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Asymptotically periodic solutions of semilinear fractional integro-differential equations

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Abstract
In this paper, we study the existence of an $\mathcal{S}$-asymptotically ω-periodic mild solution of semilinear fractional integro-differential equations in Banach space, where the nonlinear perturbation is $\mathcal{S}$-asymptotically ω-periodic or $\mathcal{S}$-asymptotically ω-periodic in the Stepanov sense. A fixed point theorem and the nonlinear Leray-Schauder alternative theorem are the main tools in carrying out our proof. Some examples are given to show the efficiency and usefulness of the main findings.

MSC: 65R05; 35B40

Keywords: $\mathcal{S}$-asymptotically ω-periodic function; fractional integro-differential equations; sectorial operator; Leray-Schauder alternative theorem

1 Introduction

The study of the existence of periodic solutions is one of the most interesting and important topics in the qualitative theory of differential equations, due to its mathematical interest as well as their applications in physics, control theory, mathematical biology, among other areas. Some contributions on the existence of periodic solutions for differential equations have been made. Mostly, the environmental change in the real word is not periodic, but approximately periodic. For this reason, in the past decades many authors studied several extensions of the concept of periodicity, such as asymptotic periodicity, almost periodicity, almost automorphy, pseudo almost periodicity, pseudo almost automorphy, *etc.* and the same concept in the Stepanov sense, one can see [1–4] for more details.

The notion of $\mathcal{S}$-asymptotic ω-periodicity, introduced by Henríquez *et al.* in [5, 6], is related to and more general than that of asymptotic periodicity. Since then, it has attracted the attention of many researchers [7–13]. Recently, in [14], the concept of $\mathcal{S}$-asymptotic ω-periodicity in the Stepanov sense, which generalizes the notion of $\mathcal{S}$-asymptotic ω-periodicity, was introduced and the applications to semilinear first-order abstract differential equations were studied.

Due to their numerous applications in several branches of science, fractional integro-differential equations have received much attention in recent years [15–19]. The properties of solutions of fractional integro-differential equations have been studied from a different point of view, *e.g.*, maximal regularity [17], positivity and contractivity [20], asymptotic equivalence [21], asymptotic periodicity [22–25], almost periodicity [26, 27], almost automorphy [28, 29] and so on. To the best of our knowledge, there is no work reported in literature on $\mathcal{S}$-asymptotic ω-periodicity for fractional integro-differential equations if the nonlinear perturbation is $\mathcal{S}$-asymptotically ω-periodic in the Stepanov sense. This is one of the key motivations of this study.

The paper is organized as follows. In Section 2, some notations and preliminary results are presented. Section 3 is divided into two parts. In the first one, Section 3.1, we investigate the existence and uniqueness of an $\mathcal{S}$-asymptotically ω-periodic mild solution of semilinear fraction integro-differential equations when the nonlinear perturbation f satisfies the Lipschitz condition. In the second part, Section 3.2, when f is a non-Lipschitz case, we explore the properties of solutions for the same equation. In Section 4, we provide some examples to illustrate the main results.

2 Preliminaries and basic results

Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|_Y)$ be two Banach spaces and $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{C}$ stand for the set of natural numbers, real numbers, nonnegative real numbers, and complex numbers, respectively. In order to facilitate the discussion below, we further introduce the following notations:

- $BC(\mathbb{R}^+, X)$ (resp. $BC(\mathbb{R}^+ \times Y, X)$): the Banach space of bounded continuous functions from $\mathbb{R}^+$ to X (resp. from $\mathbb{R}^+ \times Y$ to X) with the supremum norm.
- $C(\mathbb{R}^+, X)$ (resp. $C(\mathbb{R}^+ \times Y, X)$): the set of continuous functions from $\mathbb{R}^+$ to X (resp. from $\mathbb{R}^+ \times Y$ to X).
- $L(X, Y)$: the Banach space of bounded linear operators from X to Y endowed with the operator topology. In particular, we write $L(X)$ when $X = Y$.
- $L^p(\mathbb{R}^+, X)$: the space of all classes of equivalence (with respect to the equality almost everywhere on $\mathbb{R}^+$) of measurable functions $f : \mathbb{R} \to X$ such that $\|f\| \in L^p(\mathbb{R}^+, \mathbb{R}^+)$.
- $L^p_{loc}(\mathbb{R}^+, X)$: stand for the space of all classes of equivalence of measurable functions $f : \mathbb{R}^+ \to X$ such that the restriction of f to every bounded subinterval of $\mathbb{R}^+$ is in $L^p(\mathbb{R}^+, X)$.

2.1 Sectorial operators and Riemann-Liouville fractional derivative

Definition 2.1 [30] A closed and densely defined linear operator A is said to be sectorial of type $\widetilde{\omega}$ if there exist $0 < \theta < \pi/2$, $M > 0$, and $\widetilde{\omega} \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\widetilde{\omega} + S_\theta := \{\widetilde{\omega} + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\},$$

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda - \widetilde{\omega}|}, \quad \lambda \notin \widetilde{\omega} + S_\theta.$$

The sectorial operators are well studied in the literature, we refer to [30] for more details.

Definition 2.2 [31] Let A be a closed and linear operator with domain $D(A)$ defined on a Banach space X. We call A the generator of a solution operator if there exist $\widetilde{\omega} \in \mathbb{R}$ and a strong continuous function $S_\alpha : \mathbb{R}^+ \to L(X)$ such that $\{\lambda^\alpha : Re\lambda > \widetilde{\omega}\} \subset \rho(A)$ and

$$\lambda^{\alpha-1}(\lambda^\alpha - A)^{-1}x = \int_0^\infty e^{-\lambda t} S_\alpha(t)x\,dt, \quad Re\lambda > \widetilde{\omega}, x \in X.$$

In this case, $S_\alpha(t)$ is called the solution operator generated by A.

Note that if A is sectorial of type $\widetilde{\omega}$ with $0 < \theta < \pi(1-\alpha/2)$, then A is the generator of a solution operator given by

$$S_\alpha(t) := \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}\bigl(\lambda^\alpha - A\bigr)^{-1}\,d\lambda,$$

where γ is a suitable path lying outside the sector $\widetilde{\omega} + S_\theta$ [32]. Recently, Cuesta [32] proved that if A is a sectorial operator of type $\widetilde{\omega} < 0$ for some $0 < \theta < \pi(1-\alpha/2)$ $(1 < \alpha < 2)$, $M > 0$, then there exists a constant $C > 0$ such that

$$\bigl\|S_\alpha(t)\bigr\| \le \frac{CM}{1+|\widetilde{\omega}|t^\alpha}, \quad t \ge 0. \tag{2.1}$$

Note that

$$\int_0^\infty \frac{1}{1+|\widetilde{\omega}|t^\alpha}\,dt = \frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}$$

for $1 < \alpha < 2$, therefore $S_\alpha(t)$ is integrable on $(0,\infty)$.

In the rest of this subsection, we list some necessary basic definitions in the theory of fractional calculus.

Definition 2.3 [19] The fractional order integral of order $\alpha > 0$ with the low limit $t_0 > 0$ for a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}f(s)\,ds, \quad t > t_0, \alpha > 0,$$

provided the right-hand side is pointwise defined on $[t_0,\infty)$, where Γ is the gamma function.

Definition 2.4 [19] Riemann-Liouville derivative of order $\alpha > 0$ with the low limit $t_0 > 0$ for a function $f : [t_0,\infty) \to \mathbb{R}$ can be written as

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^t (t-s)^{n-\alpha-1}f(s)\,ds, \quad t > t_0, n-1 < \alpha < n.$$

2.2 Compactness criterion and fixed point theorem

First, we recall two useful compactness criteria.

Let $h : [0,\infty) \to [1,\infty)$ be a continuous nondecreasing function such that $h(t) \to \infty$ as $t \to \infty$. Define

$$C_h\bigl(\mathbb{R}^+, X\bigr) := \Bigl\{u \in C\bigl(\mathbb{R}^+, X\bigr) : \lim_{t\to\infty}\bigl(u(t)/h(t)\bigr) = 0\Bigr\}$$

endowed with the norm $\|u\|_h = \sup_{t\ge 0}(\|u(t)\|/h(t))$.

Lemma 2.1 [33] *A set $K \subseteq C_h(\mathbb{R}^+, X)$ is relatively compact in $C_h(\mathbb{R}^+, X)$ if it verifies the following conditions*:

(c_1) *For all $b > 0$, the set $K_b(t) := \{u|_{[0,b]} : u \in K\}$ is relatively compact in $C([0,b], X)$.*
(c_2) $\lim_{t\to\infty}(\|u(t)\|/h(t)) = 0$ *uniformly for $u \in K$.*

Lemma 2.2 (Simon's theorem [34]) *Let $F \subset L^p([0,T],X)$, F is relatively compact in $L^p([0,T],X)$ for $1 \leq p < \infty$ if and only if*

(1) *$\{\int_{t_1}^{t_2} f(t)\,dt : f \in F, \forall 0 < t_1 < t_2 < T\}$ is relatively compact in X.*

(2) *$\|\tau_h f - f\|_{L^p([0,T-h],X)} \to 0$ as $h \to 0$ uniformly for $f \in F$, where $(\tau_h f)(t) = f(t+h)$.*

Now, we recall the so-called Zima's fixed point theorem [35] and the Leray-Schauder alternative theorem [36] which will be used in the sequel.

Let $(Y, \|\cdot\|_Y, \prec, m)$ denote a Banach space of elements $y \in Y$ with a binary relation '$\prec$' and a mapping $m : Y \to Y$ such that

(i) the relation $\prec$ is transitive;

(ii) $0 \prec m(u)$ and $\|m(u)\|_Y = \|u\|_Y$ for all $u \in Y$;

(iii) the norm $\|\cdot\|_Y$ is monotonic, that is, if $0 \prec u \prec v$, then $\|u\|_Y \leq \|v\|_Y$ for all $u, v \in Y$.

Theorem 2.1 ([35] Zima's fixed point theorem) *In the Banach space considered above, let the operators $\Gamma : Y \to Y$ and $B : Y \to Y$ be given with the following properties:*

(iv) *B is a bounded linear operator with spectral radius $r(B) < 1$.*

(v) *B is increasing, that is, if $0 \prec u \prec v$, then $Bu \prec Bv$ for all $u, v \in Y$.*

(vi) *$m(\Gamma u - \Gamma v) \prec Bm(u - v)$ for all $u, v \in Y$.*

Then the equation $\Gamma u = u$ has a unique solution in Y.

Theorem 2.2 ([36] Leray-Schauder alternative theorem) *Let D be a closed convex subset of a Banach space X such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

2.3 $\mathcal{S}$-Asymptotic ω-periodicity in the Stepanov sense

For $\omega > 0$, define

$$C_0(\mathbb{R}^+, X) = \Big\{x \in BC(\mathbb{R}^+, X) : \lim_{t\to\infty} \|x(t)\| = 0\Big\}.$$

$$C_\omega(\mathbb{R}^+, X) = \big\{x \in BC(\mathbb{R}^+, X) : x \text{ is } \omega\text{-periodic}\big\}.$$

Definition 2.5 [37] A function $f \in BC(\mathbb{R}^+, X)$ is called asymptotically ω-periodic if there exist $g \in C_\omega(\mathbb{R}^+, X)$, $\varphi \in C_0(\mathbb{R}^+, X)$ such that $f = g + \varphi$. The collection of those functions is denoted by $AP_\omega(\mathbb{R}^+, X)$.

Definition 2.6 [5] A function $f \in BC(\mathbb{R}^+, X)$ is said to be $\mathcal{S}$-asymptotically periodic if there exists $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. In this case, we say that f is $\mathcal{S}$-asymptotically ω-periodic. The collection of those functions is denoted by $\mathcal{S}AP_\omega(\mathbb{R}^+, X)$.

Definition 2.7 [5] A continuous function $f : \mathbb{R}^+ \times X \to X$ is said to be uniformly $\mathcal{S}$-asymptotically ω-periodic on bounded sets if for every bounded set K of X, the set $\{f(t,x) : t \geq 0, x \in K\}$ is bounded and $\lim_{t\to\infty}(f(t+\omega, x) - f(t,x)) = 0$ uniformly in $x \in K$. Denote by $\mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$ the set of such functions.

Definition 2.8 [5] A continuous function $f : \mathbb{R}^+ \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets if for every $\varepsilon > 0$ and every bounded set $K \subseteq X$, there exist $L_\varepsilon \geq 0$ and $\delta_\varepsilon > 0$ such that $\|f(t,x) - f(t,y)\| \leq \varepsilon$ for all $t \geq t_\varepsilon$ and all $x, y \in K$ with $\|x - y\| \leq \delta_\varepsilon$.

We introduce the following composition theorem for an $\mathcal{S}$-asymptotically ω-periodic function.

Lemma 2.3 [5] *Assume that $f \in \mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$ is an asymptotically uniformly continuous on bounded sets function. Let $u \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$, then $\upsilon(\cdot) = f(\cdot, u(\cdot)) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$.*

Let $p \in [1, \infty)$. The space $BS^p(\mathbb{R}, X)$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \to X$ such that $f^b \in L^\infty(\mathbb{R}, L^p([0,1]; X))$, where f^b is the Bochner transform of f defined by $f^b(t,s) := f(t+s)$, $t \in \mathbb{R}$, $s \in [0,1]$. $BS^p(\mathbb{R}, X)$ is a Banach space with the norm [38]

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p}.$$

It is obvious that $L^p(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X) \subset L^p_{loc}(\mathbb{R}, X)$ and $BS^p(\mathbb{R}, X) \subset BS^q(\mathbb{R}, X)$ for $p \geq q \geq 1$. We denote by $BS^p_0(\mathbb{R}, X)$ the subspace of $BS^p(\mathbb{R}, X)$ consisting of functions f such that $\int_t^{t+1} \|f(s)\|^p \, ds \to 0$ as $t \to \infty$.

Definition 2.9 [14] A function $f \in BS^p(\mathbb{R}^+, X)$ is called $\mathcal{S}$-asymptotically ω-periodic in the Stepanov sense (or S^p-$\mathcal{S}$-asymptotically ω-periodic) if

$$\lim_{t \to \infty} \int_t^{t+1} \|f(s+\omega) - f(s)\|^p \, ds = 0.$$

Denote by $S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ the set of such functions.

It is easy to see that

$$C_0(\mathbb{R}^+, X) \subset AP_\omega(\mathbb{R}^+, X) \subset \mathcal{S}AP_\omega(\mathbb{R}^+, X) \subset S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X).$$

Definition 2.10 [14] A function $f : \mathbb{R}^+ \times X \to X$ is said to be uniformly $\mathcal{S}$-asymptotically ω-periodic on bounded sets in the Stepanov sense if for every bounded set $B \subseteq X$, there exist positive functions $g_B \in BS^p(\mathbb{R}^+, \mathbb{R})$ and $h_B \in BS^p_0(\mathbb{R}^+, \mathbb{R})$ such that $\|f(t,x)\| \leq g_B(t)$ for all $t \in \mathbb{R}^+$, $x \in B$ and

$$\|f(t+\omega, x) - f(t,x)\| \leq h_B(s) \quad \text{for all } s \geq 0, x \in B.$$

Denote by $S^p\mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$ the set of such functions.

Definition 2.11 [14] A function $f : \mathbb{R}^+ \times X \to X$ is said to be asymptotically uniformly continuous on bounded sets in the Stepanov sense if for every $\varepsilon > 0$ and every bounded set $B \subseteq X$, there exist $t_\varepsilon \geq 0$ and $\delta_\varepsilon > 0$ such that

$$\int_t^{t+1} \|f(s,x) - f(s,y)\|^p \, ds \leq \varepsilon^p$$

for all $t \geq t_\varepsilon$ and all $x, y \in B$ with $\|x - y\| \leq \delta_\varepsilon$.

Lemma 2.4 [14] *Assume that $f \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$ is an asymptotically uniformly continuous on bounded sets in the Stepanov sense function. Let $u \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$, then $\upsilon(\cdot) = f(\cdot, u(\cdot)) \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$.*

Lemma 2.5 *Let $\{S(t)\}_{t\geq 0} \subset L(X)$ be a strongly continuous family of bounded and linear operators such that $\|S(t)\| \leq \phi(t)$, $t \in \mathbb{R}^+$, where $\phi \in L^1(\mathbb{R}^+)$ is nonincreasing. If $f \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$, then*

$$(\Lambda f)(t) := \int_0^t S(t-s)f(s)\,ds \in \mathcal{S}AP_\omega\big(\mathbb{R}^+, X\big), \quad t \in \mathbb{R}^+.$$

Proof For $n \leq t \leq n+1$, $n \in \mathbb{N}$, one has

$$\begin{aligned}
\big\|(\Lambda f)(t)\big\| &= \int_0^t \big\|S(s)\big\| \big\|f(t-s)\big\|\,ds \\
&\leq \int_0^{n+1} \phi(s)\big\|f(t-s)\big\|\,ds \\
&= \sum_{k=0}^{n} \int_k^{k+1} \phi(s)\big\|f(t-s)\big\|\,ds \\
&\leq \sum_{k=0}^{n} \phi(k)\bigg(\int_k^{k+1} \big\|f(t-s)\big\|^p\,ds\bigg)^{1/p} \\
&\leq \big(\phi(0) + \phi(1) + \cdots + \phi(n)\big)\|f\|_{S^p} \\
&\leq \bigg(\phi(0) + \int_0^1 \phi(t)\,dt + \cdots + \int_{n-1}^{n} \phi(t)\,dt\bigg)\|f\|_{S^p} \\
&\leq \big(\phi(0) + \|\phi\|_{L^1}\big)\|f\|_{S^p},
\end{aligned}$$

that is, Λf is bounded. It is clear that Λf is continuous for each $t \in \mathbb{R}^+$, whence $\Lambda f \in BC(\mathbb{R}^+, X)$. Moreover, note that

$$\begin{aligned}
&(\Lambda f)(t+\omega) - (\Lambda f)(t) \\
&\quad = \int_0^{t+\omega} S(t+\omega-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds \\
&\quad = \int_0^{\omega} S(t+\omega-s)f(s)\,ds + \int_{\omega}^{t+\omega} S(t+\omega-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds \\
&\quad = \int_0^{\omega} S(t+\omega-s)f(s)\,ds + \int_0^t S(t-s)\big[f(s+\omega) - f(s)\big]\,ds, \\
&\quad := I(t) + J(t),
\end{aligned}$$

where

$$I(t) = \int_0^{\omega} S(t+\omega-s)f(s)\,ds, \qquad J(t) = \int_0^t S(t-s)\big[f(s+\omega) - f(s)\big]\,ds.$$

By the hypothesis of ϕ, one has

$$\|I(t)\| \leq \int_0^\omega \phi(t+\omega-s)\|f(s)\|\,ds \leq \phi(t)\int_0^\omega \|f(s)\|\,ds \to 0, \quad t\to\infty,$$

then

$$\lim_{t\to\infty}\|I(t)\|\,dt = 0.$$

On the other hand, since $f \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$, there exists $m \in \mathbb{N}$ such that

$$\left(\int_t^{t+1}\|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} < \varepsilon \quad \text{for } t \geq m.$$

For $m \leq n \leq t \leq n+1$, one has

$$\begin{aligned}
\|J(t)\| &\leq \int_0^t \|S(t-s)\|\,\|f(s+\omega)-f(s)\|\,ds \\
&\leq \int_0^n \phi(t-s)\|f(s+\omega)-f(s)\|\,ds + \int_n^t \phi(t-s)\|f(s+\omega)-f(s)\|\,ds \\
&\leq \int_0^n \phi(n-s)\|f(s+\omega)-f(s)\|\,ds + \phi(0)\int_n^t \|f(s+\omega)-f(s)\|\,ds \\
&\leq \sum_{k=0}^{n-1}\int_k^{k+1} \phi(n-s)\|f(s+\omega)-f(s)\|\,ds + \phi(0)\int_n^{n+1} \|f(s+\omega)-f(s)\|\,ds \\
&\leq \sum_{k=0}^{n-1}\phi(n-k-1)\int_k^{k+1} \|f(s+\omega)-f(s)\|\,ds + \phi(0)\int_n^{n+1} \|f(s+\omega)-f(s)\|\,ds \\
&\leq \sum_{k=0}^{n-1}\phi(n-k-1)\left(\int_k^{k+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\quad + \phi(0)\left(\int_n^{n+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&= \sum_{k=0}^{m}\phi(n-k-1)\left(\int_k^{k+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\quad + \phi(0)\left(\int_n^{n+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\quad + \sum_{k=m+1}^{n-1}\phi(n-k-1)\left(\int_k^{k+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\leq \big(\phi(n-1)+\phi(n-2)+\cdots \\
&\quad + \phi(n-m-1)\big)\max_{0\leq k\leq m}\left(\int_k^{k+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\quad + \phi(0)\left(\int_n^{n+1} \|f(s+\omega)-f(s)\|^p\,ds\right)^{1/p} \\
&\quad + \big(\phi(n-m-2)+\phi(n-m-3)+\cdots+\phi(0)\big)\varepsilon
\end{aligned}$$

$$\begin{aligned}
&\le \int_{n-m-2}^{n-1} \phi(t)\,dt \cdot \max_{0\le k\le m} \left(\int_k^{k+1} \big\| f(s+\omega) - f(s) \big\|^p \,ds \right)^{1/p} \\
&\quad + \phi(0) \left(\int_n^{n+1} \big\| f(s+\omega) - f(s) \big\|^p \,ds \right)^{1/p} + \left(\phi(0) + \int_0^{n-m-2} \phi(t)\,dt \right) \varepsilon \\
&\le \int_{n-m-2}^{n-1} \phi(t)\,dt \cdot \max_{0\le k\le m} \left(\int_k^{k+1} \big\| f(s+\omega) - f(s) \big\|^p \,ds \right)^{1/p} \\
&\quad + \phi(0) \left(\int_n^{n+1} \big\| f(s+\omega) - f(s) \big\|^p \,ds \right)^{1/p} + \big(\phi(0) + \|\phi\|_{L^1} \big) \varepsilon,
\end{aligned}$$

which implies that $\|J(t)\| \to 0$ as $t \to \infty$. So

$$\lim_{t\to\infty} \big\| (\Lambda f)(t+\omega) - (\Lambda f)(t) \big\| = 0.$$

The proof is complete. □

3 Semilinear fractional integro-differential equation

Consider the semilinear fractional integro-differential equation

$$\begin{cases} u'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Au(s)\,ds + f(t,u(t)), & t \in \mathbb{R}^+, \\ u(0) = u_0 \in X, \end{cases} \tag{3.1}$$

where $1 < \alpha < 2$, $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X and $f : \mathbb{R}^+ \times X \to X$ is an appropriate function.

Before starting our main results, we recall the definition of the mild solution to (3.1).

Definition 3.1 [23] Assume that A generates a solution operator $S_\alpha(t)$. A function $u \in BC(\mathbb{R}^+, X)$ is called a mild solution of (3.1) if

$$u(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s) f\big(s, u(s)\big)\,ds, \quad t \in \mathbb{R}^+.$$

To study (3.1), we require the following assumptions:

(H_1) A is a sectorial operator of type $\widetilde{\omega} < 0$ with $0 < \theta < \pi(1-\alpha/2)$.

(H_2) $f \in \mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$.

(H_2') $f \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$, $p \ge 1$.

(H_{31}) f satisfies the Lipschitz condition

$$\big\| f(t,u) - f(t,v) \big\| \le L_f \|u - v\|, \quad u, v \in X, t \in \mathbb{R}^+.$$

(H_{32}) f satisfies the Lipschitz condition

$$\big\| f(t,u) - f(t,v) \big\| \le L_f(t) \|u - v\|, \quad u, v \in X, t \in \mathbb{R}^+,$$

where $L_f \in BS^p(\mathbb{R}^+, \mathbb{R}^+)$.

(H_{33}) f satisfies the Lipschitz condition

$$\big\| f(t,u) - f(t,v) \big\| \le L_f(t) \|u - v\|, \quad u, v \in X, t \in \mathbb{R}^+,$$

where $L_f \in BS_0^p(\mathbb{R}^+, \mathbb{R}^+)$.

(H_4) f is asymptotically uniformly continuous on bounded sets.
(H_4') f is asymptotically uniformly continuous on bounded sets in the Stepanov sense.

3.1 Lipschitz case

In this subsection, we study the existence and uniqueness of $\mathcal{S}$-asymptotically ω-periodic mild solution of (3.1) when f satisfies the Lipschitz condition.

If $f(t,u)$ is uniformly Lipschitz continuous at u, *i.e.*, (H_{31}) holds, we reach the following claim.

Theorem 3.1 *Assume that* (H_1), (H_2) (*or* (H_2')), (H_{31}) *hold, then* (3.1) *has a unique mild solution* $u(t) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$ *if* $CM|\widetilde{\omega}|^{-1/\alpha}\pi L_f < \alpha \sin(\pi/\alpha)$.

Proof Define the operator $\mathcal{F} : \mathcal{S}AP_\omega(\mathbb{R}^+, X) \to \mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by

$$(\mathcal{F}u)(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f\bigl(s,u(s)\bigr)\,ds, \quad t \in \mathbb{R}^+. \tag{3.2}$$

By (2.1), one has $\lim_{t\to\infty} \|S_\alpha(t)u_0\| = 0$, so $S_\alpha(t)u_0 \in C_0(\mathbb{R}^+, X) \subset \mathcal{S}AP_\omega(\mathbb{R}^+, X)$. By ($H_{31}$), if ($H_2$) holds, $f(\cdot, u(\cdot)) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X) \subset S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Lemma 2.3, and if (H_2') holds, $f(\cdot, u(\cdot)) \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Lemma 2.4. Hence $\mathcal{F}$ is well defined by Lemma 2.5.

Moreover, let $u, v \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$, one has

$$\begin{aligned}
\bigl\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\bigr\| &\le \int_0^t \bigl\|S_\alpha(t-s)\bigr\| \bigl\|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr\|\,ds \\
&\le L_f \int_0^t \bigl\|S_\alpha(t-s)\bigr\| \bigl\|u(s) - v(s)\bigr\|\,ds \\
&\le L_f \|u - v\| \int_0^t \bigl\|S_\alpha(s)\bigr\|\,ds \\
&\le L_f \|u - v\| \int_0^t \frac{CM}{1 + |\widetilde{\omega}|s^\alpha}\,ds \\
&\le \frac{CM|\widetilde{\omega}|^{-1/\alpha}\pi L_f}{\alpha \sin(\pi/\alpha)} \|u - v\|,
\end{aligned}$$

by the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{S}AP_\omega(\mathbb{R}^+, X)$, which is the unique $\mathcal{S}AP_\omega$ mild solution to (3.1). □

Theorem 3.2 *Assume that* (H_1), (H_2) (*or* (H_2')), (H_{32}) *hold and*

$$CM\left(1 + \frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)}\right)\|L_f\|_{S^p} < 1, \tag{3.3}$$

then (3.1) *has a unique mild solution* $u(t) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$.

Proof Define the operator $\mathcal{F}$ as in (3.2). If (H_2) holds, then $f \in \mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X) \subset S^p\mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$. Since ($H_{32}$) holds, f is asymptotically uniformly continuous on bounded sets in the Stepanov sense, so $f(\cdot, u(\cdot)) \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Lemma 2.4. If (H_2') holds, $f(\cdot, u(\cdot)) \in S^p\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Lemma 2.4. Hence $\mathcal{F}$ is well defined by Lemma 2.5.

For $u, v \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$, one has

$$\begin{aligned}\big\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\big\| &\leq \int_0^t \big\|S_\alpha(t-s)\big\| \big\|f\big(s, u(s)\big) - f\big(s, v(s)\big)\big\| \, ds \\ &\leq \int_0^t \frac{CM}{1 + |\widetilde{\omega}|(t-s)^\alpha} L_f(s) \, ds \cdot \|u - v\|.\end{aligned}$$

- If $t = m \in \mathbb{N}$, in this case

$$\begin{aligned}\int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\, ds &= \int_0^m \frac{1}{1+|\widetilde{\omega}|(m-s)^\alpha} L_f(s)\, ds \\ &= \sum_{k=0}^{m-1} \int_k^{k+1} \frac{1}{1+|\widetilde{\omega}|(m-s)^\alpha} L_f(s)\, ds \\ &\leq \sum_{k=0}^{m-1} \frac{1}{1+|\widetilde{\omega}|(m-k-1)^\alpha} \int_k^{k+1} L_f(s)\, ds \\ &\leq \sum_{k=0}^{m-1} \frac{1}{1+|\widetilde{\omega}|(m-k-1)^\alpha} \left(\int_k^{k+1} L_f(s)^p\, ds \right)^{1/p} \\ &\leq \left[1 + \left(\int_0^1 + \int_1^2 + \cdots + \int_{m-2}^{m-1} \right) \frac{1}{1+|\widetilde{\omega}|t^\alpha}\, dt \right] \|L_f\|_{S^p} \\ &\leq \left(1 + \int_0^\infty \frac{1}{1+|\widetilde{\omega}|t^\alpha}\, dt \right) \|L_f\|_{S^p} \\ &\leq \left(1 + \frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \right) \|L_f\|_{S^p}. \end{aligned} \tag{3.4}$$

- If $t = m - h$, where $0 < h < 1$. In this general case,

$$\begin{aligned}\int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\, ds &= \int_0^{m-h} \frac{1}{1+|\widetilde{\omega}|(m-h-s)^\alpha} L_f(s)\, ds \\ &= \int_h^m \frac{1}{1+|\widetilde{\omega}|(m-s)^\alpha} L_f(s-h)\, ds \\ &= \int_0^m \frac{1}{1+|\widetilde{\omega}|(m-s)^\alpha} \widetilde{L}_f(s)\, ds \\ &\leq \left(1 + \frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \right) \|\widetilde{L}_f\|_{S^p},\end{aligned}$$

where $\widetilde{L}_f$ is defined by

$$\widetilde{L}_f(s) = \begin{cases} 0, & 0 \leq s < h, \\ L_f(s-h), & s \geq h, \end{cases}$$

then $\|\widetilde{L}_f\|_{S^p} = \|L_f\|_{S^p}$. So we infer that

$$\int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\, ds \leq \left(1 + \frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \right) \|L_f\|_{S^p}. \tag{3.5}$$

By (3.4), (3.5), one has

$$\big\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\big\| \le CM\left(1+\frac{|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha\sin(\pi/\alpha)}\right)\|L_f\|_{S^p}\|u-v\|. \tag{3.6}$$

By the Banach contraction mapping principle, $\mathcal{F}$ has a unique fixed point in $\mathcal{S}AP_\omega(\mathbb{R}^+,X)$, which is the unique $\mathcal{S}AP_\omega$ mild solution to (3.1). □

In next results, we relax condition (3.3) to study the existence and uniqueness of $\mathcal{S}AP_\omega$ mild solution of (3.1).

Theorem 3.3 *Assume that* (H_1), (H_2) (*or* (H_2')), (H_{32}) *hold and the integral* $\int_0^t L_f(s)\,ds$ *exists for all* $t\in\mathbb{R}^+$. *Then* (3.1) *has a unique mild solution* $u(t)\in\mathcal{S}AP_\omega(\mathbb{R}^+,X)$.

Proof Define an equivalent norm on $\mathcal{S}AP_\omega(\mathbb{R}^+,X)$ as $\|f\|_c=\sup_{t\in\mathbb{R}^+}\{e^{-c\lambda(t)}\|f\|\}$, where $c>MC$ and $\lambda(t)=\int_0^t L_f(\tau)\,d\tau$. Define the operator $\mathcal{F}$ as in (3.2). Let $u,v\in\mathcal{S}AP_\omega(\mathbb{R}^+,X)$, one has

$$\begin{aligned}\big\|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\big\| &\le \int_0^t \big\|S_\alpha(t-s)\big\|\big\|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big\|\,ds\\ &\le \int_0^t \frac{CM}{1+|\widetilde{\omega}|(t-s)^\alpha}L_f(s)\big\|u(s)-v(s)\big\|\,ds\\ &\le CM\|u-v\|_c\int_0^t L_f(s)e^{c\lambda(s)}\,ds\\ &= CM\|u-v\|_c\int_0^t \lambda'(s)e^{c\lambda(s)}\,ds\\ &\le \frac{CM}{c}\|u-v\|_c e^{c\lambda(t)},\end{aligned}$$

consequently,

$$\|\mathcal{F}u-\mathcal{F}v\|_c \le \frac{CM}{c}\|u-v\|_c.$$

Since $c>MC$, $\mathcal{F}$ is a contraction and then it has a unique fixed point $u(t)$, which is the unique $\mathcal{S}AP_\omega$ mild solution to (3.1). □

Theorem 3.4 *Assume that* (H_1), (H_2) (*or* (H_2')), (H_{33}) *hold, then* (3.1) *has a unique mild solution* $u(t)\in\mathcal{S}AP_\omega(\mathbb{R}^+,X)$.

Proof Define the operator $\mathcal{F}$ as in (3.2), then $\mathcal{F}$ is a map from $\mathcal{S}AP_\omega(\mathbb{R}^+,X)$ into $\mathcal{S}AP_\omega(\mathbb{R}^+,X)$. Moreover, $\mathcal{F}$ is continuous by (3.6). Define the map B on $BC(\mathbb{R}^+,\mathbb{R})$ by

$$(B\alpha)(t)=CM\int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha}L_f(s)\alpha(s)\,ds,\quad t\in\mathbb{R}^+. \tag{3.7}$$

It is clear that B is a bounded linear operator from $BC(\mathbb{R}^+,\mathbb{R})$ into $BC(\mathbb{R}^+,\mathbb{R})$.

First, we will show that B is a compact operator. For each $a \geq 0$ and each $\alpha \in BC(\mathbb{R}^+, \mathbb{R})$ with $\|\alpha\| \leq 1$, define the functions

$$w_1(\alpha)(t) = \begin{cases} CM \int_0^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\alpha(s)\,ds, & 0 \leq t \leq a, \\ CM \int_0^a \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\alpha(s)\,ds, & t \geq a, \end{cases}$$

and

$$w_2(\alpha)(t) = \begin{cases} 0, & 0 \leq t \leq a, \\ CM \int_a^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\alpha(s)\,ds, & t \geq a. \end{cases}$$

It follows from the Ascoli-Arzelá theorem in the space $C_0(\mathbb{R}^+, \mathbb{R})$ that the set $K_a = \{w_1(\alpha) : \|\alpha\| \leq 1\}$ is relatively compact in $C_0(\mathbb{R}^+, \mathbb{R})$, and therefore in $BC(\mathbb{R}^+, \mathbb{R})$.

Since $L_f \in BS_0^p(\mathbb{R}^+, \mathbb{R})$, for each $\varepsilon > 0$, take $a \geq 0$ such that for $t \geq a$,

$$\sup_{r \geq a} \left(\int_r^{r+1} L_f(s)^p\,ds \right)^{1/p} < \varepsilon.$$

For $a + m \leq t < a + m + 1$, $m \in \mathbb{N}$, one has

$$\begin{aligned}
&\bigl|w_2(\alpha)(t)\bigr| \\
&\quad = CM \int_a^{a+m} \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\bigl|\alpha(s)\bigr|\,ds + CM \int_{a+m}^t \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\bigl|\alpha(s)\bigr|\,ds \\
&\quad \leq CM \sum_{k=0}^{m-1} \int_{a+k}^{a+k+1} \frac{1}{1+|\widetilde{\omega}|(t-s)^\alpha} L_f(s)\bigl|\alpha(s)\bigr|\,ds + CM \int_{a+m}^t L_f(s)\bigl|\alpha(s)\bigr|\,ds \\
&\quad \leq CM \sum_{k=0}^{m-1} \int_{a+k}^{a+k+1} \frac{1}{1+|\widetilde{\omega}|(a+m-s)^\alpha} L_f(s)\,ds + CM \int_{a+m}^{a+m+1} L_f(s)\,ds \\
&\quad \leq CM \sum_{k=0}^{m-1} \frac{1}{1+|\widetilde{\omega}|(m-k-1)^\alpha} \int_{a+k}^{a+k+1} L_f(s)\,ds + CM \int_{a+m}^{a+m+1} L_f(s)\,ds \\
&\quad \leq CM \sum_{k=0}^{m-1} \frac{1}{1+|\widetilde{\omega}|(m-k-1)^\alpha} \left(\int_{a+k}^{a+k+1} L_f(s)^p\,ds \right)^{1/p} + CM \left(\int_{a+m}^{a+m+1} L_f(s)^p\,ds \right)^{1/p} \\
&\quad \leq CM \int_0^{m-1} \frac{1}{1+|\widetilde{\omega}|t^\alpha}\,dt \cdot \sup_{r \geq a} \left(\int_r^{r+1} L_f(s)^p\,ds \right)^{1/p} + CM \left(\int_{a+m}^{a+m+1} L_f(s)^p\,ds \right)^{1/p} \\
&\quad \leq \frac{CM|\widetilde{\omega}|^{-1/\alpha}\pi}{\alpha \sin(\pi/\alpha)} \cdot \sup_{r \geq a} \left(\int_r^{r+1} L_f(s)^p\,ds \right)^{1/p} + CM \left(\int_{a+m}^{a+m+1} L_f(s)^p\,ds \right)^{1/p},
\end{aligned}$$

then $|w_2(\alpha)t| \leq \varepsilon$. Since $B\alpha(t) = w_1(\alpha)(t) + w_2(\alpha)(t)$ for $t \in \mathbb{R}^+$, one has

$$\bigl\{B(\alpha) : \|\alpha\| \leq 1\bigr\} \subseteq K_a + \bigl\{\varphi : \varphi \in BC\bigl(\mathbb{R}^+, \mathbb{R}\bigr), \|\varphi\| \leq \varepsilon\bigr\},$$

which implies that $\{B(\alpha) : \|\alpha\| \leq 1\}$ is relatively compact, so B is a compact operator. Moreover, it follows from the Gronwall-Bellman lemma that the point spectrum $\sigma_p(B) = \{0\}$, which implies that the spectral radius of B is equal to zero since B is a compact operator.

Consider the Banach space $Y = BC(\mathbb{R}^+, \mathbb{R})$ equipped with both the relation $\prec$ and the mapping $m : BC(\mathbb{R}^+, \mathbb{R}) \to BC(\mathbb{R}^+, \mathbb{R})$ defined by: if $u, v \in BC(\mathbb{R}^+, \mathbb{R})$

$$u \prec v \quad \text{if and only if} \quad \|u(t)\| \le \|v(t)\| \quad \forall t \in \mathbb{R}^+,$$

and $(m(u))(t) = \sup_{0 \le s \le t} \|u(s)\|$. It is easy to check that conditions (i), (ii), (iii) are satisfied.

Let $u, v \in BC(\mathbb{R}^+, \mathbb{R})$, one has

$$\|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\| \le CM \int_0^t \frac{1}{1 + |\widetilde{\omega}|(t-s)^\alpha} L_f(s) \|u(s) - v(s)\| \, ds,$$

hence $m(\mathcal{F}(u) - \mathcal{F}(v)) \prec Bm(u - v)$, and B is increasing with spectral radius $r(B) < 1$. By Theorem 2.1, $\mathcal{F}$ has a unique fixed point in $BC(\mathbb{R}^+, \mathbb{R})$, which is the unique $\mathcal{S}AP_\omega$ mild solution to (3.1). □

3.2 Non-Lipschitz case

In this subsection, we study the existence of $\mathcal{S}$-asymptotically ω-periodic mild solution of (3.1) when f does not satisfy the Lipschitz condition.

The following existence result is based upon the nonlinear Leray-Schauder alternative theorem.

Theorem 3.5 *Assume that* (H_1), (H_2), (H_4) *hold* (*or* (H_1), (H_2'), (H_4') *hold*) *and satisfy the following conditions*:

- (A_1) *There exists a continuous nondecreasing function* $W : [0, +\infty) \to [0, +\infty)$ *such that* $\|f(t, u)\| \le W(\|u\|)$ *for all* $t \in \mathbb{R}^+$, $u \in X$.
- (A_2) *For each* $\nu > 0$, $\lim_{t \to \infty} \frac{1}{h(t)} \int_0^t \frac{W(\nu h(s))}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds = 0$.
- (A_3) *For each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for* $u, v \in C_h(\mathbb{R}^+, X)$, $\|u - v\|_h \le \delta$ *implies that*

$$\int_0^t \frac{\|f(s, u(s)) - f(s, v(s))\|}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds \le \varepsilon \quad \text{for all } t \in \mathbb{R}^+.$$

- (A_4) *For all* $a, b \in \mathbb{R}^+$, $a \le b$ *and* $r \ge 0$, *the set* $\{f(s, u) : a \le s \le b, u \in X, \|u\| \le r\}$ *is relatively compact in* X.
- (A_5) $\liminf_{\xi \to \infty} \frac{\xi}{\beta(\xi)} > 1$, *where* $\beta(\nu) = \|\sigma_\nu\|_h$ *and*

$$\sigma_\nu(t) := \|S_\alpha(\cdot)u_0\| + CM \int_0^t \frac{W(\nu h(s))}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds, \quad t \ge 0,$$

  C, M are constants given in (2.1).

Then (3.1) *has a mild solution* $u(t) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$.

Proof Define $\Gamma : C_h(\mathbb{R}^+, X) \to C(\mathbb{R}^+, X)$ by

$$(\Gamma u)(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s) f\big(s, u(s)\big) \, ds, \quad t \in \mathbb{R}^+.$$

Next, we prove that Γ has a fixed point in $\mathcal{S}AP_\omega(\mathbb{R}^+, X)$. We divide the proof into several steps.

(i) For $x \in C_h(\mathbb{R}^+, X)$, by (A_1), one has

$$\begin{aligned}\frac{\|\Gamma u(t)\|}{h(t)} &\le \frac{CM}{h(t)}\|u_0\| + \frac{CM}{h(t)}\int_0^t \frac{\|f(s,u)\|}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds \\ &\le \frac{CM}{h(t)}\|u_0\| + \frac{CM}{h(t)}\int_0^t \frac{W(\|u\|_h h(s))}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds.\end{aligned}$$

It follows from (A_2) that $\Gamma : C_h(\mathbb{R}^+, X) \to C_h(\mathbb{R}^+, X)$.

(ii) Γ is continuous. In fact, for each $\varepsilon > 0$, by (A_3), there exits $\delta > 0$, for $u, v \in C_h(\mathbb{R}^+, X)$ and $\|u - v\|_h \le \delta$, one has

$$\begin{aligned}\|\Gamma u - \Gamma v\| &\le \int_0^t \|S_\alpha(t-s)\| \big\| f\big(s,u(s)\big) - f\big(s,v(s)\big)\big\|\,ds \\ &\le CM\int_0^t \frac{\|f(s,u(s)) - f(s,v(s))\|}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds.\end{aligned}$$

Take into account that $h(t) \ge 1$, by (A_3)

$$\frac{\|\Gamma u - \Gamma v\|}{h(t)} \le CM\varepsilon,$$

which implies that $\|\Gamma u - \Gamma v\|_h \le CM\varepsilon$, so Γ is continuous.

(iii) Γ is completely continuous. Set $B_r(Z)$ for the closed ball with center at 0 and radius r in the space Z. Let $V = \Gamma(B_r(C_h(\mathbb{R}^+, X)))$ and $v = \Gamma(u)$ for $u \in B_r(C_h(\mathbb{R}^+, X))$.

Initially, we prove that $V_b(t)$ is a relatively compact subset of X for each $t \in [0, b]$, here $V_b(t) = \{v(t), v \in V, t \in [0, b]\}$. Since

$$v(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(s) f\big(t-s, u(t-s)\big)\,ds \in S_\alpha(t)u_0 + t\overline{c(K)},$$

where $c(K)$ denotes the convex hull of K and $K = \{S_\alpha(s)f(\xi, u) : 0 \le s \le t, 0 \le \xi \le t, \|u\| \le r\}$. Using the fact that $S_\alpha(\cdot)$ is strong continuous and (A_4), we infer that K is a relatively compact set, and $V_b(t) \subseteq S_\alpha(t)u_0 + t\overline{c(K)}$ is also a relatively compact set.

Next, we show that V_b is equicontinuous. In fact,

$$\begin{aligned}v(t+s) - v(t) &= \big(S_\alpha(t+s) - S_\alpha(t)\big)u_0 + \int_t^{t+s} S_\alpha(t+s-\xi) f\big(\xi, u(\xi)\big)\,d\xi \\ &\quad + \int_0^t \big(S_\alpha(\xi+s) - S_\alpha(\xi)\big) f\big(t-\xi, u(t-\xi)\big)\,d\xi.\end{aligned}$$

For each $\varepsilon > 0$, we can choose $\delta_1 > 0$ such that

$$\left\|\int_t^{t+s} S_\alpha(t+s-\xi) f\big(\xi, u(\xi)\big)\,d\xi\right\| \le CM\int_t^{t+s} \frac{W(rh(\xi))}{1+|\widetilde{\omega}|(t+s-\xi)^\alpha}\,d\xi \le \frac{\varepsilon}{3} \quad \text{for } s \le \delta_1.$$

Moreover, since $\{f(t-\xi, u(t-\xi)) : 0 \le \xi \le t, u \in B_r(C_h(\mathbb{R}^+, X))\}$ is a relatively compact set and $S_\alpha(\cdot)$ is strong continuous, we can choose $\delta_2 > 0$, $\delta_3 > 0$ such that

$$\big\|\big(S_\alpha(t+s) - S_\alpha(t)\big)u_0\big\| \le \frac{\varepsilon}{3} \quad \text{for } s \le \delta_2$$

and

$$\left\| \big(S_\alpha(\xi + s) - S_\alpha(\xi)\big) f\big(t - \xi, u(t - \xi)\big) \right\| \le \frac{\varepsilon}{3(t+1)} \quad \text{for } s \le \delta_3.$$

So, $\|v(t+s) - v(t)\| \le \varepsilon$ for $|s| \le \min\{\delta_1, \delta_2, \delta_3\}$ with $t + s \ge 0$ and for all $u \in B_r(C_h(\mathbb{R}^+, X))$.

Finally, by (A_2), one has

$$\frac{\|v(t)\|}{h(t)} \le \frac{CM}{h(t)} \|u_0\| + \frac{CM}{h(t)} \int_0^t \frac{W(rh(s))}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds \to 0, \quad t \to \infty,$$

and this convergence is independent of $u \in B_r(C_h(\mathbb{R}^+, X))$. Hence V satisfies (c_1), (c_2) of Lemma 2.1, which completes the proof that V is a relatively compact set in $C_h(\mathbb{R}^+, X)$.

(iv) If u^λ is a solution of the equation $u^\lambda = \lambda\Gamma(u^\lambda)$ for some $0 < \lambda < 1$, then

$$\begin{aligned}
\|u^\lambda\| &= \lambda \left\| S_\alpha(t) u_0 + \int_0^t S_\alpha(t-s) f\big(s, u^\lambda\big)\, ds \right\| \\
&\le \|S_\alpha(\cdot) u_0\| + CM \int_0^t \frac{W(\|u^\lambda\|_h h(s))}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds \\
&\le \beta\big(\|u^\lambda\|_h\big) h(t).
\end{aligned}$$

Hence, one has

$$\frac{\|u^\lambda\|_h}{\beta(\|u^\lambda\|_h)} \le 1$$

and by (A_5), we conclude that the set $\{u^\lambda : u^\lambda = \lambda\Gamma(u^\lambda), \lambda \in (0,1)\}$ is bounded.

(v) If follows from Lemmas 2.3, 2.4 and 2.5 that $\Gamma(\mathcal{S}AP_\omega(\mathbb{R}^+, X)) \subseteq \mathcal{S}AP_\omega(\mathbb{R}^+, X)$; consequently, we consider $\Gamma : \overline{\mathcal{S}AP_\omega(\mathbb{R}^+, X)} \to \overline{\mathcal{S}AP_\omega(\mathbb{R}^+, X)}$. Using (i)-(iii), we have that the map is completely continuous. By (iv) and Theorem 2.2, we deduce that Γ has a fixed point $u \in \overline{\mathcal{S}AP_\omega(\mathbb{R}^+, X)}$.

Let u_n be a sequence in $\mathcal{S}AP_\omega(\mathbb{R}^+, X)$ such that it converges to u in the norm $C_h(\mathbb{R}^+, X)$. For $\varepsilon > 0$, let $\delta > 0$ be the constant in (A_3), there exists $n_0 \in \mathbb{N}$ such that $\|u_n - u\|_h \le \delta$ for all $n \ge n_0$. For $n \ge n_0$,

$$\begin{aligned}
\|\Gamma u_n - \Gamma u\| &\le \int_0^t \|S_\alpha(t-s)\| \big\|f\big(s, u_n(s)\big) - f\big(s, u(s)\big)\big\| \, ds \\
&\le CM \int_0^t \frac{\|f(s, u_n(s)) - f(s, u(s))\|}{1 + |\widetilde{\omega}|(t-s)^\alpha} \, ds \le CM\varepsilon.
\end{aligned}$$

Hence, $(\Gamma u_n)_n$ converges to $\Gamma u = u$ uniformly in $[0, \infty)$. This implies that $u \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$ and completes the proof. □

Corollary 3.1 *Assume that* (H_1), (H_2) (*or* (H_2')) *hold and satisfy the following conditions*:

(a) $f(t, 0) = q(t)$.

(b) f *satisfies the Hölder-type condition*

$$\big\|f(t,u) - f(t,v)\big\| \le C_1 \|u - v\|^\alpha, \quad u, v \in X, t \in \mathbb{R}^+,$$

where $0 < \alpha < 1$, $C_1 > 0$ *is a constant.*

(c) *For all* $a, b \in \mathbb{R}^+$, $a \leq b$ *and* $r \geq 0$, *the set* $\{f(s,u) : a \leq s \leq b, u \in X, \|u\| \leq r\}$ *is relatively compact in* X.

Then (3.1) *has a mild solution* $u(t) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$.

Proof By (b), it is easy to see that (H_4), (H_4') hold. Let $C_0 = \|q\|$ and $W(\xi) = C_0 + C_1\xi^\alpha$, then (A_1) is satisfied. Take a function h such that $\sup_{t\in\mathbb{R}^+} \int_0^t \frac{h(s)^\alpha}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds := C_2 < \infty$, it is not difficult to see that (A_2) is satisfied. To verify (A_3), note that for each $\varepsilon > 0$, there exists $0 < \delta < (\frac{\varepsilon}{C_1C_2})^{1/\alpha}$ such that for every $u, v \in C_h(\mathbb{R}^+, X)$, $\|u - v\|_h \leq \delta$ implies that

$$\int_0^t \frac{\|f(s,u(s)) - f(s,v(s))\|}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds \leq \int_0^t \frac{C_1 h(s)^\alpha \|u-v\|_h^\alpha}{1+|\widetilde{\omega}|(t-s)^\alpha}\,ds \leq C_1C_2\delta^\alpha \leq \varepsilon \quad \text{for all } t \in \mathbb{R}^+.$$

On the other hand, (A_5) can be easily verified using the definition of W. By Theorem 3.5, (3.1) has a mild solution $u(t) \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$. □

4 Examples

In this section, we provide some examples to illustrate our main results.

Example 4.1 Consider the following fractional differential equation:

$$\begin{cases} \partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1}\left(\int_0^x \eta a(t) u(t,\xi)\,d\xi\right), & t \in \mathbb{R}^+, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & x \in [0,\pi], \end{cases} \tag{4.1}$$

where $\mu > 0$, $u_0 \in L^2[0,\pi]$, $a \in \mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{R})$. In what follows we consider $X = L^2[0,\pi]$ and let A be the operator given by

$$Au = u'' - \mu u$$

with domain

$$D(A) = \{u \in X, u'' \in X, u(0) = u(\pi) = 0\}.$$

It is well know that A is sectorial of type $\widetilde{\omega} = -\mu < 0$ [30]. Equation (4.1) can be expressed as an abstract system of the form (3.1), where $u(t)(x) = u(t,x)$ for $t \in \mathbb{R}^+$, $x \in [0,\pi]$, and $f(t,\phi)(\xi) = \eta a(t)\int_0^\xi \phi(\tau)\,d\tau$ for $t \in \mathbb{R}^+$, $\xi \in [0,\pi]$. Moreover, one has

$$\begin{aligned} &\|f(t,\phi)\|_{L^2} \leq \pi|\eta||a(t)|\|\phi\|_{L^2}, \quad t \geq 0, \phi \in X, \\ &\|f(t+\omega,\phi) - f(t,\phi)\|_{L^2} \leq \pi|\eta||a(t+\omega) - a(t)|\|\phi\|_{L^2}, \quad t \geq 0, \phi \in X, \end{aligned}$$

since $a \in \mathcal{S}AP_\omega(\mathbb{R}^+, \mathbb{R})$, we deduce that $f \in \mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$. From

$$\begin{aligned} \|f(t,\phi_1) - f(t,\phi_2)\|_{L^2} &\leq \pi|\eta||a(t)|\|\phi_1 - \phi_2\|_{L^2} \\ &\leq \pi|\eta|\|a\|\|\phi_1 - \phi_2\|_{L^2}, \quad t \geq 0, \phi_1, \phi_2 \in X, \end{aligned}$$

so (H_{31}) holds with $L_f = \pi|\eta|\|a\|$. If $|\eta|$ is small enough, (4.1) has a unique mild solution $u \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Theorem 3.1.

Example 4.2 Consider the following fractional differential equation:

$$\begin{cases} \partial_t^\alpha u(t,x) = \partial_x^2 u(t,x) - \mu u(t,x) + \partial_t^{\alpha-1} F(t,u)(x), & t \in \mathbb{R}^+, x \in [0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \geq 0, \\ u(0,x) = u_0(x), & x \in [0,\pi], \end{cases} \tag{4.2}$$

where $\mu > 0$, $u_0 \in L^2[0,\pi]$, $F(t,u)(x) = e^{-\lambda t}|\int_0^x u(t,\tau)\,d\tau|^\vartheta \sin x$, $\vartheta \in (0,1)$. Let $X = L^2[0,\pi]$, $Au = u'' - \mu u$ with domain $D(A) = \{u \in X, u'' \in X, u(0) = u(\pi) = 0\}$, so A is sectorial of type $\widetilde{\omega} = -\mu < 0$. Equation (4.2) can be rewritten as the abstract form (3.1), where

$$f(t,\phi)(\xi) = e^{-\lambda t}\left|\int_0^\xi \phi(\tau)\,d\tau\right|^\vartheta \sin\xi, \quad \vartheta \in (0,1).$$

Moreover, one has

$$\|f(t,\phi)\|_{L^2} \leq e^{-\lambda t}\frac{\pi^{\frac{\vartheta+1}{2}}}{\sqrt{\vartheta+1}}\|\phi\|_{L^2}^\vartheta \leq \frac{\pi^{\frac{\vartheta+1}{2}}}{\sqrt{\vartheta+1}}\|\phi\|_{L^2}^\vartheta, \quad t \geq 0, \phi \in X, \tag{4.3}$$

$$\|f(t+\omega,\phi) - f(t,\phi)\|_{L^2} \leq \left(e^{-\lambda(t+\omega)} + e^{-\lambda t}\right)\frac{\pi^{\frac{\vartheta+1}{2}}}{\sqrt{\vartheta+1}}\|\phi\|_{L^2}^\vartheta, \quad t \geq 0, \phi \in X, \tag{4.4}$$

$$\|f(t,\phi_1) - f(t,\phi_2)\|_{L^2} \leq e^{-\lambda t}\frac{\pi^{\frac{\vartheta+1}{2}}}{\sqrt{\vartheta+1}}\|\phi_1 - \phi_2\|_{L^2}^\vartheta, \quad t \geq 0, \phi_1, \phi_2 \in X, \tag{4.5}$$

so $f \in \mathcal{S}AP_\omega(\mathbb{R}^+ \times X, X)$ and f is asymptotically uniformly continuous on bounded sets by (4.5). By (4.3), we define W by $W(\xi) = \frac{\pi^{\frac{\vartheta+1}{2}}}{\sqrt{\vartheta+1}}\xi^\vartheta$. Let $h(t) = e^{\lambda t}$, $\lambda > 0$, $u, v \in C_h(\mathbb{R}^+, X)$, one has

$$\frac{1}{h(t)}\int_0^t \frac{W(vh(s))}{1+|\mu|(t-s)^\alpha}\,ds \leq \frac{v^\vartheta \pi^{\frac{\vartheta+3}{2}}|\mu|^{-1/\alpha}}{\sqrt{\vartheta+1}\alpha\sin(\pi/\alpha)} \cdot \frac{1}{e^{\lambda(1-\vartheta)t}} \to 0, \quad t \to \infty,$$

$$\int_0^t \frac{\|f(s,u(s)) - f(s,v(s))\|_{L^2}}{1+|\mu|(t-s)^\alpha}\,ds \leq \frac{\pi^{\frac{\vartheta+3}{2}}|\mu|^{-1/\alpha}}{\sqrt{\vartheta+1}\alpha\sin(\pi/\alpha)}\|u-v\|_h^\vartheta.$$

Hence (A_1)-(A_3) hold.

Next, we prove that the set $\{f(s,e^{\lambda s}\phi) : a \leq s \leq b, \phi \in X, \|\phi\|_{L^2} \leq r\}$ is relatively compact in $L^2[0,T]$ by Simon's theorem. In fact, one has

$$\|f(s,e^{\lambda s}\phi)\|_{L^2} \leq \frac{\pi^{\frac{\vartheta+1}{2}}r^\vartheta}{\sqrt{\vartheta+1}}, \quad \phi \in L^2[0,\pi], \|\phi\|_{L^2} \leq r.$$

Hence, for $a_1 < a_2$, $\int_{a_1}^{a_2} f(s,e^{\lambda s}\phi)(\xi)\,d\xi$ is bounded uniformly for $a \leq s \leq b$ and $\phi \in L^2[0,\pi]$, $\|\phi\|_{L^2} \leq r$. On the other hand,

$$\|f(s,e^{\lambda s}\phi)(\xi) - f(s,e^{\lambda s}\phi)(\xi')\| \leq r^{\vartheta/2}|\xi - \xi'|^{\vartheta/2} + \pi^{\vartheta/2}r^\vartheta|\xi - \xi'|,$$

therefore,

$$\int_0^{\pi-h} |f(s,e^{\lambda s}\phi)(\xi+h) - f(s,e^{\lambda s}\phi)(\xi)|^2\,d\xi \to 0, \quad \text{as } h \to 0$$

uniformly for $a \leq s \leq b$, $\phi \in L^2[0,\pi]$, $\|\phi\|_{L^2} \leq r$. So (A_4) holds by Lemma 2.2. It is not difficult to see that (A_5) holds. Whence (4.2) has a mild solution $u \in \mathcal{S}AP_\omega(\mathbb{R}^+, X)$ by Theorem 3.5.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author has made this manuscript independently. The author read and approved the final version.

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