Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Separated boundary value problems for second-order impulsive q-integro-difference equations

Chatthai Thaiprayoon[1], Jessada Tariboon[1*] and Sotiris K Ntouyas[2]

*Correspondence: jessadat@kmutnb.ac.th
[1] Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology, North Bangkok, Bangkok, Thailand
Full list of author information is available at the end of the article

Abstract
This article studies the existence and uniqueness of solutions for a boundary value problem of nonlinear second-order impulsive q_k-integro-difference equations with separated boundary conditions. Several new results are obtained by applying a variety of fixed point theorems. Some examples are presented to illustrate the results.
MSC: 26A33; 39A13; 34A37

Keywords: q_k-derivative; q_k-integral; impulsive q_k-difference equation; existence; uniqueness; fixed point theorems

1 Introduction

In this paper, we study the separated boundary value problem for impulsive q_k-integro-difference equation of the following form:

$$\begin{cases} D_{q_k}^2 x(t) = f(t, x(t), (S_{q_k}x)(t)), & t \in J := [0,T], t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1,2,\dots,m, \\ D_{q_k}x(t_k^+) - D_{q_{k-1}}x(t_k) = I_k^*(x(t_k)), & k = 1,2,\dots,m, \\ x(0) + D_{q_0}x(0) = 0, & x(T) + D_{q_m}x(T) = 0, \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^2 \to \mathbb{R}$,

$$(S_{q_k}x)(t) = \int_{t_k}^{t} \phi(t,s)x(s)\,d_{q_k}s, \quad t \in (t_k, t_{k+1}], k = 0,1,2,\dots,m, \tag{1.2}$$

$\phi : J \times J \to [0,\infty)$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,\dots,m$, $x(t_k^+) = \lim_{h\to 0} x(t_k + h)$ and $0 < q_k < 1$ for $k = 0,1,2,\dots,m$.

The notions of q_k-derivative and q_k-integral on finite intervals were introduced in [1]. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 1.1 Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, \quad t \neq t_k, \qquad D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t) \tag{1.3}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k}f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (1.3), then $D_{q_k}f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}. \tag{1.4}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 1.2 Let $f : J_k \to \mathbb{R}$ be a continuous function, we call the second-order q_k-derivative $D^2_{q_k}f$ provided $D_{q_k}f$ is q_k-differentiable on J_k with $D^2_{q_k}f = D_{q_k}(D_{q_k}f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D^n_{q_k} : J_k \to \mathbb{R}$.

The q_k-integral is defined as follows.

Definition 1.3 Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\,d_{q_k}s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \tag{1.5}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned}\int_a^t f(s)\,d_{q_k}s &= \int_{t_k}^t f(s)\,d_{q_k}s - \int_{t_k}^a f(s)\,d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \\ &\quad - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n a + \bigl(1-q_k^n\bigr)t_k\bigr).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (1.5) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)\,d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0,\infty)$.

For the basic properties of q_k-derivative and q_k-integral we refer to [1].

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [16–18].

In this paper we prove an existence and uniqueness result for the impulsive boundary value problem (1.1) by using Banach's contraction mapping principle and three existence results by applying Schaefer's, Krasnoselskii's fixed point theorems and the Leray-Schauder Nonlinear Alternative. The rest of this paper is organized as follows: In Section 2 we present an auxiliary lemma which is used to convert the impulsive boundary value problem (1.1) into an equivalent integral equation. The main results are given in Section 3, while examples illustrating the results are presented in Section 4.

2 An auxiliary lemma

Let $J = [0, T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \dots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \dots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

We now consider the following linear case:

$$\begin{cases} D_{q_k}^2 x(t) = h(t), \quad t \in J, t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) + D_{q_0} x(0) = 0, \qquad x(T) + D_{q_m} x(T) = 0, \end{cases} \tag{2.1}$$

where $h : J \to \mathbb{R}$ is a continuous function.

Lemma 2.1 *The unique solution of problem* (2.1) *is given by*

$$\begin{aligned} x(t) = {} & \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} h(r)\, d_{q_{k-1}} r\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\ & + \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) (T - t_k + 1) \\ & + \left(\frac{1-t}{T}\right) \int_{t_m}^{T} \int_{t_m}^{s} h(r)\, d_{q_m} r\, d_{q_m} s + \left(\frac{1-t}{T}\right) \int_{t_m}^{T} h(s)\, d_{q_m} s \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} h(r)\, d_{q_{k-1}} r\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) \right) (t - t_k) + \int_{t_k}^{t} \int_{t_k}^{s} h(r)\, d_{q_k} r\, d_{q_k} s, \end{aligned} \tag{2.2}$$

with $\sum_{i=a}^{b}(\cdot) = 0$ *for* $a > b$.

Proof Taking the q_0-integral for the first equation of (2.1), for $t \in J_0$, we have

$$D_{q_0} x(t) = D_{q_0} x(0) + \int_0^t h(s)\, d_{q_0} s, \tag{2.3}$$

which leads to

$$D_{q_0} x(t_1) = D_{q_0} x(0) + \int_0^{t_1} h(s)\, d_{q_0} s. \tag{2.4}$$

For $t \in J_0$ we get by q_0-integrating (2.3),

$$\begin{aligned} x(t) &= x(0) + D_{q_0} x(0) t + \int_0^t \int_0^s h(r)\, d_{q_0} r\, d_{q_0} s \\ &:= A + Bt + \int_0^t \int_0^s h(r)\, d_{q_0} r\, d_{q_0} s, \quad \text{if } A = x(0), B = D_{q_0} x(0). \end{aligned}$$

In particular, for $t = t_1$, we obtain

$$x(t_1) = A + Bt_1 + \int_0^{t_1} \int_0^s h(r)\, d_{q_0} r\, d_{q_0} s. \tag{2.5}$$

For $t \in J_1 = (t_1, t_2]$, q_1-integrating (2.1), we have

$$D_{q_1} x(t) = D_{q_1} x\left(t_1^+\right) + \int_{t_1}^t h(s)\, d_{q_1} s.$$

Using the third condition of (2.1) with (2.4), it follows that

$$D_{q_1} x(t) = B + \int_0^{t_1} h(s)\, d_{q_0} s + I_1^*\left(x(t_1)\right) + \int_{t_1}^t h(s)\, d_{q_1} s. \tag{2.6}$$

Taking q_1-integral to (2.6) for $t \in J_1$, we obtain

$$\begin{aligned} x(t) &= x\left(t_1^+\right) + \left[B + \int_0^{t_1} h(s)\, d_{q_0} s + I_1^*\left(x(t_1)\right)\right](t - t_1) \\ &\quad + \int_{t_1}^t \int_{t_1}^s h(r)\, d_{q_1} r\, d_{q_1} s. \end{aligned} \tag{2.7}$$

Applying the second equation of (2.1) with (2.5) and (2.7), we get

$$\begin{aligned} x(t) &= A + Bt_1 + \int_0^{t_1} \int_0^s h(r)\, d_{q_0} r\, d_{q_0} s + I_1\left(x(t_1)\right) \\ &\quad + \left[B + \int_0^{t_1} h(s)\, d_{q_0} s + I_1^*\left(x(t_1)\right)\right](t - t_1) + \int_{t_1}^t \int_{t_1}^s h(r)\, d_{q_1} r\, d_{q_1} s \\ &= A + Bt + \int_0^{t_1} \int_0^s h(r)\, d_{q_0} r\, d_{q_0} s + I_1\left(x(t_1)\right) \\ &\quad + \left[\int_0^{t_1} h(s)\, d_{q_0} s + I_1^*\left(x(t_1)\right)\right](t - t_1) + \int_{t_1}^t \int_{t_1}^s h(r)\, d_{q_1} r\, d_{q_1} s. \end{aligned}$$

Repeating the above process, for $t \in J_k$, we get

$$\begin{aligned} x(t) &= A + Bt \\ &\quad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^s h(r)\, d_{q_{k-1}} r\, d_{q_{k-1}} s + I_k\left(x(t_k)\right) \right) \\ &\quad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s + I_k^*\left(x(t_k)\right) \right)(t - t_k) \\ &\quad + \int_{t_k}^t \int_{t_k}^s h(r)\, d_{q_k} r\, d_{q_k} s. \end{aligned} \tag{2.8}$$

From the first boundary condition of (2.1) (*i.e.* $x(0) + D_{q_0} x(0) = 0$) and (2.8), we have

$$A + B = 0. \tag{2.9}$$

Also, the second boundary condition of (2.1) (*i.e.* $x(T) + D_{q_m}x(T) = 0$) and (2.8), yields

$$A + B(T+1) + \sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} h(r)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) + \sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} h(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k + 1) + \int_{t_m}^{T}\int_{t_m}^{s} h(r)\,d_{q_m}r\,d_{q_m}s + \int_{t_m}^{T} h(s)\,d_{q_m}s = 0. \tag{2.10}$$

From (2.9) and (2.10), we have that

$$B = -\frac{1}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} h(r)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) - \frac{1}{T}\sum_{i=1}^{m}\left(\int_{t_{k-1}}^{t_k} h(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k + 1) - \frac{1}{T}\int_{t_m}^{T}\int_{t_m}^{s} h(r)\,d_{q_m}r\,d_{q_m}s - \frac{1}{T}\int_{t_m}^{T} h(s)\,d_{q_m}s,$$

which implies

$$A = \frac{1}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} h(r)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) + \frac{1}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} h(s)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k + 1) + \frac{1}{T}\int_{t_m}^{T}\int_{t_m}^{s} h(r)\,d_{q_m}r\,d_{q_m}s + \frac{1}{T}\int_{t_m}^{T} h(s)\,d_{q_m}s.$$

Substituting constants A and B into (2.8), we obtain (2.2) as requested. □

3 Main results

In view of Lemma 2.1, we define an operator $\mathcal{A} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}(\mathcal{A}x)(t) = {} & \left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r, x(r), (S_{q_{k-1}}x)(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\ & + \left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), (S_{q_{k-1}}x)(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T - t_k + 1) \\ & + \left(\frac{1-t}{T}\right)\int_{t_m}^{T}\int_{t_m}^{s} f\big(r, x(r), (S_{q_m}x)(r)\big)\,d_{q_m}r\,d_{q_m}s \\ & + \left(\frac{1-t}{T}\right)\int_{t_m}^{T} f\big(s, x(s), (S_{q_m}x)(s)\big)\,d_{q_m}s \\ & + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r, x(r), (S_{q_{k-1}}x)(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right)\end{aligned}$$

$$+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s),(S_{q_{k-1}}x)(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(t-t_k)$$
$$+\int_{t_k}^{t}\int_{t_k}^{s} f\big(r,x(r),(S_{q_k}x)(r)\big)\,d_{q_k}r\,d_{q_k}s. \tag{3.1}$$

It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

Our first result is an existence and uniqueness result for the impulsive boundary value problem (1.1) by using the Banach contraction mapping principle.

Let $\phi_0 = \max\{\phi(t,s) : (t,s) \in J \times J\}$. Further, for convenience we set

$$\begin{aligned}\omega = {}& \frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{L_1(t_k-t_{k-1})^2}{1+q_{k-1}} + \frac{\phi_0 L_2(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}\right] \\ &+ \frac{1+T}{T}\sum_{k=1}^{m+1}\left[L_1(t_k-t_{k-1}) + \frac{\phi_0 L_2(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\ &+ \frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[L_1(t_k-t_{k-1}) + \frac{\phi_0 L_2(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\ &+ \frac{mL_3(1+2T)}{T} + \frac{mL_4(1+T)}{T} + \frac{L_4(1+2T)}{T}\sum_{k=1}^{m}(T-t_k),\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\lambda_0 = {}& \frac{M_1(1+2T)}{T}\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + \frac{M_1(1+T)}{T}\sum_{k=1}^{m+1}(t_k-t_{k-1}) \\ &+ \frac{M_1(1+2T)}{T}\sum_{k=1}^{m}(T-t_k)(t_k-t_{k-1}) + \frac{mM_2(1+2T)}{T} \\ &+ \frac{mM_3(1+T)}{T} + \frac{M_3(1+2T)}{T}\sum_{k=1}^{m}(T-t_k).\end{aligned} \tag{3.3}$$

Theorem 3.1 *Assume that*:

(H_1) *The function $f : J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and there exist constants $L_1, L_2 > 0$ such that*

$$\big|f\big(t,x,(S_{q_k}x)\big) - f\big(t,y,(S_{q_k}y)\big)\big| \le L_1|x-y| + L_2\big(S_{q_k}|x-y|\big),$$

for each $t \in J$ and $x, y \in \mathbb{R}$, $k = 0,1,2,\ldots,m$.

(H_2) *The functions $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ are continuous and there exist constants $L_3, L_4 > 0$ such that*

$$\big|I_k(x) - I_k(y)\big| \le L_3|x-y| \quad \textit{and} \quad \big|I_k^*(x) - I_k^*(y)\big| \le L_4|x-y|,$$

for each $x, y \in \mathbb{R}$, $k = 1,2,\ldots,m$.

If

$$\omega \leq \delta < 1, \tag{3.4}$$

where ω *is defined by* (3.2), *and* $\delta > 0$, *then the boundary value problem* (1.1) *has a unique solution on* J.

Proof We transform the boundary value problem (1.1) into a fixed point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined by (3.1). By using the Banach contraction mapping principle, we shall show that $\mathcal{A}$ has a fixed point which is the unique solution of the boundary value problem (1.1).

Let M_1, M_2, and M_3 be nonnegative constants such that $\sup_{t\in J}|f(t,0,0)| = M_1$, $\sup\{|I_k(0)| : k = 1,2,\ldots,m\} = M_2$, and $\sup\{|I_k^*(0)| : k = 1,2,\ldots,m\} = M_3$. By choosing a constant R as

$$R \geq \frac{\lambda_0}{1-\varepsilon},$$

where $\delta \leq \varepsilon < 1$ and λ_0 defined by (3.3), we will show that $\mathcal{A}B_R \subset B_R$, where a ball B_R is defined by $B_R = \{x \in PC(J,\mathbb{R}) : \|x\| \leq R\}$. For $x \in B_R$, we have

$$\begin{aligned}
&\|\mathcal{A}x\| \\
&\quad\leq \sup_{t\in J}\Bigg\{\left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r,x(r),(S_{q_{k-1}}x)(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\
&\qquad + \left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s),(S_{q_{k-1}}x)(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(T-t_k+1) \\
&\qquad + \left(\frac{1-t}{T}\right)\int_{t_m}^{T}\int_{t_m}^{s} f\big(r,x(r),(S_{q_m}x)(r)\big)\,d_{q_m}r\,d_{q_m}s \\
&\qquad + \left(\frac{1-t}{T}\right)\int_{t_m}^{T} f\big(s,x(s),(S_{q_m}x)(s)\big)\,d_{q_m}s \\
&\qquad + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r,x(r),(S_{q_{k-1}}x)(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\
&\qquad + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s),(S_{q_{k-1}}x)(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big)\right)(t-t_k) \\
&\qquad + \int_{t_k}^{t}\int_{t_k}^{s} f\big(r,x(r),(S_{q_k}x)(r)\big)\,d_{q_k}r\,d_{q_k}s\Bigg\} \\
&\quad\leq \left(\frac{1+T}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} \big|f\big(r,x(r),(S_{q_{k-1}}x)(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s + \big|I_k\big(x(t_k)\big)\big|\right) \\
&\qquad + \left(\frac{1+T}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} \big|f\big(s,x(s),(S_{q_{k-1}}x)(s)\big)\big|\,d_{q_{k-1}}s + \big|I_k^*\big(x(t_k)\big)\big|\right)(T-t_k+1) \\
&\qquad + \left(\frac{1+T}{T}\right)\int_{t_m}^{T}\int_{t_m}^{s} \big|f\big(r,x(r),(S_{q_m}x)(r)\big)\big|\,d_{q_m}r\,d_{q_m}s
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1+T}{T}\right)\int_{t_m}^{T}\bigl|f\bigl(s,x(s),(S_{q_m}x)(s)\bigr)\bigr|\,d_{q_m}s\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s+\bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\right)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r),(S_{q_m}x)(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s\\
\le{}&\frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl(\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)-f(r,0,0)\bigr|+\bigl|f(r,0,0)\bigr|\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&+\frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)-f(s,0,0)\bigr|+\bigl|f(s,0,0)\bigr|\bigr)\,d_{q_{k-1}}s\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)-f(s,0,0)\bigr|+\bigl|f(s,0,0)\bigr|\bigr)\,d_{q_{k-1}}s\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k\bigl(x(t_k)\bigr)-I_k(0)\bigr|+\bigl|I_k(0)\bigr|\bigr)\\
&+\frac{1+T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k^*\bigl(x(t_k)\bigr)-I_k^*(0)\bigr|+\bigl|I_k^*(0)\bigr|\bigr)\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k^*\bigl(x(t_k)\bigr)-I_k^*(0)\bigr|+\bigl|I_k^*(0)\bigr|\bigr)(T-t_k)\\
\le{}&\frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(L_1R+\phi_0L_2R\int_{t_{k-1}}^{r}d_{q_{k-1}}u+M_1\right)d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&+\frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(L_1R+\phi_0L_2R\int_{t_{k-1}}^{s}d_{q_{k-1}}r+M_1\right)d_{q_{k-1}}s\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\left(L_1R+\phi_0L_2R\int_{t_{k-1}}^{s}d_{q_{k-1}}r+M_1\right)d_{q_{k-1}}s\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(L_3R+M_2)+\frac{1+T}{T}\sum_{k=1}^{m}(L_4R+M_3)\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(L_4R+M_3)(T-t_k)\\
\le{}&\frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{L_1R(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0L_2R(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}+\frac{M_1(t_k-t_{k-1})^2}{1+q_{k-1}}\right]\\
&+\frac{1+T}{T}\sum_{k=1}^{m+1}\left[L_1R(t_k-t_{k-1})+\frac{\phi_0L_2R(t_k-t_{k-1})^2}{1+q_{k-1}}+M_1(t_k-t_{k-1})\right]\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[L_1R(t_k-t_{k-1})+\frac{\phi_0L_2R(t_k-t_{k-1})^2}{1+q_{k-1}}+M_1(t_k-t_{k-1})\right]
\end{aligned}
$$

$$\begin{aligned}&+\frac{1+2T}{T}\sum_{k=1}^{m}(L_3R+M_2)+\frac{1+T}{T}\sum_{k=1}^{m}(L_4R+M_3)+\frac{1+2T}{T}\sum_{k=1}^{m}(L_4R+M_3)(T-t_k)\\&=\omega R+\lambda_0\le(\delta+1-\varepsilon)R\le R,\end{aligned}$$

which implies that $\mathcal{A}B_R\subset B_R$.

For any $x,y\in PC(J,\mathbb{R})$ and for each $t\in J$, we have

$$\begin{aligned}&\bigl|\mathcal{A}x(t)-\mathcal{A}y(t)\bigr|\\
&\le\frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl(\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)-f\bigl(r,y(r),(S_{q_{k-1}}y)(r)\bigr)\bigr|\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&\quad+\frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)-f\bigl(s,y(s),(S_{q_{k-1}}y)(s)\bigr)\bigr|\bigr)\,d_{q_{k-1}}s\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\bigl(\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)-f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\bigr)\,d_{q_{k-1}}s\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k\bigl(x(t_k)\bigr)-I_k\bigl(y(t_k)\bigr)\bigr|\bigr)+\frac{1+T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k^*\bigl(x(t_k)\bigr)-I_k^*\bigl(y(t_k)\bigr)\bigr|\bigr)\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}\bigl(\bigl|I_k^*\bigl(x(t_k)\bigr)-I_k^*\bigl(y(t_k)\bigr)\bigr|\bigr)(T-t_k)\\
&\le\frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{L_1(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0L_2(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}\right]\|x-y\|\\
&\quad+\frac{1+T}{T}\sum_{k=1}^{m+1}\left[L_1(t_k-t_{k-1})+\frac{\phi_0L_2(t_k-t_{k-1})^2}{1+q_{k-1}}\right]\|x-y\|\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[L_1(t_k-t_{k-1})+\frac{\phi_0L_2(t_k-t_{k-1})^2}{1+q_{k-1}}\right]\|x-y\|\\
&\quad+\frac{m(1+2T)L_3}{T}\|x-y\|+\frac{m(1+T)L_4}{T}\|x-y\|+\frac{(1+2T)L_4\|x-y\|}{T}\sum_{k=1}^{m}(T-t_k)\\
&=\omega\|x-y\|,\end{aligned}$$

which implies that $\|\mathcal{A}x-\mathcal{A}y\|\le\omega\|x-y\|$. As $\omega<1$, $\mathcal{A}$ is a contraction. Therefore, by the Banach contraction mapping principle, we find that $\mathcal{A}$ has a fixed point which is the unique solution of problem (1.1). This completes the proof. □

The second existence result is based on Schaefer's fixed point theorem.

Theorem 3.2 *Assume that*:

(H_3) $f:J\times\mathbb{R}^2\to\mathbb{R}$ *is a continuous function and there exists a constant* $N_1>0$ *such that*

$$\bigl|f\bigl(t,x,(S_{q_k}x)\bigr)\bigr|\le N_1,$$

for each $t\in J$ *and all* $x\in\mathbb{R}$, $k=0,1,2,\dots,m$.

(H_4) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $N_2, N_3 > 0$ *such that*

$$\left|I_k(x)\right| \le N_2 \quad \text{and} \quad \left|I_k^*(x)\right| \le N_3,$$

for all $x \in \mathbb{R}$, $k = 1, 2, \ldots, m$.

Then the boundary value problem (1.1) *has at least one solution on* J.

Proof We will use Schaefer's fixed point theorem to prove that $\mathcal{A}$, defined by (3.1), has a fixed point. We divide the proof into four steps.

Step 1: *Continuity of* $\mathcal{A}$.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J, \mathbb{R})$. Since f is a continuous function on $J \times \mathbb{R}^2$ and I_k, I_k^* are continuous functions on $\mathbb{R}$ for $k = 1, 2, \ldots$, we have

$$f\big(t, x_n(t), (S_{q_k} x_n)(t)\big) \to f\big(t, x(t), (S_{q_k} x)(t)\big),$$

and $I_k(x_n(t_k)) \to I_k(x(t_k))$, $I_k^*(x_n(t_k)) \to I_k^*(x(t_k))$ for $k = 1, 2, \ldots$, as $n \to \infty$.

Then, for each $t \in J$, we get

$$\begin{aligned}
&\big|(\mathcal{A}x_n)(t) - (\mathcal{A}x)(t)\big| \\
&\quad= \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \big| f\big(r, x_n(r), (S_{q_{k-1}} x_n)(r)\big) \right. \\
&\qquad \left. - f\big(r, x(r), (S_{q_{k-1}} x)(r)\big)\big| \, d_{q_{k-1}} r \, d_{q_{k-1}} s + \big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big| \right) \\
&\qquad + \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \big| f\big(s, x_n(s), (S_{q_{k-1}} x_n)(s)\big) - f\big(s, x(s), (S_{q_{k-1}} x)(s)\big)\big| \, d_{q_{k-1}} s \right. \\
&\qquad \left. + \big|I_k^*\big(x_n(t_k)\big) - I_k^*\big(x(t_k)\big)\big| \right) (T - t_k + 1) \\
&\qquad + \left(\frac{1-t}{T}\right) \int_{t_m}^{T} \int_{t_m}^{s} \big| f\big(r, x_n(r), (S_{q_m} x_n)(r)\big) - f\big(r, x(r), (S_{q_m} x)(r)\big)\big| \, d_{q_m} r \, d_{q_m} s \\
&\qquad + \left(\frac{1-t}{T}\right) \int_{t_m}^{T} \big| f\big(s, x_n(s), (S_{q_m} x_n)(s)\big) - f\big(s, x(s), (S_{q_m} x)(s)\big)\big| \, d_{q_m} s \\
&\qquad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} \big| f\big(r, x_n(r), (S_{q_{k-1}} x_n)(r)\big) - f\big(r, x(r), (S_{q_{k-1}} x)(r)\big)\big| \, d_{q_{k-1}} r \, d_{q_{k-1}} s \right. \\
&\qquad \left. + \big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big| \right) \\
&\qquad + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \big| f\big(s, x_n(s), (S_{q_{k-1}} x_n)(s)\big) - f\big(s, x(s), (S_{q_{k-1}} x)(s)\big)\big| \, d_{q_{k-1}} s \right. \\
&\qquad \left. + \big|I_k^*\big(x_n(t_k)\big) - I_k^*\big(x(t_k)\big)\big| \right) (t - t_k) \\
&\qquad + \int_{t_k}^{t} \int_{t_k}^{s} \big| f\big(r, x_n(r), (S_{q_k} x_n)(r)\big) - f\big(r, x(r), (S_{q_k} x)(r)\big)\big| \, d_{q_k} r \, d_{q_k} s,
\end{aligned}$$

which gives $\|\mathcal{A}x_n - \mathcal{A}x\| \to 0$ as $n \to \infty$. This means that $\mathcal{A}$ is continuous.

Step 2: *$\mathcal{A}$ maps bounded sets into bounded sets in $PC(J,\mathbb{R})$.*

So, let us prove that for any $r > 0$, there exists a positive constant ρ such that for each $x \in B_r = \{x \in PC(J,\mathbb{R}) : \|x\| \le r\}$, we have $\|\mathcal{A}x\| \le \rho$. For any $x \in B_r$, we have

$$
\begin{aligned}
\bigl|(\mathcal{A}x)(t)\bigr| \le{}& \frac{|1-t|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)\\
&+\frac{|1-t|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s+\bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\biggr)(T-t_k+1)\\
&+\frac{|1-t|}{T}\int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r),(S_{q_m}x)(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s\\
&+\frac{|1-t|}{T}\int_{t_m}^{T}\bigl|f\bigl(s,x(s),(S_{q_m}x)(s)\bigr)\bigr|\,d_{q_m}s\\
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)\\
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s+\bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\biggr)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r),(S_{q_m}x)(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s\\
\le{}& \frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}N_1\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}N_1\,d_{q_{k-1}}s\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}N_1\,d_{q_{k-1}}s+\frac{1+2T}{T}\sum_{k=1}^{m}N_2\\
&+\frac{1+T}{T}\sum_{k=1}^{m}N_3+\frac{1+2T}{T}\sum_{k=1}^{m}N_3(T-t_k)\\
\le{}& \frac{1+2T}{T}\sum_{k=1}^{m+1}\biggl[\frac{N_1(t_k-t_{k-1})^2}{1+q_{k-1}}\biggr]+\frac{1+T}{T}\sum_{k=1}^{m+1}\bigl[N_1(t_k-t_{k-1})\bigr]\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\bigl[N_1(t_k-t_{k-1})\bigr]+\frac{m(1+2T)N_2}{T}\\
&+\frac{m(1+T)N_3}{T}+\frac{(1+2T)N_3}{T}\sum_{k=1}^{m}(T-t_k)\\
:={}& \rho.
\end{aligned}
$$

Hence, we deduce that $\|\mathcal{A}x\| \le \rho$.

Step 3: *$\mathcal{A}$ maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.*

Let $\tau_1, \tau_2 \in J_i$ for some $i \in \{0,1,2,\dots,m\}$, $\tau_1 < \tau_2$, B_r be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $x \in B_r$. Then we have

$$
\begin{aligned}
&\bigl|(\mathcal{A}x)(\tau_2)-(\mathcal{A}x)(\tau_1)\bigr|\\
&\quad\le \frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),(S_{q_{k-1}}x)(s)\right)\right| d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)(T-t_k+1)\\
&+\frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\int_{t_m}^{s}\left|f\left(r,x(r),(S_{q_m}x)(r)\right)\right| d_{q_m}r\,d_{q_m}s\\
&+\frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\left|f\left(s,x(s),(S_{q_m}x)(s)\right)\right| d_{q_m}s\\
&+|\tau_2-\tau_1|\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),(S_{q_{k-1}}x)(s)\right)\right| d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)\\
&+\left|\int_{t_i}^{\tau_2}\int_{t_i}^{s}\left|f\left(r,x(r),(S_{q_i}x)(r)\right)\right| d_{q_i}r\,d_{q_i}s-\int_{t_i}^{\tau_1}\int_{t_i}^{s}\left|f\left(r,x(r),(S_{q_i}x)(r)\right)\right| d_{q_i}r\,d_{q_i}s\right|\\
\le{}&\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left[\frac{N_1(t_k-t_{k-1})^2}{1+q_{k-1}}+N_2\right]\\
&+\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left[N_1(t_k-t_{k-1})+N_3\right](T-t_k+1)+\frac{|\tau_2-\tau_1|}{T}\left[\frac{N_1(T-t_m)^2}{1+q_m}\right]\\
&+\frac{|\tau_2-\tau_1|}{T}\left[N_1(T-t_m)\right]+|\tau_2-\tau_1|\sum_{k=1}^{i}\left[N_1(t_k-t_{k-1})+N_3\right]\\
&+\frac{|\tau_2-\tau_1|N_1}{1+q_i}(\tau_2+\tau_1+2t_i).
\end{aligned}
$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality (which is independent of x) tends to zero. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we deduce that $\mathcal{A}: PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *We show that the set*

$$E=\left\{x\in PC(J,\mathbb{R}): x=\theta\mathcal{A}x \text{ for some } 0<\theta<1\right\}$$

is bounded.

Let $x\in E$. Then $x(t)=\theta(\mathcal{A}x)(t)$ for some $0<\theta<1$. Thus, for each $t\in J$, by using the computations of Step 2, we have that

$$\|\mathcal{A}x\|\le\rho.$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we conclude that $\mathcal{A}$ has a fixed point which is a solution of the impulsive q_k-integro-difference boundary value problem (1.1). □

The third existence result for the impulsive boundary value problem (1.1) is based on Krasnoselskii's fixed point theorem.

Lemma 3.3 (Krasnoselskii's fixed point theorem) [19] *Let M be a closed, bounded, convex and nonempty subset of a Banach space X. Let A, B be the operators such that*

(a) *$Ax+By\in M$ whenever $x,y\in M$*;
(b) *A is a compact and continuous*;

(c) *B is a contraction mapping.*

Then there exists $z \in M$ such that $z = Az + Bz$.

For convenience we put

$$\Lambda = \frac{1+2T}{T}\sum_{k=1}^{m+1}\frac{(t_k - t_{k-1})^2}{1+q_{k-1}} + \frac{1+T}{T}\sum_{k=1}^{m+1}(t_k - t_{k-1}) + \frac{1+2T}{T}\sum_{k=1}^{m}(T - t_k)(t_k - t_{k-1}), \tag{3.5}$$

and

$$\lambda_1 = \frac{m(1+2T)N_1}{T} + \frac{m(1+T)N_2}{T} + \frac{(1+2T)N_2}{T}\sum_{k=1}^{m}(T - t_k). \tag{3.6}$$

Theorem 3.4 *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*:

(A$_1$) $|f(t,x,y)| \le \mu(t)$, $\forall (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}$ *and* $\mu \in C(J,\mathbb{R}^+)$.

(A$_2$) *There exist constants* $N_1, N_2 > 0$ *such that* $|I_k(x)| \le N_1$ *and* $|I_k^*(x)| \le N_2$, $\forall x \in \mathbb{R}$, *for* $k = 1,2,\ldots,m$.

(A$_3$) *There exist constants* $K_1, K_2 > 0$ *such that* $|I_k(x) - I_k(y)| \le K_1|x - y|$ *and* $|I_k^*(x) - I_k^*(y)| \le K_2|x - y|$, $\forall x, y \in \mathbb{R}$, *for* $k = 1,2,\ldots,m$.

If

$$\frac{m(1+2T)K_1}{T} + \frac{m(1+T)K_2}{T} + \frac{(1+2T)K_2}{T}\sum_{k=1}^{m}(T - t_k) < 1, \tag{3.7}$$

then boundary value problem (1.1) *has at least one solution on* J.

Proof We define $\sup_{t\in J}|\mu(t)| = \|\mu\|$ and choose a suitable constant ρ as

$$\rho \ge \|\mu\|\Lambda + \lambda_1,$$

where Λ and λ_1 are defined by (3.5) and (3.6), respectively. We define the operators Φ and Ψ on $B_\rho = \{x \in PC(J,\mathbb{R}) : \|x\| \le \rho\}$ as

$$\begin{aligned}(\Phi x)(t) &= \left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r, x(r), (S_{q_{k-1}}x)(r)\big)\, d_{q_{k-1}}r\, d_{q_{k-1}}s\right) \\ &\quad + \left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), (S_{q_{k-1}}x)(s)\big)\, d_{q_{k-1}}s\right)(T - t_k + 1) \\ &\quad + \left(\frac{1-t}{T}\right)\int_{t_m}^{T}\int_{t_m}^{s} f\big(r, x(r), (S_{q_m}x)(r)\big)\, d_{q_m}r\, d_{q_m}s \\ &\quad + \left(\frac{1-t}{T}\right)\int_{t_m}^{T} f\big(s, x(s), (S_{q_m}x)(s)\big)\, d_{q_m}s\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f\big(r, x(r), (S_{q_{k-1}}x)(r)\big)\, d_{q_{k-1}} r\, d_{q_{k-1}} s \right) \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s), (S_{q_{k-1}}x)(s)\big)\, d_{q_{k-1}} s \right)(t - t_k) \\
&+ \int_{t_k}^{t} \int_{t_k}^{s} f\big(r, x(r), (S_{q_k}x)(r)\big)\, d_{q_k} r\, d_{q_k} s
\end{aligned}
$$

and

$$
\begin{aligned}
(\Psi x)(t) &= \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} I_k\big(x(t_k)\big) + \left(\frac{1-t}{T}\right) \sum_{k=1}^{m} I_k^*\big(x(t_k)\big)(T - t_k + 1) \\
&\quad + \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \sum_{0<t_k<t} I_k^*\big(x(t_k)\big)(t - t_k).
\end{aligned}
$$

For $x, y \in B_\rho$, we have

$$
\begin{aligned}
\|\Phi x + \Psi y\| &\le \|\mu\| \left\{ \frac{1+2T}{T} \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \frac{1+T}{T} \sum_{k=1}^{m+1} (t_k - t_{k-1}) \right. \\
&\quad \left. + \frac{1+2T}{T} \sum_{k=1}^{m} (T - t_k)(t_k - t_{k-1}) \right\} \\
&\quad + \frac{m(1+2T)N_1}{T} + \frac{m(1+T)N_2}{T} + \frac{(1+2T)N_2}{T} \sum_{k=1}^{m} (T - t_k) \\
&= \|\mu\| \Lambda + \lambda_1 \\
&\le \rho.
\end{aligned}
$$

Thus, $\Phi x + \Psi y \in B_\rho$.

For $x, y \in PC(J, \mathbb{R})$, from (A_3), we have

$$
\begin{aligned}
\|\Psi x - \Psi y\| &\le \frac{1+2T}{T} \sum_{k=1}^{m} \big| I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big) \big| + \frac{1+T}{T} \sum_{k=1}^{m} \big| I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big) \big| \\
&\quad + \frac{1+2T}{T} \sum_{k=1}^{m} \big| I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big) \big| (T - t_k) \\
&\le \frac{m(1+2T)K_1 \|x - y\|}{T} + \frac{m(1+T)K_2 \|x - y\|}{T} \\
&\quad + \frac{(1+2T)K_2 \|x - y\|}{T} \sum_{k=1}^{m} (T - t_k),
\end{aligned}
$$

which implies, by (3.7), that Ψ is a contraction mapping.

Continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_ρ as

$$
\|\Phi x\| \le \|\mu\| \Lambda.
$$

Now we prove the compactness of the operator Φ.

We define $\sup_{(t,x)\in J\times B_\rho} |f(t,x)| = \overline{f} < \infty$, $\tau_1, \tau_2 \in (t_i, t_{i+1})$ for some $i \in \{0,1,\ldots,m\}$ with $\tau_1 < \tau_2$ and consequently we get

$$\begin{aligned}
&\bigl|(\Phi x)(\tau_2) - (\Phi x)(\tau_1)\bigr| \\
&\quad\le \frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s\biggr) \\
&\qquad + \frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s\biggr)(T-t_k+1) \\
&\qquad + \frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r),(S_{q_m}x)(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s \\
&\qquad + \frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\bigl|f\bigl(s,x(s),(S_{q_m}x)(s)\bigr)\bigr|\,d_{q_m}s \\
&\qquad + |\tau_2-\tau_1|\sum_{k=1}^{i}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s\biggr) \\
&\qquad + \biggl|\int_{t_i}^{\tau_2}\int_{t_i}^{s}\bigl|f\bigl(r,x(r),(S_{q_i}x)(r)\bigr)\bigr|\,d_{q_i}r\,d_{q_i}s - \int_{t_i}^{\tau_1}\int_{t_i}^{s}\bigl|f\bigl(r,x(r),(S_{q_i}x)(r)\bigr)\bigr|\,d_{q_i}r\,d_{q_i}s\biggr| \\
&\quad\le \frac{|\tau_2-\tau_1|\overline{f}}{T}\sum_{k=1}^{m}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + \frac{|\tau_2-\tau_1|\overline{f}}{T}\sum_{k=1}^{m}(t_k-t_{k-1})(T-t_k+1) \\
&\qquad + \frac{|\tau_2-\tau_1|\overline{f}}{T}\frac{(T-t_m)^2}{1+q_m} + \frac{|\tau_2-\tau_1|\overline{f}}{T}(T-t_m) + |\tau_2-\tau_1|\overline{f}\sum_{k=1}^{i}(t_k-t_{k-1}) \\
&\qquad + \frac{|\tau_2-\tau_1|\overline{f}}{1+q_i}(\tau_2+\tau_1+2t_i),
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, Φ is equicontinuous. So Φ is relatively compact on B_ρ. Hence, by the Arzelá-Ascoli theorem, Φ is compact on B_ρ. Thus all the assumptions of Lemma 3.3 are satisfied. So the boundary value problem (1.1) has at least one solution on J. The proof is completed. □

Our final, fourth existence result is based on the Leray-Schauder Nonlinear Alternative.

Lemma 3.5 (Nonlinear alternative for single valued maps) [20] *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\theta \in (0,1)$ with $u = \theta F(u)$.*

Theorem 3.6 *Assume that*:

(A_4) *There exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a continuous function $p : J \to \mathbb{R}^+$ such that*

$$\bigl|f(t,x,y)\bigr| \le p(t)\psi\bigl(|x|\bigr) + |y| \quad \textit{for each } (t,x,y) \in J \times \mathbb{R} \times \mathbb{R}.$$

(A_5) *There exist continuous nondecreasing functions* $\varphi_1, \varphi_2 : [0,\infty) \to (0,\infty)$ *such that*

$$\bigl|I_k(x)\bigr| \le \varphi_1\bigl(|x|\bigr) \quad \text{and} \quad \bigl|I_k^*(x)\bigr| \le \varphi_2\bigl(|x|\bigr),$$

for all $x \in \mathbb{R}$, $k = 1, 2, \dots, m$.

(A_6) *There exists a constant* $M^* > 0$ *such that*

$$\frac{M^*}{p_0\psi(M^*)Q_1 + \phi_0 M^* Q_2 + Q_3} > 1,$$

where $p_0 = \max\{p(t) : t \in J\}$, $\phi_0 = \max\{\phi(t,s) : (t,s) \in J \times J\}$ *and*

$$\begin{aligned} Q_1 &= \frac{1+2T}{T}\sum_{k=1}^{m+1}\frac{(t_k - t_{k-1})^2}{1+q_{k-1}} + \frac{1+T}{T}\sum_{k=1}^{m+1}(t_k - t_{k-1}) \\ &\quad + \frac{1+2T}{T}\sum_{k=1}^{m}(T - t_k)(t_k - t_{k-1}), \\ Q_2 &= \frac{1+2T}{T}\sum_{k=1}^{m+1}\frac{(t_k - t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2} + \frac{1+T}{T}\sum_{k=1}^{m+1}\frac{(t_k - t_{k-1})^2}{1+q_{k-1}} \\ &\quad + \frac{1+2T}{T}\sum_{k=1}^{m}(T - t_k)\frac{(t_k - t_{k-1})^2}{1+q_{k-1}}, \\ Q_3 &= \frac{m(1+2T)\varphi_1(M^*)}{T} + \frac{m(1+T)\varphi_2(M^*)}{T} + \frac{(1+2T)\varphi_2(M^*)}{T}\sum_{k=1}^{m}(T - t_k). \end{aligned}$$

Then the impulsive boundary value problem (1.1) *has at least one solution on* J.

Proof First we show that $\mathcal{A}$ *maps bounded sets (balls) into bounded sets in* $PC(J,\mathbb{R})$. For a positive number $\overline{\rho}$, let $B_{\overline{\rho}} = \{x \in PC(J,\mathbb{R}) : \|x\| \le \overline{\rho}\}$ be a bounded ball in $PC(J,\mathbb{R})$. Then for $t \in J$ we have

$$\begin{aligned} &\bigl|(\mathcal{A}x)(t)\bigr| \\ &\quad \le \frac{|1-t|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) \\ &\qquad + \frac{|1-t|}{T}\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\biggr)(T - t_k + 1) \\ &\qquad + \frac{|1-t|}{T}\int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r),(S_{q_m}x)(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s \\ &\qquad + \frac{|1-t|}{T}\int_{t_m}^{T}\bigl|f\bigl(s,x(s),(S_{q_m}x)(s)\bigr)\bigr|\,d_{q_m}s \\ &\qquad + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r),(S_{q_{k-1}}x)(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) \\ &\qquad + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s),(S_{q_{k-1}}x)(s)\bigr)\bigr|\,d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|\biggr)(T - t_k) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_m}^{T}\int_{t_m}^{s}\big|f\big(r,x(r),(S_{q_m}x)(r)\big)\big|\,d_{q_m}r\,d_{q_m}s \\
&\leq \frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{r}d_{q_{k-1}}u\right)d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&\quad + \frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{s}d_{q_{k-1}}r\right)d_{q_{k-1}}s \\
&\quad + \frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{s}d_{q_{k-1}}r\right)d_{q_{k-1}}s \\
&\quad + \frac{1+2T}{T}\sum_{k=1}^{m}\varphi_1\big(\|x\|\big)+\frac{1+T}{T}\sum_{k=1}^{m}\varphi_2\big(\|x\|\big)+\frac{1+2T}{T}\sum_{k=1}^{m}\varphi_2\big(\|x\|\big)(T-t_k) \\
&\leq \frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{p_0\psi(\|x\|)(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0\|x\|(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}\right] \\
&\quad + \frac{1+T}{T}\sum_{k=1}^{m+1}\left[p_0\psi\big(\|x\|\big)(t_k-t_{k-1})+\frac{\phi_0\|x\|(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\
&\quad + \frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[p_0\psi\big(\|x\|\big)(t_k-t_{k-1})+\frac{\phi_0\|x\|(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\
&\quad + \frac{m(1+2T)\varphi_1(\|x\|)}{T}+\frac{m(1+T)\varphi_2(\|x\|)}{T} \\
&\quad + \frac{(1+2T)\varphi_2(\|x\|)}{T}\sum_{k=1}^{m}(T-t_k) \\
&\leq \frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{p_0\psi(\overline{\rho})(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}\right] \\
&\quad + \frac{1+T}{T}\sum_{k=1}^{m+1}\left[p_0\psi(\overline{\rho})(t_k-t_{k-1})+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\
&\quad + \frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[p_0\psi(\overline{\rho})(t_k-t_{k-1})+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^2}{1+q_{k-1}}\right] \\
&\quad + \frac{m(1+2T)\varphi_1(\overline{\rho})}{T}+\frac{m(1+T)\varphi_2(\overline{\rho})}{T}+\frac{(1+2T)\varphi_2(\overline{\rho})}{T}\sum_{k=1}^{m}(T-t_k) \\
&:= K.
\end{aligned}
$$

Hence, we deduce that $\|\mathcal{A}x\| \leq K$.

Next we show that $\mathcal{A}$ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$.

Let $\tau_1, \tau_2 \in J_i$ for some $i \in \{0,1,2,\dots,m\}$, $\tau_1 < \tau_2$, $B_{\overline{\rho}}$ be a bounded set of $PC(J,\mathbb{R})$ as in the previous step, and let $x \in B_{\overline{\rho}}$. Then we have

$$
\begin{aligned}
&\big|(\mathcal{A}x)(\tau_2)-(\mathcal{A}x)(\tau_1)\big| \\
&\quad \leq \frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f\big(r,x(r),(S_{q_{k-1}}x)(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s+\big|I_k\big(x(t_k)\big)\big|\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),(S_{q_{k-1}}x)(s)\right)\right|d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)(T-t_k+1)\\
&+\frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\int_{t_m}^{s}\left|f\left(r,x(r),(S_{q_m}x)(r)\right)\right|d_{q_m}r\,d_{q_m}s\\
&+\frac{|\tau_2-\tau_1|}{T}\int_{t_m}^{T}\left|f\left(s,x(s),(S_{q_m}x)(s)\right)\right|d_{q_m}s\\
&+|\tau_2-\tau_1|\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),(S_{q_{k-1}}x)(s)\right)\right|d_{q_{k-1}}s+\left|I_k^*\left(x(t_k)\right)\right|\right)\\
&+\left|\int_{t_i}^{\tau_2}\int_{t_i}^{s}\left|f\left(r,x(r),(S_{q_i}x)(r)\right)\right|d_{q_i}r\,d_{q_i}s-\int_{t_i}^{\tau_1}\int_{t_i}^{s}\left|f\left(r,x(r),(S_{q_i}x)(r)\right)\right|d_{q_i}r\,d_{q_i}s\right|\\
\le{}&\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left[\frac{p_0\psi(\overline{\rho})(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}+\varphi_1(\overline{\rho})\right]\\
&+\frac{|\tau_2-\tau_1|}{T}\sum_{k=1}^{m}\left[p_0\psi(\overline{\rho})(t_k-t_{k-1})+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi_2(\overline{\rho})\right](T-t_k+1)\\
&+\frac{|\tau_2-\tau_1|}{T}\left[\frac{p_0\psi(\overline{\rho})(T-t_m)^2}{1+q_m}+\frac{\phi_0\overline{\rho}(T-t_m)^3}{1+q_m+q_m^2}\right]\\
&+\frac{|\tau_2-\tau_1|}{T}\left[p_0\psi(\overline{\rho})(T-t_m)+\frac{\phi_0\overline{\rho}(T-t_m)^2}{1+q_m}\right]\\
&+|\tau_2-\tau_1|\sum_{k=1}^{i}\left[p_0\psi(\overline{\rho})(t_k-t_{k-1})+\frac{\phi_0\overline{\rho}(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi_2(\overline{\rho})\right]\\
&+\frac{|\tau_2-\tau_1|p_0\psi(\overline{\rho})}{1+q_i}(\tau_2+\tau_1+2t_i)\\
&+\frac{|\tau_2-\tau_1|\phi_0\overline{\rho}}{1+q_i+q_i^2}\left(\tau_2^2+\tau_2\tau_1+\tau_1^2+3(\tau_2+\tau_1)t_i+3t_i^2\right).
\end{aligned}
$$

The right-hand side of the above inequality, which is independent of x, tends to zero as $\tau_1\to\tau_2$. From all above and by the Arzelá-Ascoli theorem $\mathcal{A}:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

The result will follow from the Leray-Schauder nonlinear alternative (Lemma 3.5) once we have proved the boundedness of the set of all solutions to the equations $x(t)=\theta(\mathcal{A}x)(t)$ for some $0<\theta<1$.

Let x be a solution. Thus, for each $t\in J$, we have

$$
\begin{aligned}
&(\mathcal{A}x)(t)\\
&=\theta\left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f\left(r,x(r),(S_{q_{k-1}}x)(r)\right)d_{q_{k-1}}r\,d_{q_{k-1}}s+I_k\left(x(t_k)\right)\right)\\
&\quad+\theta\left(\frac{1-t}{T}\right)\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}f\left(s,x(s),(S_{q_{k-1}}x)(s)\right)d_{q_{k-1}}s+I_k^*\left(x(t_k)\right)\right)(T-t_k+1)\\
&\quad+\theta\left(\frac{1-t}{T}\right)\int_{t_m}^{T}\int_{t_m}^{s}f\left(r,x(r),(S_{q_m}x)(r)\right)d_{q_m}r\,d_{q_m}s
\end{aligned}
$$

$$
\begin{aligned}
&+\theta\left(\frac{1-t}{T}\right)\int_{t_m}^{T} f\big(s,x(s),(S_{q_m}x)(s)\big)\,d_{q_m}s\\
&+\theta\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r,x(r),(S_{q_{k-1}}x)(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s+I_k\big(x(t_k)\big)\right)\\
&+\theta\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s),(S_{q_{k-1}}x)(s)\big)\,d_{q_{k-1}}s+I_k^*\big(x(t_k)\big)\right)(t-t_k)\\
&+\theta\int_{t_k}^{t}\int_{t_k}^{s} f\big(r,x(r),(S_{q_k}x)(r)\big)\,d_{q_k}r\,d_{q_k}s.
\end{aligned}
$$

This implies by (A_4) and (A_5) that for each $t\in J$, we have

$$
\begin{aligned}
\big|(\mathcal{A}x)(t)\big| &\le \frac{\theta(1+2T)}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big(p(t)\psi\big(|x|\big)+|S_{q_{k-1}}x|\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&\quad+\frac{\theta(1+T)}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\big(p(t)\psi\big(|x|\big)+|S_{q_{k-1}}x|\big)\,d_{q_{k-1}}s\\
&\quad+\frac{\theta(1+2T)}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\big(p(t)\psi\big(|x|\big)+|S_{q_{k-1}}x|\big)\,d_{q_{k-1}}s\\
&\quad+\frac{\theta(1+2T)}{T}\sum_{k=1}^{m}\varphi_1\big(|x|\big)+\frac{\theta(1+T)}{T}\sum_{k=1}^{m}\varphi_2\big(|x|\big)\\
&\quad+\frac{\theta(1+2T)}{T}\sum_{k=1}^{m}\varphi_2\big(|x|\big)(T-t_k)\\
&\le \frac{1+2T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{r}d_{q_{k-1}}u\right)d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&\quad+\frac{1+T}{T}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{s}d_{q_{k-1}}r\right)d_{q_{k-1}}s\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\int_{t_{k-1}}^{t_k}\left(p_0\psi\big(\|x\|\big)+\phi_0\|x\|\int_{t_{k-1}}^{s}d_{q_{k-1}}r\right)d_{q_{k-1}}s\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}\varphi_1\big(\|x\|\big)+\frac{1+T}{T}\sum_{k=1}^{m}\varphi_2\big(\|x\|\big)\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}\varphi_2\big(\|x\|\big)(T-t_k)\\
&\le \frac{1+2T}{T}\sum_{k=1}^{m+1}\left[\frac{p_0\psi(\|x\|)(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{\phi_0\|x\|(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}\right]\\
&\quad+\frac{1+T}{T}\sum_{k=1}^{m+1}\left[p_0\psi\big(\|x\|\big)(t_k-t_{k-1})+\frac{\phi_0\|x\|(t_k-t_{k-1})^2}{1+q_{k-1}}\right]\\
&\quad+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\left[p_0\psi\big(\|x\|\big)(t_k-t_{k-1})+\frac{\phi_0\|x\|(t_k-t_{k-1})^2}{1+q_{k-1}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{m(1+2T)\varphi_1(\|x\|)}{T}+\frac{m(1+T)\varphi_2(\|x\|)}{T}\\
&+\frac{(1+2T)\varphi_2(\|x\|)}{T}\sum_{k=1}^{m}(T-t_k)\\
=\;& p_0\psi\big(\|x\|\big)\Bigg[\frac{1+2T}{T}\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\frac{1+T}{T}\sum_{k=1}^{m+1}(t_k-t_{k-1})\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)(t_k-t_{k-1})\Bigg]\\
&+\phi_0\|x\|\Bigg[\frac{1+2T}{T}\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^3}{1+q_{k-1}+q_{k-1}^2}+\frac{1+T}{T}\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\\
&+\frac{1+2T}{T}\sum_{k=1}^{m}(T-t_k)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\Bigg]+\frac{m(1+2T)\varphi_1(\|x\|)}{T}\\
&+\frac{m(1+T)\varphi_2(\|x\|)}{T}+\frac{(1+2T)\varphi_2(\|x\|)}{T}\sum_{k=1}^{m}(T-t_k)\\
=\;& p_0\psi\big(\|x\|\big)Q_1+\phi_0\|x\|Q_2+Q_3.
\end{aligned}
$$

Consequently, we have

$$
\frac{\|x\|}{p_0\psi(\|x\|)Q_1+\phi_0\|x\|Q_2+Q_3}\le 1.
$$

In view of (A_6), there exists M^* such that $\|x\|\neq M^*$. Let us set

$$
U=\big\{x\in PC(J,\mathbb{R}):\|x\|<M^*\big\}.
$$

Note that the operator $\mathcal{A}:\overline{U}\to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $x\in\partial U$ such that $x=\theta\mathcal{A}x$ for some $\theta\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.5), we deduce that $\mathcal{A}$ has a fixed point $x\in\overline{U}$ which is a solution of the problem (1.1). This completes the proof. □

4 Examples

In this section, we will give examples to illustrate our main results.

Example 4.1 Consider the following boundary value problem for the nonlinear second-order impulsive q_k-integro-difference equation:

$$
\begin{cases}
D^2_{\frac{k+1}{k+4}}x(t)=\frac{t}{e^t(t+3)^2}\frac{|x(t)|}{|x(t)|+3}+\frac{t^2}{36}\int_{t_k}^{t}\frac{\sin\pi(t-s)}{3}x(s)\,d_{q_k}s, & t\in J, t\neq t_k,\\
\Delta x(t_k)=\frac{|x(t_k)|}{9(k+8)+|x(t_k)|}, \quad k=1,2,\dots,9,\\
D_{\frac{k+1}{k+4}}x(t_k^+)-D_{\frac{k}{k+3}}x(t_k)=\frac{|x(t_k)|}{8(k+7)+|x(t_k)|}, \quad k=1,2,\dots,9,\\
x(0)+D_{\frac{1}{4}}x(0)=0, \qquad x(1)+D_{\frac{10}{13}}x(1)=0.
\end{cases}
\tag{4.1}
$$

Here $J=[0,1]$, $t_k=k/10$, $q_k=(k+1)/(k+4)$ for $k=0,1,2,\dots,9$, $m=9$, $T=1$, $f(t,x,S_{q_k}x)=(t|x|)/(e^t(t+3)^2(|x|+3))+(t^2/36)\int_{t_k}^{t}((\sin\pi(t-s))/3)x(s)\,d_{q_k}s$, $I_k(x)=|x|/(9(k+8)+|x|)$ and

$I_k^*(x) = |x|/(8(k+7)+|x|)$. Since

$$\begin{aligned}
&\bigl|f(t,x)-f(t,y)\bigr| \le \frac{1}{27}|x-y| + (1/36)\bigl(S_{q_k}|x-y|\bigr),\\
&\bigl|I_k(x)-I_k(y)\bigr| \le \frac{1}{81}|x-y| \quad \text{and} \quad \bigl|I_k^*(x)-I_k^*(y)\bigr| \le \frac{1}{64}|x-y|,
\end{aligned}$$

(H_1) and (H_2) are satisfied with $L_1 = (1/27)$, $L_2 = (1/36)$, $L_3 = (1/81)$, $L_4 = (1/64)$. We can show that

$$\omega \approx 0.9587977316 < 1.$$

Hence, by Theorem 3.1, boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following boundary value problem for the nonlinear second-order impulsive q_k-integro-difference equation:

$$\begin{cases}
D^2_{\frac{2k+3}{3k+6}}x(t) = \frac{4t^2}{(1+x^2)^2} + \frac{t^2\sin t}{t|\int_{t_k}^t \frac{\cos^2\pi(t-s)}{3}x(s)\,d_{q_k}s|+1}, & t\in J, t\neq t_k,\\
\Delta x(t_k) = \frac{k\sin\pi t}{k+t|x(t_k)|}, \quad k=1,2,\ldots,9,\\
D_{\frac{2k+3}{3k+6}}x(t_k^+) - D_{\frac{2k+1}{3k+3}}x(t_k) = \frac{4k\cos^2 t}{2k+|x(t_k)|\sin t}, \quad k=1,2,\ldots,9,\\
x(0)+D_{\frac{1}{2}}x(0)=0, \qquad x(1)+D_{\frac{7}{11}}x(1)=0.
\end{cases} \tag{4.2}$$

Here $J=[0,1]$, $t_k = k/10$, $q_k = (2k+3)/(3k+6)$ for $k=0,1,2,\ldots,9$, $m=9$, $T=1$, $f(t,x,S_{q_k}x) = ((4t^2)/(1+x^2)^2)+(t^2\sin t)/(t|\int_{t_k}^t((\cos^2\pi(t-s))/3)x(s)\,d_{q_k}s|+1)$, $I_k(x) = (k\sin\pi t)/(k+t|x(t_k)|)$ and $I_k^*(x) = (4k\cos^2 t)/(2k+|x(t_k)|\sin t)$. We can show that

$$\begin{aligned}
&\bigl|f\bigl(t,x,(S_{q_k}x)\bigr)\bigr| = \left|\frac{4t^2}{(1+x^2)^2} + \frac{t^2\sin t}{t|\int_{t_k}^t \frac{\cos^2\pi(t-s)}{3}x(s)\,d_{q_k}s|+1}\right| \le 5 = N_1,\\
&\bigl|I_k(x)\bigr| = \left|\frac{k\sin\pi t}{k+t|x(t_k)|}\right| \le 1 = N_2, \quad \text{and} \quad \bigl|I_k^*(x)\bigr| = \left|\frac{4k\cos^2 t}{2k+|x(t_k)|\sin t}\right| \le 2 = N_3.
\end{aligned}$$

Hence, by Theorem 3.2, boundary value problem (4.2) has at least one solution on $[0,1]$.

Example 4.3 Consider the following nonlinear second-order impulsive q_k-difference equation with separated boundary condition:

$$\begin{cases}
D^2_{\frac{k+1}{3k+2}}x(t) = \frac{t\sin\pi t}{(t+1)^2}\frac{|x|}{2|x|+1} + \frac{\cos\frac{\pi t^2}{2}}{|\int_{t_k}^t \frac{e^{t-s}}{t+e^{t-s}}x(s)\,d_{q_k}s|}, & t\in J, t\neq t_k,\\
\Delta x(t_k) = \frac{|x(t_k)|}{7(k+8)+|x(t_k)|}, \quad k=1,2,\ldots,9,\\
D_{\frac{k+1}{3k+2}}x(t_k^+) - D_{\frac{k}{3k-1}}x(t_k) = \frac{|x(t_k)|}{8(k+7)+|x(t_k)|}, \quad k=1,2,\ldots,9,\\
x(0)+D_{\frac{1}{2}}x(0)=0, \qquad x(1)+D_{\frac{10}{29}}x(1)=0,
\end{cases} \tag{4.3}$$

where $J=[0,1]$, $t_k = k/10$, $q_k = (k+1)/(3k+2)$ for $k=0,1,2,\ldots,9$, $m=9$, $T=1$, $f(t,x) = (t\sin\pi t/(t+1)^2)(|x|/(2|x|+1)) + (\cos(\pi t^2/2))/(|\int_{t_k}^t(e^{t-s}/(t+e^{t-s}))x(s)\,d_{q_k}s|)$, $I_k(x) = |x(t_k)|/(7(k+8)+|x(t_k)|)$ and $I_k^*(x) = |x(t_k)|/(8(k+7)+|x(t_k)|)$. Since

$$\bigl|I_k(x)-I_k(y)\bigr| \le (1/63)|x-y| \quad \text{and} \quad \bigl|I_k^*(x)-I_k^*(y)\bigr| \le (1/64)|x-y|,$$

(A_3) is satisfied with $L_1 = (1/63)$, $L_2 = (1/64)$. It is easy to verify that $|f(t,x)| \leq \mu(t) \equiv (t\sin\pi t)/(t+1)^2 + e^{-t}\cos(\pi t^2/2)$, $I_k(x) \leq N_1 = 1$, and $I_k^*(x) \leq N_2 = 1$ for all $t \in [0,1]$, $x \in \mathbb{R}$, $k = 1,\ldots,m$. Thus (A_1) and (A_2) are satisfied. We can show that

$$\frac{m(1+2T)K_1}{T} + \frac{m(1+T)K_2}{T} + \frac{(1+2T)K_2}{T}\sum_{k=1}^{m}(T-t_k) \approx 0.9207589286 < 1.$$

Hence, by Theorem 3.4, boundary value problem (4.3) has at least one solution on $[0,1]$.

Example 4.4 Consider the following nonlinear second-order impulsive q_k-difference equation with separated boundary condition:

$$\begin{cases} D^2_{\frac{k+2}{k+4}}x(t) = \frac{x}{x^2+6e^t+42} + \frac{1+\sin\pi t}{e^t+7} - \cos t\int_{t_k}^{t}\frac{(t-s)\sin t}{e^{t-s}+2}x(s)\,d_{q_k}s, \quad t \in J, t \neq t_k, \\ \Delta x(t_k) = \frac{\sin\pi x(t_k)}{8\pi(k+7)}, \quad k = 1,2,\ldots,9, \\ D_{\frac{k+2}{k+4}}x(t_k^+) - D_{\frac{k+1}{k+3}}x(t_k) = \frac{x(t_k)\cos^2 kt}{8(k+11)}, \quad k = 1,2,\ldots,9, \\ x(0) + D_{\frac{1}{2}}x(0) = 0, \qquad x(1) + D_{\frac{11}{13}}x(1) = 0. \end{cases} \tag{4.4}$$

Here $J = [0,1]$, $t_k = k/10$, $q_k = (k+2)/(k+4)$ for $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $f(t,x) = (x/(x^2+6e^t+42)) + (1+\sin\pi t)/(e^t+7) - \cos t\int_{t_k}^{t}((t-s)\sin t/(e^{t-s}+2))x(s)\,d_{q_k}s$, $I_k(x) = (\sin\pi x)/(8\pi(k+7))$, and $I_k^*(x) = (x\cos^2 kt)/(8(k+11))$. Clearly,

$$\begin{aligned} |f(t,x)| &= \left|\frac{x}{x^2+6e^t+42} + \frac{1+\sin\pi t}{e^t+7} - \cos t\int_{t_k}^{t}\frac{(t-s)\sin t}{e^{t-s}+2}x(s)\,d_{q_k}s\right| \\ &\leq \left(\frac{2}{e^t+7}\right)\frac{|x+1|}{6} + \left|\int_{t_k}^{t}\frac{(t-s)\sin t}{e^{t-s}+2}x(s)\,d_{q_k}s\right|, \\ |I_k(x)| &= \left|\frac{\sin\pi x}{8\pi(k+7)}\right| \leq \frac{|x|}{64} \quad\text{and}\quad |I_k^*(x)| = \left|\frac{x\cos^2 kt}{8(k+11)}\right| \leq \frac{|x|}{96}. \end{aligned}$$

Choosing $p(t) = \frac{2}{e^t+7}$, $\psi(|x|) = \frac{|x+1|}{6}$, $\varphi_1(|u|) = \frac{|x|}{64}$, and $\varphi_2(|x|) = \frac{|x|}{96}$, we obtain

$$\frac{M^*}{0.9854645864M^* + 0.1469016279} > 1$$

which implies that $M^* > 10.10646356$. Hence, by Theorem 3.6, boundary value problem (4.4) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this article. They read and approved the final manuscript.

Authors' information
Sotiris K Ntouyas is a member of Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group at King Abdulaziz University, Jeddah, Saudi Arabia.

Author details
[1]Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology, North Bangkok, Bangkok, Thailand. [2]Department of Mathematics, University of Ioannina, Ioannina, 451 10, Greece.

Acknowledgements
The research of C Thaiprayoon and J Tariboon is supported by King Mongkut's University of Technology North Bangkok, Thailand.

Received: 3 December 2013 Accepted: 28 February 2014 Published: 17 Mar 2014

References

1. Tariboon, J, Ntouyas, SK: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. **2013**, 282 (2013)
2. Kac, V, Cheung, P: Quantum Calculus. Springer, New York (2002)
3. Bangerezako, G: Variational q-calculus. J. Math. Anal. Appl. **289**, 650-665 (2004)
4. Dobrogowska, A, Odzijewicz, A: Second order q-difference equations solvable by factorization method. J. Comput. Appl. Math. **193**, 319-346 (2006)
5. Gasper, G, Rahman, M: Some systems of multivariable orthogonal q-Racah polynomials. Ramanujan J. **13**, 389-405 (2007)
6. Ismail, MEH, Simeonov, P: q-difference operators for orthogonal polynomials. J. Comput. Appl. Math. **233**, 749-761 (2009)
7. Bohner, M, Guseinov, GS: The h-Laplace and q-Laplace transforms. J. Math. Anal. Appl. **365**, 75-92 (2010)
8. El-Shahed, M, Hassan, HA: Positive solutions of q-difference equation. Proc. Am. Math. Soc. **138**, 1733-1738 (2010)
9. Ahmad, B: Boundary-value problems for nonlinear third-order q-difference equations. Electron. J. Differ. Equ. **94**, 1-7 (2011)
10. Ahmad, B, Alsaedi, A, Ntouyas, SK: A study of second-order q-difference equations with boundary conditions. Adv. Differ. Equ. **2012**, 35 (2012)
11. Ahmad, B, Ntouyas, SK, Purnaras, IK: Existence results for nonlinear q-difference equations with nonlocal boundary conditions. Commun. Appl. Nonlinear Anal. **19**, 59-72 (2012)
12. Ahmad, B, Nieto, JJ: On nonlocal boundary value problems of nonlinear q-difference equations. Adv. Differ. Equ. **2012**, 81 (2012)
13. Ahmad, B, Ntouyas, SK: Boundary value problems for q-difference inclusions. Abstr. Appl. Anal. **2011**, 292860 (2011)
14. Zhou, W, Liu, H: Existence solutions for boundary value problem of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2013**, 113 (2013)
15. Yu, C, Wang, J: Existence of solutions for nonlinear second-order q-difference equations with first-order q-derivatives. Adv. Differ. Equ. **2013**, 124 (2013)
16. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
17. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
18. Benchohra, M, Henderson, J, Ntouyas, SK: Impulsive Differential Equations and Inclusions, vol. 2. Hindawi Publishing Corporation, New York (2006)
19. Krasnoselskii, MA: Two remarks on the method of successive approximations. Usp. Mat. Nauk **10**, 123-127 (1955)
20. Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003)

10.1186/1687-1847-2014-88
Cite this article as: Thaiprayoon et al.: **Separated boundary value problems for second-order impulsive q-integro-difference equations.** *Advances in Difference Equations* 2014, 2014:88