$

For $a \leq s \leq t \leq b$,

$$\frac{G(t,s)}{G(s,s)} = \frac{\varphi(t)}{\varphi(s)} \geq \frac{\varphi(t)}{\varphi(b)} = \frac{\beta}{\varphi(a)}.$$

From the definitions of φ and θ, we obtain $\min_{t\in[a,b]} G(t,s) \geq \Gamma G(s,s)$. Hence,

$$\begin{aligned} y(t) &= \int_a^b G(t,s) f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\quad + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\geq \int_a^b \Gamma G(s,s) f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\quad + \Gamma\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\bigl(s, y(s), y^{\triangle}(s)\bigr)\nabla s \\ &\geq \Gamma \max_{t\in[a,b]} y(t). \end{aligned}$$

The proof is complete. □

Lemma 2.5 *For* $t,s \in [a,b]$, $G_t^{[\triangle]}(t,s) \leq \frac{\alpha}{\beta} G(s,s)$.

Proof One can easily see that the inequality holds. □

Let $\mathbb{B} = C[a,b]$ be a Banach space with the norm

$$\|y\| = \max\Bigl\{\max_{t\in[a,b]} \bigl|y(t)\bigr|, \max_{t\in[a,b]} \bigl|y^{[\triangle]}(t)\bigr|\Bigr\}.$$

Define the cone $P \subset \mathbb{B}$ by

$$P = \Big\{ y \in \mathbb{B} : y(t) \geq 0, y \text{ is symmetric on } [a,b], \min_{t\in[a,b]} y(t) \geq \Gamma \max_{t\in[a,b]} y(t) \Big\}.$$

We define an operator $T : P \to \mathbb{B}$,

$$\begin{aligned}(Ty)(t) = {} & \int_a^b G(t,s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \\ & + \frac{1}{\alpha} \sum_{i=1}^{m-2} \alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s. \end{aligned} \tag{2.6}$$

Lemma 2.6 *Let* (C1), (C2) *hold. Then* $T : P \to P$ *is completely continuous.*

Proof For all $y \in P$, by (C1), (C2) and (2.6), $(Ty)(t) \geq 0$, for all $t \in [a,b]$. Furthermore, by (2.5) and $\min_{t\in[a,b]} G(t,s) \geq \Gamma G(s,s)$.

$$\begin{aligned} Ty(t) \geq {} & \int_a^b \Gamma G(s,s) f\big(s, y(s), y^{\triangle}\big) \nabla s \\ & + \frac{1}{\alpha} \sum_{i=1}^{m-2} \alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \\ \geq {} & \Gamma \Bigg(\int_a^b G(s,s) f\big(s, y(s), y^{\triangle}\big) \nabla s \\ & + \frac{1}{\alpha} \sum_{i=1}^{m-2} \alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \Bigg) \\ \geq {} & \Gamma \max_{t\in[a,b]} (Ty)(t), \end{aligned}$$

i.e., $\min_{t\in[a,b]} (Ty)(t) \geq \Gamma \max_{t\in[a,b]} (Ty)(t)$.

Noticing that $p(t)$, $y(t)$, $f(s, y(s), y^{\triangle}(s))$ are symmetric on $[a,b]$, and by Lemma 2.2, we have

$$\begin{aligned} Ty(b+a-t) = {} & \int_a^b G(b+a-t, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \\ & + \frac{1}{\alpha} \sum_{i=1}^{m-2} \alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \\ = {} & \int_b^a \big(G(b+a-t, b+a-s) f\big(b+a-s, y(b+a-s), y^{\triangle}(b+a-s)\big)\big) \\ & \times \nabla(b+a-s) + \frac{1}{\alpha} \sum_{i=1}^{m-2} \alpha_i \int_a^b G_t{}^{[\triangle]}(\eta_i, s) f\big(s, y(s), y^{\triangle}(s)\big) \nabla s \\ = {} & Ty(t), \end{aligned}$$

i.e., $Ty(b+a-t) = Ty(t)$ for $t \in [a,b]$. Therefore, $Ty(t)$ is symmetric on $[a,b]$.

So, $Ty \in P$ and then $Ty \subset P$. Next, by standard methods and the Arzela-Ascoli theorem, one can easily prove that the operator T is completely continuous. □

3 Main results

We are now ready to apply the fixed point theorem due to Bai and Ge [25] to the operator T in order to get sufficient conditions for the existence of multiple positive solutions to the problem (1.1).

Suppose that $\mu, \gamma : P \to [0, \infty)$ are two nonnegative continuous convex functionals satisfying

$$\|y\| \leq M \max\{\mu(y), \gamma(y)\}, \quad \text{for all } y \in P, \tag{3.1}$$

where M is a positive constant, and

$$\Omega = \{y \in P : \mu(y) < r, \gamma(y) < L\} \neq \emptyset, \quad \text{for any } r > 0, L > 0. \tag{3.2}$$

With (3.1) and (3.2), Ω is a bounded nonempty open subset in P.

Let $r > c > 0$, $L > 0$ be given. Let $\mu, \gamma : P \to [0, \infty)$ be two nonnegative continuous convex functionals satisfying (3.1) and (3.2), and ρ be a nonnegative continuous concave functional on the cone P. Define bounded convex sets

$$\begin{aligned}
&P(\mu, r; \gamma, L) = \{y \in P : \mu(y) < r, \gamma(y) < L\},\\
&\overline{P}(\mu, r; \gamma, L) = \{y \in P : \mu(y) \leq r, \gamma(y) \leq L\},\\
&P(\mu, r; \gamma, L; \rho, c) = \{y \in P : \mu(y) < r, \gamma(y) < L, \rho(y) > c\},\\
&\overline{P}(\mu, r; \gamma, L; \rho, c) = \{y \in P : \mu(y) \leq r, \gamma(y) \leq L, \rho(y) \geq c\}.
\end{aligned}$$

To prove our results, we need the following fixed point theorem due to Bai and Ge in [25].

Theorem 3.1 [25] *Let $\mathbb{B}$ be a Banach space, $P \subset \mathbb{B}$ be a cone and $r_2 \geq d > e > r_1 > 0$, $L_2 \geq L_1 > 0$ be given. Assume that μ, γ are nonnegative continuous convex functionals on P, such that* (3.1) *and* (3.2) *are satisfied, ρ is a nonnegative continuous concave functional on P, such that $\rho(y) \leq \mu(y)$ for all $y \in \overline{P}(\mu, r_2; \gamma, L_2)$ and let $T : \overline{P}(\mu, r_2; \gamma, L_2) \to \overline{P}(\mu, r_2; \gamma, L_2)$ be a completely continuous operator. Suppose*

(A1) $\{y \in \overline{P}(\mu, d; \gamma, L_2; \rho, e) : \rho(y) > e\} \neq \emptyset$, *$\rho(Ty) > e$ for $y \in \overline{P}(\mu, d; \gamma, L_2; \rho, e)$,*
(A2) *$\mu(Ty) < r_1$, $\gamma(Ty) < L_1$, for all $y \in \overline{P}(\mu, r_1; \gamma, L_1)$,*
(A3) *$\rho(Ty) > e$, for all $y \in \overline{P}(\mu, r_2; \gamma, L_2; \rho, e)$ with $\mu(Ty) > e$.*

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}(\mu, r_2; \gamma, L_2)$ with

$$\begin{aligned}
&y_1 \in P(\mu, r_1; \gamma, L_1), \qquad y_2 \in \{\overline{P}(\mu, r_2; \gamma, L_2; \rho, e) : \rho(y) > e\},\\
&y_3 \in \overline{P}(\mu, r_2; \gamma, L_2) \setminus (\overline{P}(\mu, r_2; \gamma, L_2; \rho, e) \cup \overline{P}(\mu, r_1; \gamma, L_1)).
\end{aligned}$$

Define nonnegative continuous functionals μ, γ, and ρ by

$$\mu(y) = \max_{t \in [a,b]} |y(t)|, \qquad \gamma(y) = \max_{t \in [a,b]} |y^{[\Delta]}(t)|, \qquad \rho(y) = \min_{t \in [a,b]} |y(t)|, \quad \text{for } y \in P.$$

Then on the cone P, ρ is a concave functional, and μ and γ are convex functionals satisfying (3.1) and (3.2).

For the sake of convenience, we introduce the following notations:

$$A = \int_a^b G(s,s)\nabla s + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t^{[\triangle]}(\eta_i,s)\nabla s,$$

$$B = \frac{\alpha}{\beta}\int_a^b G(s,s)\nabla s.$$

Theorem 3.2 *Assume that* (C1), (C2) *hold and that there exist constants* $r_2 \geq \frac{e}{\Gamma} > e > r_1 > 0$, $L_2 \geq L_1 > 0$ *such that* $\frac{e}{\Gamma A} \leq \min\{\frac{r_2}{A}, \frac{L_2}{B}\}$ *and the following conditions hold*:

(S1) $f(t,u,v) < \min\{\frac{r_1}{A}, \frac{L_1}{B}\}$, *for* $(t,u,v) \in [a,b] \times [0,r_1] \times [-L_1, L_1]$,

(S2) $f(t,u,v) > \frac{e}{\Gamma A}$, *for* $(t,u,v) \in [a,b] \times [e, \frac{e}{\Gamma}] \times [-L_2, L_2]$,

(S3) $f(t,u,v) \leq \min\{\frac{r_2}{A}, \frac{L_2}{B}\}$, *for* $(t,u,v) \in [a,b] \times [0,r_2] \times [-L_2, L_2]$.

Then the BVP (1.1) *has at least three symmetric positive solutions* y_1, y_2, y_3 *satisfying*

$$\max_{t\in[a,b]} y_1(t) \leq r_1, \qquad \max_{t\in[a,b]} \left|y_1^{\triangle}(t)\right| \leq L_1;$$

$$e < \max_{t\in[a,b]} y_2(t) \leq r_2, \qquad \max_{t\in[a,b]} \left|y_2^{\triangle}(t)\right| \leq L_2;$$

$$\max_{t\in[a,b]} y_3(t) \leq \frac{e}{\Gamma}, \qquad \max_{t\in[a,b]} \left|y_3^{\triangle}(t)\right| \leq L_2.$$

Proof The problem (1.1) has a solution $y = y(t)$ if and only if y satisfies the operator equation $y = Ty$. Thus we set out to verify that the operator T satisfies all conditions of Theorem 3.1. The proof is divided into four steps.

Step 1. First we show that

$$T : \overline{P}(\mu, r_2; \gamma, L_2) \to \overline{P}(\mu, r_2; \gamma, L_2).$$

In fact, if $y \in \overline{P}(\mu, r_2; \gamma, L_2)$, then there is $\mu(y) \leq r_2$, $\gamma(y) \leq L_2$. From assumption (S3),

$$\begin{aligned}
\mu(Ty) &= \max_{t\in[a,b]}\left|\int_a^b G(t,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s \right.\\
&\quad \left. + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t^{[\triangle]}(\eta_i,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s\right|\\
&\leq \int_a^b G(s,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t^{[\triangle]}(\eta_i,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s\\
&\leq \frac{r_2}{A}\left[\int_a^b G(s,s)\nabla s + \frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i \int_a^b G_t^{[\triangle]}(\eta_i,s)\nabla s\right]\\
&= r_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\gamma(Ty) &= \max_{t\in[a,b]}\left|\int_a^b G_t^{[\triangle]}(t,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s\right|\\
&\leq \frac{\alpha}{\beta}\int_a^b G(s,s) f\big(s,y(s),y^{\triangle}(s)\big)\nabla s
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{L_2}{B}\frac{\alpha}{\beta}\int_a^b G(s,s)\nabla s \\
&= L_2.
\end{aligned}$$

So $T:\overline{P}(\mu,r_2;\gamma,L_2)\to\overline{P}(\mu,r_2;\gamma,L_2)$ holds.

Step 2. We show that condition (A1) in Theorem 3.1 holds.

We choose $y(t)=\frac{e}{2}(1+\frac{1}{\Gamma})$. It is easy to see that $y(t)\in\overline{P}(\mu,\frac{e}{\Gamma};\gamma,L_2;\rho,e)$ and $\rho(y)>e$ and so $\{y\in\overline{P}(\mu,\frac{e}{\Gamma};\gamma,L_2;\rho,e):\rho(y)>e\}\neq\emptyset$. Hence if $y\in\overline{P}(\mu,\frac{e}{\Gamma};\gamma,L_2;\rho,e)$, then $e\leq y(t)\leq\frac{e}{\Gamma}$ for $t\in[a,b]$. From assumption (S2), we have $f(t,y(t),y^{\triangle}(t))>\frac{e}{\Gamma A}$ for $t\in[a,b]$,

$$\begin{aligned}
\rho(Ty) &= \min_{t\in[a,b]}\left|\int_a^b G(t,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t^{[\triangle]}(\eta_i,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s\right| \\
&\geq \Gamma\int_a^b G(s,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t^{[\triangle]}(\eta_i,s)f\big(s,y(s),y^{\triangle}(s)\big)\nabla s \\
&\geq \Gamma\frac{e}{\Gamma A}\left[\int_a^b G(s,s)\nabla s+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t^{[\triangle]}(\eta_i,s)\nabla s\right] \\
&= e.
\end{aligned}$$

Consequently, condition (A1) of Theorem 3.1 is satisfied.

Step 3. We now show that condition (A2) in Theorem 3.1 is satisfied. In the same way as in Step 1, if $y\in\overline{P}(\mu,r_1;\gamma,L_1)$, then assumption (S1) yields $f(t,y(t),y^{\triangle}(t))<\min(\frac{r_1}{A},\frac{L_1}{B})$ for $(t,u,v)\in[a,b]\times[0,r_1]\times[-L_1,L_1]$. Therefore, condition (A2) of Theorem 3.1 is satisfied.

Step 4. Finally, we show that condition (A3) in Theorem 3.1 is also satisfied. Suppose that $y\in\overline{P}(\mu,r_2;\gamma,L_2;\rho,\frac{e}{\Gamma})$ with $\mu(Ty)>\frac{e}{\Gamma}$. Then

$$\rho(Ty)=\min_{t\in[a,b]}(Ty)(t)\geq\Gamma\max_{t\in[a,b]}(Ty)(t)>\frac{\Gamma e}{\Gamma}=e.$$

Consequently, by Theorem 3.1, the problem (1.1) has at least three positive solutions y_1, y_2, y_3 in $\overline{P}(\mu,r_2;\gamma,L_2)$ with

$$\begin{aligned}
&y_1\in P(\mu,r_1;\gamma,L_1),\qquad y_2\in\big\{\overline{P}(\mu,r_2;\gamma,L_2;\rho,e):\rho(y)>e\big\}, \\
&y_3\in\overline{P}(\mu,r_2;\gamma,L_2)\setminus\big(\overline{P}(\mu,r_2;\gamma,L_2;\rho,e)\cup\overline{P}(\mu,r_1;\gamma,L_1)\big).
\end{aligned}$$

The proof is complete. □

4 Example

To illustrate how our main result can be used in practice we present an example.

Example 4.1 Let $\mathbb{T}=[0,1]$, $a=0$, $b=1$, $\alpha_1=\frac{1}{10}$, $\alpha_2=\frac{1}{5}$, $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, $\eta_1=\frac{1}{3}$, $\eta_2=\frac{3}{4}$, $\xi_1=\frac{2}{3}$, $\xi_2=\frac{1}{4}p(t)=1$, $m=4$, in the boundary value problem (1.1). Now we consider the following problem:

$$\begin{cases}
-y''(t)=f(t,y(t),y'(t)), & t\in(0,1), \\
-\frac{1}{2}y(0)+\frac{1}{3}y'(0)=\frac{1}{10}y'(\frac{2}{3})+\frac{1}{5}y'(\frac{1}{4}), \\
\frac{1}{2}y(1)+\frac{1}{3}y'(1)=\frac{1}{10}y'(\frac{1}{3})+\frac{1}{5}y'(\frac{3}{4}),
\end{cases}\tag{4.1}$$

where

$$f\big(t,y(t),y^{\triangle}(t)\big)=\begin{cases}1.8y(t), & y\in[0,1],\\ 18.2y(t)-16.4, & y\in(1,2),\\ 20, & y\in[2,10).\end{cases}$$

Set $r_1=1$, $e=2$, $r_2=10$, $L_1=23$, $L_2=46$, $\mathbb{T}=[0,1]$, then $\Delta s=\nabla s=ds$. We get $\Gamma=\frac{2}{5}$; then $\frac{e}{\Gamma}=5$ and $D=\frac{7}{12}$ and by a simple calculation

$$\begin{aligned}G(s,s)&=\frac{1}{D}\theta(s)\varphi(s)=\frac{1}{D}\left(\beta+\alpha\int_a^s\frac{d\tau}{p(\tau)}\right)\left(\beta+\alpha\int_s^b\frac{d\tau}{p(\tau)}\right)\\&=\frac{12}{7}\left(\frac{1}{3}+\frac{1}{2}s\right)\left(\frac{1}{3}+\frac{1}{2}(1-s)\right),\end{aligned}$$

$$G_t^{[\triangle]}(\eta_i,s)=p(t)G_t^{\triangle}(\eta_i,s)=G_t^{\triangle}(\eta_i,s)=G_t'(\eta_i,s),$$

$$G_t'(\eta_i,s)=\frac{1}{D}\begin{cases}\theta'(t)\varphi(s), & a\le\eta_i\le s\le b,\\ \theta(s)\varphi'(t), & a\le s\le\eta_i\le b\end{cases}=\frac{1}{D}\begin{cases}\alpha\varphi(s), & a\le\eta_i\le s\le b,\\ -\alpha\theta(s), & a\le s\le\eta_i\le b.\end{cases}$$

Thus,

$$B=\frac{\alpha}{\beta}\int_a^b G(s,s)\nabla s=\frac{3}{2}\int_0^1\frac{12}{7}\left(\frac{1}{3}+\frac{1}{2}s\right)\left(\frac{1}{3}+\frac{1}{2}(1-s)\right)ds=\frac{23}{28},$$

$$\begin{aligned}A&=\int_a^b G(s,s)\nabla s+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t^{[\triangle]}(\eta_i,s)\nabla s\\&=\int_a^b G(s,s)\,ds+\frac{1}{\alpha}\sum_{i=1}^{m-2}\alpha_i\int_a^b G_t'(\eta_i,s)\,ds\\&=\int_a^b G(s,s)\,ds+\frac{1}{\alpha}\alpha_1\int_0^1 G_t'(\eta_1,s)\,ds+\frac{1}{\alpha}\alpha_2\int_0^1 G_t'(\eta_2,s)\,ds\\&=\int_0^1\frac{12}{7}\left(\frac{1}{3}+\frac{1}{2}s\right)\left(\frac{1}{3}+\frac{1}{2}(1-s)\right)ds\\&\quad+2\frac{1}{10}\int_0^1 G_t'\left(\frac{1}{3},s\right)ds+2\frac{1}{5}\int_0^1 G_t'\left(\frac{3}{4},s\right)ds\\&=\int_0^1\frac{12}{7}\left(\frac{1}{3}+\frac{1}{2}s\right)\left(\frac{1}{3}+\frac{1}{2}(1-s)\right)ds\\&\quad+\frac{17}{10}\left[\int_0^{\frac{1}{3}}\frac{-1}{2}\left(\frac{1}{3}+\frac{s}{2}\right)ds+\int_{\frac{1}{3}}^1\frac{1}{2}\left(\frac{1}{3}+\frac{1-s}{2}\right)ds\right]\\&\quad+\frac{17}{5}\left[\int_0^{\frac{3}{4}}\frac{-1}{2}\left(\frac{1}{3}+\frac{s}{2}\right)ds+\int_{\frac{3}{4}}^1\frac{1}{2}\left(\frac{1}{3}+\frac{1-s}{2}\right)ds\right]\\&=\frac{101}{210}.\end{aligned}$$

Hence we have

$$f\big(t,y(t),y'(t)\big)<\frac{210}{101},\quad\text{for } y\in[0,1],$$

$$f\big(t, y(t), y'(t)\big) > \frac{1{,}050}{101}, \quad \text{for } y \in [2,5],$$
$$f\big(t, y(t), y'(t)\big) < \frac{2{,}100}{101}, \quad \text{for } y \in [0,10].$$

Then the conditions of Theorem 3.2 are satisfied. Then by Theorem 3.2, the BVP (1.1) has at least three symmetric positive solutions y_1, y_2, y_3, with

$$\max_{t\in[a,b]} y_1(t) \le 1, \qquad \max_{t\in[a,b]} \big|y_1'(t)\big| \le 23;$$
$$2 < \max_{t\in[a,b]} y_2(t) \le 10, \qquad \max_{t\in[a,b]} \big|y_2'(t)\big| \le 46;$$
$$\max_{t\in[a,b]} y_3(t) \le 5, \qquad \max_{t\in[a,b]} \big|y_3'(t)\big| \le 46.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read and approved the final manuscript.

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