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On the oscillation of solutions for a class of second-order nonlinear stochastic difference equations

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Abstract
In this paper, we investigate the asymptotical behavior for a partial sum sequence of independent random variables, and we derive a law of the iterated logarithm type. It is worth to point out that the partial sum sequence needs not to be an independent increment process. As an application of the theory established, we also give a sufficient criterion on the almost sure oscillation of solutions for a class of second-order stochastic difference equation of neutral type.

Keywords: second-order nonlinear stochastic difference equations; almost sure oscillation

1 Introduction

To date, the asymptotic behavior of the solutions to deterministic difference equations has been discussed in many papers. Among them there are many papers about the oscillation of the solutions to deterministic difference equations. In a related field, the asymptotic behavior of the solutions to stochastic difference equation was discussed in many papers, and there have been also very fruitful achievements. However, there is little known about the oscillation of the solutions of stochastic difference equations. Recently Appleby and Rodkina [1] and Appleby *et al.* [2] first investigated the oscillation of the solutions of first-order nonlinear stochastic difference equations. In [1], the authors considered the following equation:

$$X(n+1) = X(n) - f\big(X(n)\big) + \sigma(n)\xi(n+1), \quad n = 0, 1, \ldots. \tag{1.1}$$

The solution of (1.1) can be expressed as

$$X(n,\omega) = X_0 - \sum_{i=0}^{n-1} f\big(X(i,\omega)\big) + \sum_{i=0}^{n-1} \sigma(i)\xi(i+1,\omega), \tag{1.2}$$

where $(\xi(n))_{n\geq 0}$ is a sequence of independent and identically distributed random variables. Note that the sequence $S_n^* =: \sum_{i=0}^{n} \sigma(i)\xi(i+1)$ $(n = 0, 1, \ldots)$ has the independent increment property, and as a result the authors can analyze the limit behavior of system (1.1) by the law of the iterated logarithm and they obtain a beautiful result, *i.e.*, the solution of (1.1) is an almost sure oscillation under some sufficient conditions. Motivated by [1], in this paper we investigate the oscillation of the solution for the following second-order nonlinear stochastic difference equation:

$$\Delta\bigl(r(k)\Delta X(k)\bigr) + f(k)F\bigl(X(k)\bigr) = \xi(k+2), \quad k = 0,1,\ldots. \tag{1.3}$$

Here $\Delta X(k) = X(k+1) - X(k)$ is the forward difference operator. This equation can be viewed as a stochastic analog of the following classical deterministic difference equations:

$$\Delta\bigl(r(k)\Delta X(k)\bigr) + f(k)F\bigl(X(k)\bigr) = 0 \tag{1.4}$$

or

$$\Delta\bigl(r(k)\Delta X(k)\bigr) + f(k)F\bigl(X(k)\bigr) = g(k). \tag{1.5}$$

The solution of (1.3) can be expressed as

$$X(n+1) = X(1) + \sum_{k=1}^{n} \frac{V(k)}{r(k)} + \sum_{k=1}^{n} \left\{ \sum_{i=k}^{n} \frac{1}{r(i)} \right\} \xi(k+1), \tag{1.6}$$

where $V(k)$ is determined by $\Delta V(k) = -f(k)F(X(k))$ and $V(0) = r(0)(X(1) - X(0))$. The proof of (1.6) is to be given in Section 4. Denoting $S_n = \sum_{k=1}^{n}\{\sum_{i=k}^{n} \frac{1}{r(i)}\}\xi(k+1)$ for any $n \in \mathbb{N}_1$, it implies $S_n - S_{n-1} = \frac{1}{r(n)}\sum_{i=1}^{n}\xi(i+1)$. It is obvious that $S_n - S_{n-1}$ is not independent of S_{n-1} even though $(\xi(k))$ is a sequence of independent random variables. That is to say that $(S_n)_{n\geq 1}$ does not have the independent increment property, which means that we do not directly adopt law of the iterated logarithm to S_n to obtain their limit behavior. However, under some restrictions we can use a roundabout way to analyze the limit behavior on $(S_n)_{n\geq 1}$ by the law of the iterated logarithm, then we give some sufficient conditions on the almost sure oscillation for (1.3). These results and proofs are deferred to the following sections.

2 Definitions and assumptions

Throughout this paper, the following notation, definitions, and assumptions are needed. $\mathbb{N}$ and $\mathbb{R}$ denote, respectively, the positive integer numbers and real numbers. Let $\mathbb{N}_a = \{a, a+1, \ldots\}$ for every $a \in \mathbb{N} \cup \{0\}$. $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a complete probability space. $\{\xi(n)\}_{n\in\mathbb{N}_1}$ is a random variable sequence defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose that filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is naturally generated, namely that $\mathcal{F}_n = \sigma\{\xi(1), \xi(2), \ldots, \xi(n)\}$. We use the standard abbreviations 'a.s.' and 'i.i.d.' instead of 'almost surely' and 'independent identically distribution', respectively. For simplicity, we denote $\log_2 \cdot =: \log\log \cdot$ throughout this paper.

For (1.3), the following elementary assumptions are needed.

(A2.1) $r(n) > 0$ for every $n \in \mathbb{N}_0$,

(A2.2) $f(n) \geq 0$ for every $n \in \mathbb{N}_0$,

(A2.3) F is assumed to be Borel measurable and to obey $uF(u) > 0$ for $u \neq 0$, and $F(0) = 0$,

(A2.4) $\{\xi(n)\}_{n\in\mathbb{N}_2}$ is assumed to be independent identically distributed random variable sequence defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and, moreover, $\mathbb{E}\xi(n) = 0$, $\mathbb{E}\xi^2(n) = 1$.

Definition 2.1 $\{X(n)\}_{n\in\mathbb{N}_0}$ is called a solution of (1.3) with initial values $X(0)$, $X(1)$. $\{X(n)\}_{n\in\mathbb{N}_0}$ is constituted by $X(n,\omega)$ and $X(0)$, $X(1)$, where $X(n,\omega)$ is obtained by $n-1$ steps of iteration of (1.3) with initial values $X(0)$, $X(1)$.

Definition 2.2 The solution $\{X(n)\}_{n\in\mathbb{N}_0}$ of (1.3) is said to be a.s. oscillatory if

$$\mathbb{P}\bigl\{X(n) < 0 \text{ i.o.}\bigr\} = 1, \qquad \mathbb{P}\bigl\{X(n) > 0 \text{ i.o.}\bigr\} = 1,$$

where 'i.o.' stands for infinitely often.

Definition 2.3 Equation (1.3) is said to be a.s. oscillatory if its any solution is a.s. oscillation.

3 Law of the iterated logarithm

The classical Kolmogorov law of the iterated logarithm is an effective tool in studying the limit behavior of partial sum of independent random variable sequence (see [3]). In 1973, Chow and Teicher [4] generalized the classical results and obtained the following law of the iterated logarithm for weighted averages.

Theorem 3.1 (Iterated logarithm laws of weighted averages) *If* $\{X_n, n\geq 1\}$ *are i.i.d. random variables with* $\mathbb{E}X_n = 0$, $\mathbb{E}X_n^2 = 1$ *and* $\{a_n, n\geq 1\}$ *are real constants satisfying*

(i) $a_n^2/\sum_1^n a_j^2 \leq C/n$, $n\geq 1$,

(ii) $\sum_1^n a_j^2 \to \infty$

for some C *in* $(0,\infty)$, *then*

$$\mathbb{P}\left\{\overline{\lim_{n\to\infty}} \frac{\sum_{j=1}^n a_j X_j}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} = 1\right\} = 1$$

and

$$\mathbb{P}\left\{\underline{\lim_{n\to\infty}} \frac{\sum_{j=1}^n a_j X_j}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} = -1\right\} = 1.$$

On the above results, notice that $\sum_1^n a_j X_j - \sum_1^{n-1} a_j X_j = a_n X_n$ is independent of $\sum_1^{n-1} a_j X_j$.

Now we establish a new result about law of iterated logarithm type. Suppose that $r:\mathbb{N}_1\to\mathbb{R}$ satisfies $r(n)>0$ for every $n\in\mathbb{N}_1$, and $\{\xi(n)\}_{n\in N_1}$ is an i.i.d. random variable sequence defined on $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{E}\xi(n)=0$, $\mathbb{E}\xi^2(n)=1$. For $n\in\mathbb{N}_1$, set

$$\begin{aligned}
a_n &= \sum_{j=1}^n \frac{1}{r(j)},\\
S_n &= \sum_{k=1}^n \left\{\sum_{j=k}^n \frac{1}{r(j)}\right\}\xi(k+1),\\
S_n(1) &= a_n \sum_{j=1}^n \xi(j+1),\\
S_n(2) &= \sum_{j=1}^n a_j \xi(j+2).
\end{aligned} \tag{3.1}$$

Here we appoint $\sum_{j=k_2}^{k_1}(\cdot) = 0$ if $k_1 < k_2$. It is obvious that $S_n - S_{n-1}$ is not independent of S_{n-1}, but we have

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\left\{\sum_{j=k}^{n}\frac{1}{r(j)}\right\}\xi(k+1) \\ &= \sum_{k=1}^{n}(a_n - a_{k-1})\xi(k+1) \\ &= \sum_{k=1}^{n}a_n\xi(k+1) - \sum_{k=2}^{n}a_{k-1}\xi(k+1) \\ &= a_n\sum_{k=1}^{n}\xi(k+1) - \sum_{k=1}^{n-1}a_k\xi(k+2) \\ &= S_n(1) - S_{n-1}(2). \end{aligned} \tag{3.2}$$

Note that $\{a_n\}$ is a monotony increasing sequence, hence we give the following hypothesis:

(C.1) There exist constants $\alpha \geq 0$ and $d > 0$ such that $\lim_{n\to\infty} a_n/n^\alpha = d$.

Lemma 3.2 *If* (C.1) *holds, then*

$$\lim_{n\to\infty}\frac{na_n^2}{\sum_{j=1}^{n}a_j^2} = 1 + 2\alpha, \tag{3.3}$$

$$\lim_{n\to\infty}\frac{\log_2\sum_1^n a_j^2}{\log_2 n} = 1, \tag{3.4}$$

$$\lim_{n\to\infty}\frac{\sum_1^n a_j^2\log_2\sum_1^n a_j^2}{\sum_1^{n-1}a_j^2\log_2\sum_1^{n-1}a_j^2} = 1. \tag{3.5}$$

Proof Set $O_n = \frac{a_n}{n^\alpha} - d$, for every $n \geq 1$, we have

$$a_n = (O_n + d)n^\alpha. \tag{3.6}$$

In view of (C.1), for any fixed positive integer number m, there exists $N > 0$ such that $|O_n| \leq \frac{d}{m}$ for every $n > N$. As n is sufficiently large, we have

$$\sum_{j=1}^{N}a_j^2 + \left(\frac{m-1}{m}\right)^2 d^2\sum_{j=N+1}^{n}j^{2\alpha} \leq \sum_{j=1}^{n}a_j^2 \leq \sum_{j=1}^{N}a_j^2 + \left(\frac{m+1}{m}\right)^2 d^2\sum_{j=N+1}^{n}j^{2\alpha}. \tag{3.7}$$

It is clear that

$$\sum_{j=1}^{n}j^{2\alpha}/n^{2\alpha+1} \to \frac{1}{2\alpha+1}, \quad n\to\infty.$$

Hence

$$\frac{\sum_{j=N+1}^{n}j^{2\alpha}}{n^{2\alpha+1}} \to \frac{1}{2\alpha+1}, \quad n\to\infty. \tag{3.8}$$

In view of (3.6) and (3.7), as n is sufficiently large, we get

$$\frac{n^{2\alpha+1}(O_n+d)^2}{\sum_{j=1}^N a_j^2 + (\frac{m+1}{m})^2 d^2 \sum_{j=N+1}^n j^{2\alpha}} \le \frac{na_n^2}{\sum_{j=1}^n a_j^2} \le \frac{n^{2\alpha+1}(O_n+d)^2}{\sum_{j=1}^N a_j^2 + (\frac{m-1}{m})^2 d^2 \sum_{j=N+1}^n j^{2\alpha}}. \tag{3.9}$$

Letting $n \to \infty$ in the above formula (3.9), and combining (3.8) and (C.1), we have

$$\frac{2\alpha+1}{(\frac{m+1}{m})^2} \le \underline{\lim}_{n\to\infty} \frac{na_n^2}{\sum_{j=1}^n a_j^2} \le \overline{\lim}_{n\to\infty} \frac{na_n^2}{\sum_{j=1}^n a_j^2} \le \frac{2\alpha+1}{(\frac{m-1}{m})^2}. \tag{3.10}$$

Setting $m \to \infty$ in the above inequalities, we obtain (3.3).

Let $A = (1+2\alpha)^{-1}$, $P_n = \frac{\sum_{j=1}^n a_j^2}{na_n^2} - A$, $n \in \mathbb{N}_1$. According to (3.3), we have $P_n \to 0$ $(n \to \infty)$ and $\sum_{j=1}^n a_j^2 = (A+P_n)na_n^2$ for every $n \in \mathbb{N}_1$. Therefore

$$\begin{aligned} \frac{\log_2 \sum_{j=1}^n a_j^2}{\log_2 n} &= \frac{\log_2 (A+P_n)na_n^2}{\log_2 n} \\ &= \frac{\log(\log n + \log a_n^2 + \log(A+P_n))}{\log_2 n} \\ &= \frac{\log\{(1+2\alpha + \frac{\log(d+O_n)^2 + \log(A+P_n)}{\log n})\log n\}}{\log_2 n} \\ &= \frac{\log_2 n + \log(1+2\alpha + \frac{\log(d+O_n)^2 + \log(A+P_n)}{\log n})}{\log_2 n} \\ &= 1 + \frac{\log(1+2\alpha + \frac{\log(d+O_n)^2 + \log(A+P_n)}{\log n})}{\log_2 n}. \end{aligned}$$

Taking the limit on both sides of the above equation, result (3.4) follows.

Equation (3.5) can be proved similarly. □

Lemma 3.3 (Law of the iterated logarithm on S_n defined by (3.1)) *If* (C.1) *holds, then*

$$\mathbb{P}\left\{(1+2\alpha)^{1/2} - 1 \le \overline{\lim}_{n\to\infty} \frac{S_n}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} \le (1+2\alpha)^{1/2} + 1\right\} = 1 \tag{3.11}$$

and

$$\mathbb{P}\left\{-(1+2\alpha)^{1/2} - 1 \le \underline{\lim}_{n\to\infty} \frac{S_n}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} \le -(1+2\alpha)^{1/2} + 1\right\} = 1. \tag{3.12}$$

Here α *is described as* (C.1).

Proof According to (3.1), it is clear that $\xi(2), \ldots, \xi(j+1), \ldots$ is an i.i.d. random variable sequence and, moreover, $\mathbb{E}\xi(j+1) = 0$, $\mathbb{E}\xi^2(j+1) = 1$. Setting $a_n = 1$, $X_n = \xi(n+1)$ for any $n \in \mathbb{N}_1$, it is obvious that $\sum_{j=1}^n \xi(j+1) = \sum_{j=1}^n a_j X_j$, which satisfies the conditions of Theorem 3.1. In $S_n(2)$, letting $X_j = \xi(j+2)$, $\forall j \in \mathbb{N}_1$, it is clear that $\sum_{j=1}^n a_j^2 \to \infty$ $(n \to \infty)$ by (C.1). By (3.3), it is found that there is $c > 0$ such that $\frac{a_n^2}{\sum_{j=1}^n a_j^2} \le \frac{c}{n}$. It is also known that $S_n(2) =$

$\sum_{j=1}^{n} a_j\xi(j+2) = \sum_{j=1}^{n} a_j X_j$ which satisfies the conditions of Theorem 3.1. Hence

$$\mathbb{P}\left\{\overline{\lim_{n\to\infty}} \frac{\sum_{j=1}^{n}\xi(j+1)}{(2n\log_2 n)^{1/2}} = 1\right\} = 1, \tag{3.13}$$

$$\mathbb{P}\left\{\underline{\lim_{n\to\infty}} \frac{\sum_{j=1}^{n}\xi(j+1)}{(2n\log_2 n)^{1/2}} = -1\right\} = 1, \tag{3.14}$$

$$\mathbb{P}\left\{\overline{\lim_{n\to\infty}} \frac{S_n(2)}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} = 1\right\} = 1, \tag{3.15}$$

$$\mathbb{P}\left\{\underline{\lim_{n\to\infty}} \frac{S_n(2)}{(2\sum_1^n a_j^2 \log_2 \sum_1^n a_j^2)^{1/2}} = -1\right\} = 1. \tag{3.16}$$

So there is $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0^c) = 0$ such that all equalities of $\{\cdot\}$ of the left side of (3.13)-(3.16) hold on Ω_0. Therefore by (3.13), for $\omega \in \Omega_0$ there exists $\{n_k(\omega)\} \subset \{n\}$ such that

$$\lim_{k\to\infty} \frac{\sum_{j=1}^{n_k(\omega)}\xi(j+1,\omega)}{(2n_k(\omega)\log_2 n_k(\omega))^{1/2}} = 1. \tag{3.17}$$

By (3.15) and (3.16), it is clear that

$$\left\{\frac{S_{n_k(\omega)-1}(2)(\omega)}{(2\sum_1^{n_k(\omega)-1} a_j^2 \log_2 \sum_1^{n_k(\omega)-1} a_j^2)^{1/2}}\right\}$$

is a bounded sequence. Therefore there is a $\{n_{k_l}(\omega)\} \subset \{n_k(\omega)\}$ and $\beta \in [-1,1]$ such that

$$\lim_{l\to\infty} \frac{S_{n_{k_l}-1}(2)(\omega)}{(2\sum_1^{n_{k_l}-1} a_j^2 \log_2 \sum_1^{n_{k_l}-1} a_j^2)^{1/2}} = \beta. \tag{3.18}$$

By (3.2)-(3.5) and (3.17)-(3.18), we get

$$\begin{aligned}
&\lim_{l\to\infty} \frac{S_{n_{k_l}}(\omega)}{(2\sum_1^{n_{k_l}} a_j^2 \log_2 \sum_1^{n_{k_l}} a_j^2)^{1/2}} \\
&\quad = \lim_{l\to\infty}\left\{\frac{a_{n_{k_l}}\sum_1^{n_{k_l}}\xi(j+1)}{(2\sum_1^{n_{k_l}} a_j^2 \log_2 \sum_1^{n_{k_l}} a_j^2)^{1/2}} - \frac{\sum_1^{n_{k_l}-1} a_j\xi(j+2)}{(2\sum_1^{n_{k_l}} a_j^2 \log_2 \sum_1^{n_{k_l}} a_j^2)^{1/2}}\right\} \\
&\quad = \lim_{l\to\infty}\left\{\frac{(1+2\alpha)^{1/2}\sum_1^{n_{k_l}}\xi(j+1)}{(2n_{k_l}\log_2 n_{k_l})^{1/2}} - \frac{\sum_1^{n_{k_l}-1} a_j\xi(j+2)}{(2\sum_1^{n_{k_l}-1} a_j^2 \log_2 \sum_1^{n_{k_l}-1} a_j^2)^{1/2}}\right\} \\
&\quad = (1+2\alpha)^{1/2} - \beta.
\end{aligned}$$

Hence (3.11) holds.

Equation (3.12) can be proved similarly. □

To proceed the study, we give another assumption:

(C.2) There exist $\alpha, d \in (0,\infty)$ such that $\lim_{n\to\infty} a_n/n^\alpha = d$.

Note that $\alpha \neq 0$ in (C.2). It is obvious that the conclusions of Lemma 3.2 and Lemma 3.3 are also right when (C.2) replaces (C.1).

4 The main results

In this section, we give the main results on the oscillation of the solution of (1.3).

Let $\{X(k)\}_{k\in\mathbb{N}_0}$ be an any solution of (1.3) with arbitrary initial values $X(0), X(1) \in \mathbb{R}$. Set

$$\begin{aligned} &V(0) = r(0)\big(X(1) - X(0)\big), \\ &\Delta V(k) = -f(k)F\big(X(k)\big), \quad \forall k \in \mathbb{N}_0 \end{aligned}$$

and

$$D(k) = \sum_{j=0}^{k} \xi(j+2), \quad \forall k \in \mathbb{N}_0.$$

By (1.3), one obtains

$$\Delta\big(r(k)\Delta X(k)\big) = \Delta V(k) + \Delta D(k-1) = \Delta\big(V(k) + D(k-1)\big), \quad \forall k \in \mathbb{N}_0.$$

Hence

$$r(k)\triangle X(k) = V(k) + D(k-1) + c.$$

Let $k = 0$ in the above equation, and one has

$$c = r(0)\big(X(1) - X(0)\big) - V(0) = 0.$$

Therefore

$$\Delta X(k) = \frac{V(k)}{r(k)} + \frac{1}{r(k)}D(k-1).$$

So for any $n \in \mathbb{N}_1$, one has

$$\begin{aligned} X(n+1) - \left\{X(1) + \sum_{k=1}^{n} \frac{V(k)}{r(k)}\right\} &= \sum_{k=1}^{n} \frac{1}{r(k)} D(k-1) \\ &= \sum_{k=1}^{n} \frac{1}{r(k)} \sum_{j=0}^{k-1} \xi(j+2) \\ &= \sum_{k=1}^{n} \left\{\sum_{j=k}^{n} \frac{1}{r(j)}\right\} \xi(k+1) \\ &= S_n. \end{aligned} \tag{4.1}$$

Theorem 4.1 *Suppose that* (1.3) *satisfies, respectively,* (A2.1)-(A2.4), *then* (1.3) *is an almost sure oscillation under condition* (C.2).

Proof Suppose that the result is not right, then (1.3) must have a solution, denoted as $\{X(n)\}_{n\in\mathbb{N}_0}$, and it is not an almost sure oscillation. That is to say at least one is not true between $\mathbb{P}\{X(n) < 0 \text{ i.o.}\} = 1$ and $\mathbb{P}\{X(n) > 0 \text{ i.o.}\} = 1$.

1. Firstly, we assume that $\mathbb{P}\{X(n) < 0 \text{ i.o.}\} < 1$ holds. For this case, it implies that there is $\Omega_1 \subset \Omega$ with $\mathbb{P}(\Omega_1) > 0$ such that the following equation holds:

$$X(n,\omega) \geq 0, \quad \forall \omega \in \Omega_1 \tag{4.2}$$

as $n \geq N(\omega)$. Here $N(\omega) \in \mathbb{N}_1$. By virtue of (3.3) of Lemma 3.2, we have

$$\lim_{n\to\infty} \frac{a_n^2}{\sum_{j=1}^n a_j^2} = 0.$$

By Lemma 3.3, we obtain

$$\overline{\lim_{n\to\infty}} \frac{S_n(\omega)}{(\sum_1^n a_j^2)^{1/2}} = \infty, \quad \text{a.s.,} \qquad \underline{\lim_{n\to\infty}} \frac{S_n(\omega)}{(\sum_1^n a_j^2)^{1/2}} = -\infty, \quad \text{a.s.}$$

Therefore there exists $\Omega_2 \in \Omega$ with $\mathbb{P}(\Omega_2^c) = 0$ such that for any $\omega \in \Omega_2$

$$\overline{\lim_{n\to\infty}} \frac{S_n(\omega)}{a_n} = \infty, \qquad \underline{\lim_{n\to\infty}} \frac{S_n(\omega)}{a_n} = -\infty. \tag{4.3}$$

Setting $\Omega_3 = \Omega_1 \cap \Omega_2$, it is obvious that $\mathbb{P}(\Omega_3) > 0$ and (4.2) and (4.3) are also true for any $\omega \in \Omega_3$. For any $k \in \mathbb{N}_1$, we have

$$V(k) = V(0) - \sum_{j=0}^{k-1} f(j)F\bigl(X(j,\omega)\bigr).$$

So for any $\omega \in \Omega_3$, we have

$$\begin{aligned}
X(n+1,\omega) &= S_n(\omega) + X(1) + \sum_{k=1}^{n} \frac{V(k)}{r(k)} \\
&= S_n(\omega) + X(1) + \sum_{k=1}^{N(\omega)} \frac{V(k)}{r(k)} + \sum_{k=N(\omega)+1}^{n} \frac{V(0) - \sum_{j=0}^{k-1} f(j)F(X(j,\omega))}{r(k)} \\
&= S_n(\omega) + X(1) + \sum_{1}^{N(\omega)} \frac{V(k)}{r(k)} + V(0) \sum_{k=N(\omega)+1}^{n} \frac{1}{r(k)} \\
&\quad - \sum_{k=N(\omega)+1}^{n} \frac{\sum_{j=0}^{N(\omega)-1} f(j)F(X(j,\omega)) + \sum_{j=N(\omega)}^{k-1} f(j)F(X(j,\omega))}{r(k)}
\end{aligned}$$

as $n > N(\omega)$. Hence

$$\begin{aligned}
&X(n+1,\omega) + \sum_{k=N(\omega)+1}^{n} \frac{\sum_{j=N(\omega)}^{k-1} f(j)F(X(j,\omega))}{r(k)} \\
&\quad = S_n(\omega) + X(1) + \sum_{1}^{N(\omega)} \frac{V(k)}{r(k)} + V(0)(a_n - a_{N(\omega)}) - \sum_{k=N(\omega)+1}^{n} \frac{\sum_{j=0}^{N(\omega)-1} f(j)F(X(j,\omega))}{r(k)}
\end{aligned}$$

$$= S_n(\omega) + X(1) + \sum_{1}^{N(\omega)} \frac{V(k)}{r(k)} + \left(V(0) - \sum_{j=0}^{N(\omega)-1} f(j)F\bigl(X(j,\omega)\bigr)\right) \cdot (a_n - a_{N(\omega)})$$

$$= a_n \left\{ \frac{S_n(\omega)}{a_n} + \frac{X(1) + \sum_{1}^{N(\omega)} \frac{V(k)}{r(k)}}{a_n} + \left(V(0) - \sum_{j=0}^{N(\omega)-1} f(j)F\bigl(X(j,\omega)\bigr)\right) \cdot \left(1 - \frac{a_{N(\omega)}}{a_n}\right)\right\}. \tag{4.4}$$

Therefore the left-hand side of (4.4) is nonnegative.

On the right-hand side of (4.4), we have

$$\frac{X(1) + \sum_{1}^{N(\omega)} \frac{V(k)}{r(k)}}{a_n} \to 0, \qquad \frac{a_{N(\omega)}}{a_n} \to 0 \quad (n \to \infty)$$

due to $a_n \to \infty$ $(n \to \infty)$. Therefore we find that it is an oscillation by (4.3). This is a contradiction.

2. Secondly, we assume that $\mathbb{P}\{X(n) > 0 \text{ i.o.}\} < 1$ holds. We may get a contradiction for case 2 similar to case 1. Thus we finish the proof of Theorem 4.1. □

Remark 1 If the condition (C.1) replaces the condition (C.2) of Theorem 4.1 and the other conditions are not changed, then the conclusions of Theorem 4.1 cannot be guaranteed to be right as $\alpha = 0$.

The example is as follows.

Example 1 Take $r(k) = 2^k, f(k) = 1$, $k \in \mathbb{N}_0$,

$$F(u) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{if } u = 0, \\ -1 & \text{if } u < 0 \end{cases} \tag{4.5}$$

in (1.3), then (1.3) becomes the following special equation:

$$\Delta\bigl(2^k \Delta X(k)\bigr) + F\bigl(X(k)\bigr) = \xi(k+2), \quad k \in \mathbb{N}_0, \tag{4.6}$$

here $\{\xi(k+2)\}_{k\in\mathbb{N}_0}$ is assumed to satisfy (A2.4) and be locally bounded, *i.e.*, there is $h > 0$ and $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega' > 0)$ such that $|\xi(n,\omega)| \le h$, $\forall \omega \in \Omega'$, $n \in \mathbb{N}_2$.

It is clear that r, f satisfy, respectively, (A2.1) and (A2.2), and F satisfies (A2.3), and $\sum_{j=1}^n \frac{1}{r(j)} \to 1$ $(n \to \infty)$, *i.e.*, $a_n =: \sum_{j=1}^n \frac{1}{r(j)}$ satisfies (C.1) but it does not satisfy (C.2).

Now we illustrate that (4.6) is not an a.s. oscillation. Let $\{X(n)\}_{n\in\mathbb{N}_0}$ be a solution of (4.4) with initial values $X(0)$, $X(1)$, then we have

$$X(n+1,\omega) = X(1) + \sum_{k=1}^{n} \frac{V(k,\omega)}{2^k} + \sum_{k=1}^{n} \left(\sum_{i=k}^{n} \frac{1}{2^i}\right) \xi(k+1,\omega)$$

$$= X(1) + \sum_{k=1}^{n} \frac{V(k,\omega)}{2^k} + \sum_{k=1}^{n} \frac{\xi(k+1,\omega)}{2^{k-1}} - \frac{1}{2^n} \sum_{k=1}^{n} \xi(k+1,\omega) \tag{4.7}$$

for every $\omega \in \Omega$. Here $V(k)$ is determined by $\Delta V(k) = -F(X(k))$, $k \in \mathbb{N}_0$ and $V(0) = X(1) - X(0)$.

About the terms of (4.7), the following assertions are right.

(i) $\frac{1}{2^n}\sum_{k=1}^{n}\xi(k+1) \to 0$, a.s. $(n \to \infty)$.

(ii) There is finite value measurable function $h(\omega)$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\sum_{k=1}^{n}\frac{\xi(k+1)}{2^{k-1}} \to h(\omega)$, a.s. $(n \to \infty)$.

(iii) $V(1) - 1 \le \sum_{k=1}^{\infty}\frac{V(k)}{2^k} \le V(1) + 1$.

Proof of the assertions

(i) Setting $a_n = 1$, $X(n) = \xi(n+1)$, $\forall n \in \mathbb{N}_1$, because $\{\xi(k)\}$ is an i.i.d. random variable sequence and, moreover, $\mathbb{E}\xi(k) = 0$, $\mathbb{E}\xi^2(k) = 1$, then the $\{X_k\}_{k\ge 1}$ have the same properties. It is obvious that

$$\frac{a_n^2}{\sum_1^n a_j^2} = \frac{1}{n}, \qquad \sum_1^n a_j^2 = n \to \infty.$$

By Theorem 3.1, we have

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\frac{\sum_1^n \xi(k+1)}{(2n\log_2 n)^{1/2}} = \overline{\lim_{n\to\infty}}\frac{\sum_1^n a_k X_k}{(2\sum_1^n a_k^2 \log_2 \sum_1^n a_k^2)^{1/2}} = 1, \quad \text{a.s.},\\
&\underline{\lim}_{n\to\infty}\frac{\sum_1^n \xi(k+1)}{(2n\log_2 n)^{1/2}} = -1, \quad \text{a.s.}
\end{aligned}$$

So we have

$$\frac{\sum_1^n \xi(k+1)}{2^n} = \frac{(2n\log_2 n)^{1/2}}{2^n}\cdot\frac{\sum_1^n \xi(k+1)}{(2n\log_2 n)^{1/2}} \to 0 \quad (n \to \infty).$$

(ii) Setting $\eta(k) = \frac{\xi(k+1)}{2^{k-1}}$, $\forall k \in \mathbb{N}_1$, we obviously have $\mathbb{E}\eta(k) = 0$, $\operatorname{Var}(\eta(k)) = \frac{1}{4^{k-1}}$, $\forall k \in \mathbb{N}_1$. Therefore $\sum_{k=1}^{\infty}\operatorname{Var}(\eta(k)) < \infty$. It is obvious that $\{\eta(k)\}$ is an independent random variable sequence, then the conclusion holds by [5, Lemma 1, p.444] or [3, §17.3, p.248].

(iii) By (4.5) and $\Delta V(k) = -F(X(k))$, we have

$$V(1) - k + 1 \le V(k) \le V(1) + k - 1, \quad \forall k \in \mathbb{N}_1.$$

Hence

$$\begin{aligned}
\big(V(1)+1\big)\sum_{k=1}^{n}\frac{1}{2^k} - \sum_{k=1}^{n}\frac{k}{2^k} &\le \sum_{k=1}^{n}\frac{V(k)}{2^k}\\
&\le \big(V(1)-1\big)\sum_{k=1}^{n}\frac{1}{2^k} + \sum_{k=1}^{n}\frac{k}{2^k}, \quad \forall n \in \mathbb{N}_1. \qquad (4.8)
\end{aligned}$$

It is obvious that $\sum_{k=1}^{\infty}\frac{1}{2^k} = 1$, $\sum_{k=1}^{\infty}\frac{k}{2^k} = 2$. So we get

$$V(1) - 1 \le \sum_{k=1}^{\infty}\frac{V(k)}{2^k} \le V(1) + 1.$$

By (4.7) and the above assertions (i)-(iii), we obtain

$$\begin{aligned}\lim_{n\to\infty} X(n+1,\omega) &= X(1) + \sum_{k=1}^{\infty} \frac{V(k)}{2^k} + h(\omega) \\ &\geq X(1) + V(1) - 1 + h(\omega) \\ &= X(1) + \bigl(X(1) - X(0) - F\bigl(X(0)\bigr)\bigr) - 1 + h(\omega) \\ &\geq 2X(1) - X(0) - 2 + h(\omega) \end{aligned} \tag{4.9}$$

for every $\omega \in \Omega$. Here $h(\omega)$ and $(X(0), X(1))$ are mutually independent.

We choose $X(0), X(1)$ satisfying $2X(1) > X(0) + 2 + 2h$. Due to $|\xi(n,\omega)| \leq h$ for any $\omega \in \Omega'$, $n \in \mathbb{N}_2$ and

$$\sum_{k=1}^{n} \frac{\xi(k+1,\omega)}{2^{k-1}} \to h(\omega), \quad \text{a.s.}$$

as $n \to \infty$, one obtains

$$\bigl|h(\omega)\bigr| = \left|\sum_{k=1}^{\infty} \frac{\xi(k+1,\omega)}{2^{k-1}}\right| \leq h \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2h.$$

Thus $\lim_{n\to\infty} X(n+1) > 0$ on Ω' by (4.9). Therefore $\{X(n)\}$ is not an almost sure oscillation, and consequently, (4.6) is not almost surely oscillatory by Definition 2.3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in this paper. They read and approved the final manuscript.

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