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Oscillation of second-order nonlinear neutral dynamic equations with distributed deviating arguments on time scales

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Abstract
This paper concerns second-order nonlinear neutral dynamic equations with distributed deviating arguments on time scales of the form

$$(r(t)((y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma})^{\Delta}+\int_a^b f(t,y(\delta(t,\xi)))\Delta\xi=0,$$

where $\gamma>0$ is a quotient of odd positive integers. By using the generalized Riccati technique and integral averaging techniques, we derive new oscillation criteria for the above equations, which generalize and improve some existing results in the literature.

Keywords: neutral dynamic equations on time scales; distributed deviating arguments; oscillation; generalized Riccati technique

1 Introduction

In this paper, we consider second-order nonlinear neutral dynamic equations with distributed deviating arguments of the following form:

$$\big(r(t)\big(\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{\Delta}\big)^{\gamma}\big)^{\Delta}+\int_a^b f\big(t,y\big(\delta(t,\xi)\big)\big)\Delta\xi=0 \tag{1.1}$$

on a time scale $\mathbb{T}$ satisfying $\inf\mathbb{T}=t_0$ and $\sup\mathbb{T}=\infty$. Throughout this paper, we assume the following:

(H1) $\gamma>0$ is a quotient of odd positive integers, $0<a<b$, $\tau(t)\in C_{rd}(\mathbb{T},\mathbb{T})$ such that $\tau(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\infty$, $\delta(t,\xi)\in C_{rd}(\mathbb{T}\times[a,b],\mathbb{T})$ such that $\lim_{t\to\infty}\delta(t,\xi)=\infty$;

(H2) $r(t)\in C_{rd}(\mathbb{T},(0,\infty))$ such that $\int_{t_0}^{\infty}(\frac{1}{r(t)})^{\frac{1}{\gamma}}\Delta t=\infty$, and $p(t)\in C_{rd}(\mathbb{T},[0,1))$;

(H3) $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$, and there exists a function $q(t)\in C_{rd}(\mathbb{T},[0,\infty))$ such that $|f(t,u)|\geq q(t)|u|^{\gamma}$.

Oscillation of some second-order nonlinear delay dynamic equations on time scales has been discussed; see [1–18] and the references therein. Recently, there has been much research activity concerning the oscillation of second-order nonlinear neutral delay dynamic equation

$$\big(r(t)\big(\big(y(t)+p(t)y\big(\tau(t)\big)\big)^{\Delta}\big)^{\gamma}\big)^{\Delta}+f\big(t,y\big(\delta(t)\big)\big)=0, \quad t\in\mathbb{T}.$$

We refer the reader to [1–4].

In 2010, Thandapani and Piramanantham [5] discussed oscillation of the second-order nonlinear neutral delay dynamic equation with distributed deviating arguments

$$\big(r(t)\big(x(t)+p(t)x(t-\tau)\big)^{\Delta}\big)^{\Delta}+\int_a^b q(t,\xi)f\big(x\big(g(t,\xi)\big)\big)\Delta\xi=0, \quad t\in\mathbb{T},$$

where $g(t,\xi)$ is strictly increasing with respect to t and decreasing with respect to ξ, and $f\in C(\mathbb{R},\mathbb{R})$ with $uf(u)>0$ for $u\neq 0, f(-u)=-f(u)$.

In 2011, Candan [6] discussed the oscillation of Eq. (1.1) for $\delta(t,\xi)\le t$ and $\delta(t,\xi)>t$, respectively, where $\gamma>0$ is a quotient of odd positive integers. In [6], $\delta(t,\xi)$ is decreasing with respect to ξ, $0<p(t)<1$ is increasing and $f\in C(\mathbb{T}\times\mathbb{R},\mathbb{R})$ with $uf(t,u)>0$ for all $u\neq 0$. There exists a positive function $q(t)$ defined on $\mathbb{T}$ such that $|f(t,u)|\ge q(t)|u|^{\beta}$, where $\beta>0$ is a ratio of odd positive integers. In 2013, Candan [7] established other oscillation criteria of Eq. (1.1) for $\delta(t,\xi)\le t$, where $\gamma\ge 1$ is a quotient of odd positive integers, β (in [6]) is equal to γ, $r^{\Delta}(t)>0$, and $\delta(t,\xi)$ is decreasing with respect to ξ.

The purpose of this paper is to establish new oscillation criteria of Eq. (1.1) for $\gamma>0$, a quotient of odd positive integers, where functions $p(t)$ and $r(t)$ may not be monotonic, $\delta(t,\xi)$ may not be decreasing with respect to ξ. Hence, our results will generalize and improve those in [6, 7] and others.

By a solution of Eq. (1.1), we mean a nontrivial real-valued function $y(t)$ such that $y(t)+p(t)y(\tau(t))\in C^1_{rd}[\tau_1^*(t_0),\infty)$, $r(t)((y(t)+p(t)y(\tau(t)))^{\Delta})^{\gamma}\in C_{rd}[\tau_1^*(t_0),\infty)$ and satisfies Eq. (1.1). Our attention is restricted to those solutions of Eq. (1.1) that satisfy $\sup\{|y(t)|:t\ge t_y\}>0$ for any $t_y\ge t_0$. A solution $y(t)$ of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

This paper is organized as follows. After this introduction, we introduce some basic lemmas in Section 2. In Section 3, we present the main results. In Section 4, we illustrate the versatility of our results by two examples.

2 Some preliminaries

In this section, we present several technical lemmas which will be used in the proofs of the main results. For convenience, we use the notation $(x(\sigma(t)))^{\gamma}=(x^{\sigma}(t))^{\gamma}$ and set

$$x(t):=y(t)+p(t)y\big(\tau(t)\big). \tag{2.1}$$

Then Eq. (1.1) becomes

$$\big(r(t)\big(x^{\Delta}(t)\big)^{\gamma}\big)^{\Delta}+\int_a^b f\big(t,y\big(\delta(t,\xi)\big)\big)\Delta\xi=0.$$

For $t,T\in\mathbb{T}$ with $t>T$, we define

$$\beta(t,T)=\int_T^t \frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s, \quad \text{and} \quad g_{\xi}(t,T)=\begin{cases}\frac{\beta(\delta(t,\xi),T)}{\beta(t,T)}, & \delta(t,\xi)<t,\\ 1, & \delta(t,\xi)\ge t;\end{cases} \tag{2.2}$$

$$Q(t,T) = q(t)\int_a^b \bigl[1 - p\bigl(\delta(t,\xi)\bigr)\bigr]^\gamma g_\xi^\gamma(t,T)\Delta\xi.$$

For $D = \{(t,s) \in \mathbb{T}^2 : t \geq s \geq 0\}$, we define

$$\mathcal{H} = \bigl\{H(t,s) \in C_{rd}^1\bigl(D,[0,\infty)\bigr) : H(t,t) = 0, H(t,s) > 0 \text{ and } H_s^\Delta(t,s) \geq 0 \text{ for } t > s \geq 0\bigr\},$$
$$C(t,s) = H_s^\Delta(t,s)z^\sigma(s) + H(t,s)z^\Delta(s) \quad \text{for } H(t,s) \in \mathcal{H},$$

where $z \in C_{rd}^1(\mathbb{T},(0,\infty))$ is to be given in Theorems 3.1 and 3.2, and $z_+^\Delta(t) = \max\{z^\Delta(t), 0\}$.
First of all, we give the following lemma.

Lemma 2.1 *Let conditions* (H1)-(H3) *hold. If $y(t)$ is an eventually positive solution of Eq.* (1.1), *then there exists $T \in \mathbb{T}$ sufficiently large such that $x(t) > 0$, $x^\Delta(t) \geq 0$, $(r(t)(x^\Delta(t))^\gamma)^\Delta \leq 0$, $x(t) \geq r^{\frac{1}{\gamma}}(t)x^\Delta(t)\beta(t,T)$, and $x(\delta(t,\xi)) \geq g_\xi(t,T)x(t)$ for $t \in [T,\infty)_\mathbb{T}$.*

Proof Since $y(t)$ is an eventually positive solution of Eq. (1.1), then by (H1) there exists $T \in [t_0,\infty)_\mathbb{T}$ such that

$$\delta(t,\xi) > T, \qquad y(t) > 0, \qquad y\bigl(\tau(t)\bigr) > 0 \quad \text{and} \quad y\bigl(\delta(t,\xi)\bigr) > 0 \quad \text{for } t \geq T.$$

From (2.1) and (H2), we see that $x(t)$ is also positive and satisfies $x(t) \geq y(t)$. Also by Eq. (1.1) and (H3), we have that $x(t)$ satisfies

$$\bigl(r(t)\bigl(x^\Delta(t)\bigr)^\gamma\bigr)^\Delta \leq -\int_a^b q(t)y^\gamma\bigl(\delta(t,\xi)\bigr)\Delta\xi \leq 0 \quad \text{for } t \geq T,$$

which implies that $r(t)(x^\Delta(t))^\gamma$ is decreasing on $[T,\infty)_\mathbb{T}$. So we can get

$$\begin{aligned} x(t) &= x(T) + \int_T^t \frac{(r(s)(x^\Delta(s))^\gamma)^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)}\Delta s \\ &\geq r^{\frac{1}{\gamma}}(t)x^\Delta(t)\int_T^t \frac{1}{r^{\frac{1}{\gamma}}(s)}\Delta s := r^{\frac{1}{\gamma}}(t)x^\Delta(t)\beta(t,T). \end{aligned}$$

We claim that $r(t)(x^\Delta(t))^\gamma \geq 0$ on $[T,\infty)_\mathbb{T}$. Assume not, there is $t_1 \in [T,\infty)_\mathbb{T}$ such that $r(t_1)(x^\Delta(t_1))^\gamma < 0$. Since $r(t)(x^\Delta(t))^\gamma \leq r(t_1)(x^\Delta(t_1))^\gamma$ for $t \geq t_1$, we have

$$x^\Delta(t) \leq r^{\frac{1}{\gamma}}(t_1)x^\Delta(t_1)\bigl(1/r(t)\bigr)^{1/\gamma}.$$

Integrating the inequality above from t_1 to t ($\geq T$), by (H2) we get

$$x(t) \leq x(t_1) + r^{\frac{1}{\gamma}}(t_1)x^\Delta(t_1)\int_{t_1}^t \bigl(1/r(s)\bigr)^{1/\gamma}\Delta s \to -\infty \quad (t \to \infty),$$

and this contradicts the fact that $x(t) > 0$ for all $t \geq T$. Thus we have $r(t)(x^\Delta(t))^\gamma \geq 0$ on $[T,\infty)_\mathbb{T}$ and so $x^\Delta(t) \geq 0$ on $[T,\infty)_\mathbb{T}$.

Let $t \geq T$ be fixed such that $\delta(t,\xi) \geq T$. We consider the two cases $\delta(t,\xi) < t$ and $\delta(t,\xi) \geq t$, respectively.

Case I: $\delta(t,\xi) < t$. Noting that $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} \leq 0$, we have

$$x(t) - x\big(\delta(t,\xi)\big) = \int_{\delta(t,\xi)}^{t} \frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \leq \big(r\big(\delta(t,\xi)\big)\big(x^{\Delta}\big(\delta(t,\xi)\big)\big)^{\gamma}\big)^{\frac{1}{\gamma}} \int_{\delta(t,\xi)}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

It follows that

$$\frac{x(t)}{x(\delta(t,\xi))} \leq 1 + \frac{(r(\delta(t))(x^{\Delta}(\delta(t,\xi)))^{\gamma})^{\frac{1}{\gamma}}}{x(\delta(t,\xi))} \int_{\delta(t,\xi)}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}.$$

Since $\delta(t,\xi) \geq T$ for $t \in [T,\infty)$,

$$x\big(\delta(t,\xi)\big) > \int_{T}^{\delta(t,\xi)} \frac{(r(s)(x^{\Delta}(s))^{\gamma})^{\frac{1}{\gamma}}}{r^{\frac{1}{\gamma}}(s)} \Delta s \geq \big(r\big(\delta(t,\xi)\big)\big(x^{\Delta}\big(\delta(t,\xi)\big)\big)^{\gamma}\big)^{\frac{1}{\gamma}} \int_{T}^{\delta(t,\xi)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)},$$

which implies that

$$\frac{(r(\delta(t,\xi))(x^{\Delta}(\delta(t,\xi)))^{\gamma})^{\frac{1}{\gamma}}}{x(\delta(t,\xi))} < \frac{1}{\int_{T}^{\delta(t,\xi)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}.$$

Thus

$$\frac{x(t)}{x(\delta(t,\xi))} < 1 + \frac{\int_{\delta(t,\xi)}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_{T}^{\delta(t,\xi)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}} \leq \frac{\int_{T}^{t} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}{\int_{T}^{\delta(t,\xi)} \frac{\Delta s}{r^{\frac{1}{\gamma}}(s)}}.$$

Case II: $\delta(t,\xi) \geq t$. Noting that $x^{\Delta}(t) \geq 0$ and from the definition of $g_{\xi}(t,T)$ defined in (2.2), we have

$$x\big(\delta(t,\xi)\big) \geq g_{\xi}(t,T)x(t).$$

□

Remark 2.1 By $x(t) \geq y(t)$ on $[t_1,\infty)_{\mathbb{T}}$, $x^{\Delta}(t) > 0$ and $\tau(t) \leq t$, we get

$$y(t) = x(t) - p(t)x\big(\tau(t)\big) \geq \big(1 - p(t)\big)x(t).$$

Then from Eq. (1.1), $x(\delta(t,\xi)) \geq g_{\xi}(t,T)x(t)$, (H2) and (H3), we conclude that

$$\begin{aligned} 0 &\geq \big(r(t)\big(x^{\Delta}(t)\big)^{\gamma}\big)^{\Delta} \\ &\quad + x^{\gamma}(t)q(t)\int_{a}^{b} \big[1 - p\big(\delta(t,\xi)\big)\big]^{\gamma} g_{\xi}^{\gamma}(t,T)\Delta\xi, \quad t \geq t_1, \xi \in [a,b]. \end{aligned} \tag{2.3}$$

Lemma 2.2 ([2]) *Let $g(u) = Bu - Au^{\frac{\gamma+1}{\gamma}}$, where $A > 0$ and B are constants, γ is a positive number. Then g attains its maximum value on $[0,\infty)$ at $u^* = (\frac{B\gamma}{A(\gamma+1)})^{\gamma}$, and*

$$\max_{u\in[0,\infty)} g = g\big(u^*\big) = \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}}.$$

3 Main results

In this section, we establish our main results.

Theorem 3.1 *Let* $\gamma > 0$. *Assume that* (H1)-(H3) *hold. Furthermore, for sufficiently large* $T \in \mathbb{T}$, *one of the following conditions is satisfied*:

(a) *either* $\int_t^\infty Q(s,T)\Delta s = \infty$, *or*

$$\int_t^\infty Q(s,T)\Delta s < \infty \quad \text{and} \quad \beta^\gamma(t,T)\int_t^\infty Q(s,T)\Delta s > 1 \quad \text{for all } t > T,$$

(b) *there exists* $z \in C^1_{rd}(\mathbb{T},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[z(s)Q(s,T) - \frac{z_+^\Delta(s)}{\beta^\gamma(s,T)}\right]\Delta s = \infty,$$

(c) *there exists* $z \in C^1_{rd}(\mathbb{T},(0,\infty))$ *such that*

$$\limsup_{t\to\infty}\int_T^t \left[z(s)Q(s,T) - \frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(s)(z^\Delta(s))^{\gamma+1}}{z^\gamma(s)}\right]\Delta s = \infty,$$

(d) *there exist* $z \in C^1_{rd}(\mathbb{T},(0,\infty))$ *and* $H \in \mathcal{H}$ *such that*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left[H(t,s)z(s)Q(s,T) - \frac{C^{\gamma+1}(t,s)}{H^\gamma(t,s)(\gamma+1)^{\gamma+1}z^\gamma(s)}\right]\Delta s = \infty.$$

Then every solution $y(t)$ *of Eq.* (1.1) *is oscillatory.*

Proof Suppose to the contrary that Eq. (1.1) has a nonoscillatory solution $y(t)$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (H1)-(H3), there exists $T \in [t_0,\infty)_\mathbb{T}$ such that for $t \geq T$, $y(\tau(t)) > 0$, $y(\delta(t,\xi)) > 0$, and Lemma 2.1 holds.

The rest of the proof is divided into four parts corresponding to conditions (a)-(d), respectively.

Part I: Assume condition (a) holds.

Let $\phi(t) := r(t)(x^\Delta(t))^\gamma$. Then $\phi(t) \geq 0$ and $\phi^\Delta(t) \leq 0$ for $t \geq T$, and $\lim_{t\to\infty}\phi(t) = \zeta \geq 0$. From (2.3), we have

$$\phi^\Delta(t) + x^\gamma(t)q(t)\int_a^b \left[1 - p\big(\delta(t,\xi)\big)\right]^\gamma g_\xi^\gamma(t,T)\Delta\xi \leq 0. \tag{3.1}$$

Integrating both sides of (3.1) from t to ∞, we obtain

$$\zeta - \phi(t) + \int_t^\infty Q(s,T)x^\gamma(s)\Delta s \leq 0.$$

In view of $x^\Delta(t) \geq 0$, we have reached a contradiction if $\int_t^\infty Q(s,T)\Delta s = \infty$. If $\int_t^\infty Q(s, T)\Delta s < \infty$, then

$$\phi(t) \geq \int_t^\infty Q(s,T)x^\gamma(s)\Delta s \geq x^\gamma(t)\int_t^\infty Q(s,T)\Delta s.$$

By Lemma 2.1, we obtain

$$\beta^{\gamma}(t,T)\int_t^{\infty} Q(s,T)\Delta s \le 1,$$

which is a contradiction to condition (a). Therefore, every solution $y(t)$ of Eq. (1.1) is oscillatory.

Part II: Assume condition (b) holds. Define

$$w(t) := \frac{z(t)r(t)(x^{\Delta}(t))^{\gamma}}{x^{\gamma}(t)} \quad \text{for } t \ge T. \tag{3.2}$$

Then $w(t) > 0$. From (2.3), we have

$$\begin{aligned} w^{\Delta}(t) &= \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\Delta}\left(\frac{z(t)}{x^{\gamma}(t)}\right) + \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma}\left(\frac{z(t)}{x^{\gamma}(t)}\right)^{\Delta} \\ &\le -z(t)Q(t,T) + \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma}\left[\frac{z^{\Delta}(t)x^{\gamma}(t) - z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}\right] \\ &\le -z(t)Q(t,T) + \frac{z_+^{\Delta}(t)(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}}{(x^{\sigma}(t))^{\gamma}} - \frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))}. \end{aligned} \tag{3.3}$$

When $\gamma \ge 1$, using $x^{\Delta}(t) > 0$ and the Keller?s chain rule, we get

$$\begin{aligned} \bigl(x^{\gamma}(t)\bigr)^{\Delta} &= \gamma\left[\int_0^1 \bigl(x(t) + h\mu(t)x^{\Delta}(t)\bigr)^{\gamma-1}\,dh\right]x^{\Delta}(t) \\ &\ge \gamma x^{\Delta}(t)\int_0^1 \bigl((1-h)x(t) + hx(t)\bigr)^{\gamma-1}\,dh = \gamma x^{\gamma-1}(t)x^{\Delta}(t). \end{aligned} \tag{3.4}$$

When $0 < \gamma < 1$, using $x^{\Delta}(t) > 0$ and the Keller?s chain rule, we obtain

$$\bigl(x^{\gamma}(t)\bigr)^{\Delta} \ge \gamma x^{\Delta}(t)\int_0^1 \bigl((1-h)x^{\sigma}(t) + hx^{\sigma}(t)\bigr)^{\gamma-1}\,dh = \gamma\bigl(x^{\sigma}(t)\bigr)^{\gamma-1}x^{\Delta}(t). \tag{3.5}$$

Noting that $r(t) > 0$ and from (3.4), (3.5), and Lemma 2.1, we obtain

$$\frac{(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma}z(t)(x^{\gamma}(t))^{\Delta}}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \ge 0.$$

Since $(r(t)(x^{\Delta}(t))^{\gamma})^{\Delta} \le 0$ and $t \le \sigma(t)$, we have

$$r\bigl(\sigma(t)\bigr)(x^{\Delta}\bigl(\sigma(t)\bigr)^{\gamma} \le r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}. \tag{3.6}$$

Hence from (3.6) and Lemma 2.1 and noting that $x^{\Delta}(t) \ge 0$, we have

$$w^{\Delta}(t) \le -z(t)Q(t,T) + \frac{z_+^{\Delta}(t)}{\beta^{\gamma}(t,T)}.$$

Integrating the above inequality from T to t for $t \ge T$, we get

$$\int_T^t \left[z(s)Q(s,T) - \frac{z_+^{\Delta}(s)}{\beta^{\gamma}(s,T)}\right]\Delta s \le w(T) - w(t) < w(T).$$

Taking limsup on both sides as $t \to \infty$, we obtain a contradiction to condition (b). Therefore, every solution $y(t)$ of Eq. (1.1) is oscillatory.

Part III: Assume condition (c) holds.

When $\gamma \geq 1$, from (3.3) and (3.4) we have

$$\begin{aligned} w^{\Delta}(t) &\leq -z(t)Q(t,T) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma} \frac{z(t)\gamma x^{\gamma-1}(t)x^{\Delta}(t)}{x^{\gamma}(t)x^{\gamma}(\sigma(t))} \\ &\leq -z(t)Q(t,T) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma} \frac{z(t)\gamma x^{\Delta}(t)}{x^{\gamma+1}(\sigma(t))}. \end{aligned} \tag{3.7}$$

From (3.6) we get

$$\begin{aligned} -\bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma} \frac{z(t)\gamma x^{\Delta}(t)}{x^{\gamma+1}(\sigma(t))} &\leq -\bigl(r^{\sigma}(t)\bigr)^{\frac{\gamma+1}{\gamma}} \bigl(x^{\Delta}\bigl(\sigma(t)\bigr)\bigr)^{\gamma+1} \frac{z(t)\gamma}{r^{\frac{1}{\gamma}}(t)x^{\gamma+1}(\sigma(t))} \\ &= -\frac{z(t)\gamma}{z^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(t)} w^{\frac{\gamma+1}{\gamma}}\bigl(\sigma(t)\bigr). \end{aligned}$$

Then

$$w^{\Delta}(t) \leq -z(t)Q(t,T) + \frac{z^{\Delta}(t)}{z^{\sigma}(t)} w^{\sigma}(t) - \frac{z(t)\gamma}{z^{\frac{\gamma+1}{\gamma}}(\sigma(t))r^{\frac{1}{\gamma}}(t)} w^{\frac{\gamma+1}{\gamma}}\bigl(\sigma(t)\bigr). \tag{3.8}$$

When $0 < \gamma < 1$, by (3.3) and (3.5) we have

$$w^{\Delta}(t) \leq -z(t)Q(t,T) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \bigl(r(t)\bigl(x^{\Delta}(t)\bigr)^{\gamma}\bigr)^{\sigma} \frac{z(t)\gamma (x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}}.$$

By (3.6) we have

$$\begin{aligned} \frac{-(r(t)(x^{\Delta}(t))^{\gamma})^{\sigma} z(t)\gamma (x^{\sigma}(t))^{\gamma-1}x^{\Delta}(t)}{x^{\gamma}(t)(x^{\sigma}(t))^{\gamma}} &= -\frac{(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}((x^{\Delta}(t))^{\sigma})^{\gamma+1}z(t)\gamma x^{\Delta}(t)}{x^{\gamma}(t)x^{\sigma}(t)(r^{\sigma}(t))^{\frac{1}{\gamma}}(x^{\Delta}(t))^{\sigma}} \\ &\leq -\frac{(r^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}((x^{\Delta}(t))^{\sigma})^{\gamma+1}z(t)\gamma x^{\Delta}(t)}{x^{\gamma}(t)x^{\sigma}(t)r^{\frac{1}{\gamma}}(t)x^{\Delta}(t)} \\ &\leq -\frac{z(t)\gamma}{(z^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}r^{\frac{1}{\gamma}}(t)}\bigl(w^{\sigma}(t)\bigr)^{\frac{\gamma+1}{\gamma}}. \end{aligned}$$

It follows that

$$w^{\Delta}(t) \leq -z(t)Q(t,T) + w^{\sigma}(t)\frac{z^{\Delta}(t)}{z^{\sigma}(t)} - \frac{z(t)\gamma}{(z^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}r^{\frac{1}{\gamma}}(t)}\bigl(w^{\sigma}(t)\bigr)^{\frac{\gamma+1}{\gamma}}, \tag{3.9}$$

which is the same as (3.8). Let

$$B = \frac{z^{\Delta}(t)}{z^{\sigma}(t)}, \qquad A = \frac{z(t)\gamma}{(z^{\sigma}(t))^{\frac{\gamma+1}{\gamma}}r^{\frac{1}{\gamma}}(t)}, \qquad u = w^{\sigma}(t).$$

Then by Lemma 2.2 and (3.9) we obtain that for all $t \geq T$,

$$w^{\Delta}(t) \leq -z(t)Q(t,T) + \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(t)(z^{\Delta}(t))^{\gamma+1}}{z^{\gamma}(t)}.$$

Integrating the above inequality from T to t for $\geq T$, we get

$$\int_T^t \left[z(s)Q(s,T) - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(s)(z^{\Delta}(s))^{\gamma+1}}{z^{\gamma}(s)} \right] \Delta s \leq w(T) - w(t) < w(T).$$

By taking limsup on both sides as $t \to \infty$, we obtain a contradiction to condition (c). Therefore, every solution $y(t)$ of Eq. (1.1) is oscillatory.

Part IV: Assume condition (d) holds.

From (3.8) and (3.9) we have that for $H \in \mathcal{H}_*$ and $t \geq T$,

$$\begin{aligned} \int_T^t H(t,s)z(s)Q(s,T)\Delta s \leq &- \int_T^t H(t,s)w^{\Delta}(s)\Delta s + \int_T^t H(t,s)w^{\sigma}(s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)}\Delta s \\ &- \int_T^t H(t,s)\frac{z(s)\gamma}{(z^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} r^{\frac{1}{\gamma}}(s)} \left(w^{\sigma}(s)\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

By integration by parts we obtain

$$-\int_T^t H(t,s)w^{\Delta}(s)\Delta s = H(t,T)w(T) + \int_T^t H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s.$$

It follows that

$$\begin{aligned} \int_T^t H(t,s)z(s)Q(s,T)\Delta s \leq &\, H(t,T)w(T) + \int_T^t \left[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)} \right] w^{\sigma}(s)\Delta s \\ &- \int_T^t \frac{H(t,s)z(s)\gamma}{(z^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} r^{\frac{1}{\gamma}}(s)} \left(w^{\sigma}(s)\right)^{\frac{\gamma+1}{\gamma}} \Delta s. \end{aligned}$$

Let

$$B = H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)}, \qquad A = \frac{H(t,s)z(s)\gamma}{(z^{\sigma}(s))^{\frac{\gamma+1}{\gamma}} r^{\frac{1}{\gamma}}(s)}, \qquad u = w^{\sigma}(s),$$

by Lemma 2.2 we obtain that for all $t \geq T$,

$$\begin{aligned} \int_T^t H(t,s)z(s)Q(s,T)\Delta s \leq &\, H(t,T)w(T) \\ &+ \int_T^t \frac{[H_s^{\Delta}(t,s) + H(t,s)\frac{z^{\Delta}(s)}{z^{\sigma}(s)}]^{\gamma+1} r(s)(z^{\sigma}(s))^{\gamma+1}}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}z^{\gamma}(s)}\Delta s. \end{aligned}$$

That is,

$$\frac{1}{H(t,T)} \int_T^t \left[H(t,s)z(s)Q(s,T) - \frac{[C(t,s)]^{\gamma+1} r(s)}{H^{\gamma}(t,s)(\gamma+1)^{\gamma+1}z^{\gamma}(s)} \right] \Delta s \leq w(T).$$

By taking limsup on both sides as $t \to \infty$, we obtain a contradiction to condition (d). Therefore, every solution $y(t)$ of Eq. (1.1) is oscillatory.

The proof is complete. □

The results in the next theorem hold only for $\gamma \geq 1$.

Theorem 3.2 *Let* $\gamma \geq 1$. *Assume that* (H1)-(H3) *hold. Furthermore, for sufficiently large* $T \in \mathbb{T}$, *there exists* $z \in C^1_{rd}(\mathbb{T}, (0,\infty))$ *such that one of the following conditions is satisfied*:

(a)

$$\limsup_{t\to\infty} \int_T^t \left[z(s)Q(s,T) - \frac{(z^\Delta(s))^2 r^{\frac{1}{\gamma}}(s)}{4\gamma z(s)\beta^{\gamma-1}(s,T)} \right] \Delta s = \infty,$$

(b) *there exists* $H \in \mathcal{H}$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)z(s)Q(s,T) - \frac{C^2(t,s) r^{\frac{1}{\gamma}}(s)}{4\gamma z(s)H(t,s)\beta^{\gamma-1}(s,T)} \right] \Delta s = \infty.$$

Then every solution $y(t)$ *of Eq.* (1.1) *is oscillatory.*

Proof Suppose to the contrary that Eq. (1.1) has a nonoscillatory solution $y(t)$. Without loss of generality, we may assume that $y(t)$ is eventually positive. Then, by (H1)-(H3) there exists $T \in [t_0,\infty)_{\mathbb{T}}$ such that for $t \geq T$, $y(t) > 0$, $y(\delta(t,\xi)) > 0$, $y(\tau(t)) > 0$, and Lemma 2.1 holds.

The rest of the proof is divided into two parts corresponding to conditions (a) and (b), respectively.

Part I: Assume condition (a) holds.

Define $w(t)$ as in (3.2). By $x^\Delta(t) \geq 0$, $\sigma(t) \geq t$, (3.3), and (3.4), we obtain

$$\begin{aligned} w^\Delta(t) &\leq -z(t)Q(t,T) + w^\sigma(t)\frac{z^\Delta(t)}{z^\sigma(t)} - \left(r(t)\left(x^\Delta(t)\right)^\gamma\right)^\sigma \frac{z(t)\gamma x^{\gamma-1}(t)x^\Delta(t)}{x^\gamma(t)(x^\sigma(t))^\gamma} \\ &\leq -z(t)Q(t,T) + w^\sigma(t)\frac{z^\Delta(t)}{z^\sigma(t)} - \frac{z(t)\gamma x^{\gamma-1}(t)x^\Delta(t)}{(z^\sigma(t))^2(r(t)(x^\Delta(t))^\gamma)^\sigma}\left(w^\sigma(t)\right)^2. \end{aligned}$$

From (3.6) and Lemma 2.1, we get

$$\begin{aligned} w^\Delta(t) &\leq -z(t)Q(t,T) + w^\sigma(t)\frac{z^\Delta(t)}{z^\sigma(t)} - \frac{z(t)\gamma}{(z^\sigma(t))^2 r(t)} \frac{x^{\gamma-1}(t)}{(x^\Delta(t))^{\gamma-1}}\left(w^\sigma(t)\right)^2 \\ &\leq -z(t)Q(t,T) + w^\sigma(t)\frac{z^\Delta(t)}{z^\sigma(t)} - \frac{z(t)\gamma\beta^{\gamma-1}(t,T)}{(z^\sigma(t))^2 r^{\frac{1}{\gamma}}(t)}\left(w^\sigma(t)\right)^2. \end{aligned} \tag{3.10}$$

By completing the square for $w^\sigma(t)$ on the right-hand side, we have

$$w^\Delta(t) \leq -z(t)Q(t,T) + \frac{(z^\Delta(t))^2 r^{\frac{1}{\gamma}}(t)}{4\gamma z(t)\beta^{\gamma-1}(t,T)}.$$

Integrating the above inequality from T to t for $t \geq T$, we get

$$\int_T^t \left[z(s)Q(s,T) - \frac{(z^\Delta(s))^2 r^{\frac{1}{\gamma}}(s)}{4\gamma z(s)\beta^{\gamma-1}(s,T)} \right] \Delta s \leq w(T) - w(t) < w(T).$$

Taking limsup on both sides as $t \to \infty$, we obtain a contradiction to condition (a). Therefore, every solution $y(t)$ of Eq. (1.1) is oscillatory.

Part II: Assume condition (b) holds.

Based on (3.10), the proof is similar to those of Part IV of Theorem 3.1 and Part I of Theorem 3.2, and hence is omitted.

The proof is complete. □

4 Examples

In this section, we give two examples to illustrate our main results.

Example 4.1 Consider the equation

$$\left(\frac{1}{t}\left(\left(y(t)+\frac{1}{t}y\big(\tau(t)\big)\right)^{\Delta}\right)^{\frac{1}{3}}\right)^{\Delta}+\int_a^b \frac{1}{(t-1)^{\frac{1}{3}}}y^{\frac{1}{3}}\big(\delta(t,\xi)\big)\Delta\xi=0,\quad t\in\mathbb{T},\tag{4.1}$$

where $\delta(t,\xi)\ge t$, $\tau(t)\le t$ and $\mathbb{T}=[2,\infty)_{\mathbb{T}}$. Here we have

(i) $\gamma=\frac{1}{3}$, $r(t)=p(t)=\frac{1}{t}$, and $q(t)=\frac{1}{(t-1)^{\frac{1}{3}}}$;

(ii) $\int_2^{\infty} r^{-\frac{1}{\gamma}}(s)\Delta s=\int_2^{\infty} s^3\Delta s=\infty$, $g_\xi(t,T)=1$;

(iii) $\int_a^b[1-p(\delta(t,\xi))]^{\gamma}g_\xi^{\gamma}(t,T)\Delta\xi=\int_a^b[1-\frac{1}{\delta(t,\xi)}]^{\frac{1}{3}}\Delta\xi\ge\int_a^b[1-\frac{1}{t}]^{\frac{1}{3}}\Delta\xi=[1-\frac{1}{t}]^{\frac{1}{3}}(b-a)$.

Hence (H1)-(H3) hold. With $z(t)=1$, we see that for sufficiently large $T\in\mathbb{T}$,

$$\begin{aligned}\limsup_{t\to\infty}\int_T^t Q(s,T)\Delta s &\ge \limsup_{t\to\infty}[b-a]\int_T^t \frac{1}{(s-1)^{\frac{1}{3}}}\left(1-\frac{1}{s}\right)^{\frac{1}{3}}\Delta s\\ &\ge \limsup_{t\to\infty}[b-a]\int_T^t \frac{1}{(s-1)^{\frac{1}{3}}}(s-1)^{\frac{1}{3}}\frac{1}{s^{\frac{1}{3}}}\Delta s\\ &\ge \limsup_{t\to\infty}[b-a]\int_T^t \frac{1}{s^{\frac{1}{3}}}\Delta s=\infty.\end{aligned}$$

Hence condition (c) of Theorem 3.1 is satisfied.

By Theorem 3.1, every solution $y(t)$ of Eq. (4.1) is oscillatory.

Example 4.2 Consider the equation

$$\left(\frac{1}{(t+\sigma(t))^3}\big(\big(y(t)+Ay\big(\tau(t)\big)\big)^{\Delta}\big)^3\right)^{\Delta}+\int_a^b t^2y^{\gamma}\big(\delta(t-\xi)\big)\Delta\xi=0,\quad t\in\mathbb{T},\tag{4.2}$$

where $1>A\ge 0$, $\tau(t)\le t$ and $\mathbb{T}=[1,\infty)_{\mathbb{T}}$. Here we have

(i) $\gamma=3$, $r(t)=\frac{1}{(t+\sigma(t))^3}$, $p(t)=A$, $\delta(t,\xi)=t-\xi<t$, and $q(t)=t^2$;

(ii) $\int(s+\sigma(s))\Delta s=t^2+c$, $\int_1^{\infty} r^{-\frac{1}{\gamma}}(s)\Delta s=\infty$,
$\beta(\delta(t,\xi),T)=\int_T^{t-\xi}(s+\sigma(s))\Delta s>\int_T^{t-\xi}s\Delta s>T(t-\xi-T)$, and
$\beta(t,T)=\int_T^t(s+\sigma(s))\Delta s=t^2-T^2$;

(iii) $\int_a^b[1-p(\delta(t,\xi))]^{\gamma}g_\xi^{\gamma}(t,T)\Delta\xi>\frac{T^3[1-A]^3}{(t^2-T^2)^3}\int_a^b(t-\xi-T)^3\Delta\xi>\frac{T^3[1-A]^3}{(t^2-T^2)^3}(t-b-T)^3(b-a)$.

Hence (H1)-(H3) hold. With $z(t) = 1$, we see that for sufficiently large $T \in \mathbb{T}$,

$$\begin{aligned}\limsup_{t\to\infty} \int_T^t Q(s,T)\Delta s &\geq \limsup_{t\to\infty} T^3[1-A]^3(b-a)\int_T^t s^2 \frac{1}{(s^2-T^2)^3}(s-b-T)^3\Delta s \\ &\geq \limsup_{t\to\infty} T^3[1-A]^3(b-a)\int_T^t \frac{1}{s}\left(1-\frac{b}{s}-\frac{T}{s}\right)^3 \Delta s = \infty.\end{aligned}$$

Hence condition (a) of Theorem 3.2 is satisfied.

By Theorem 3.2, every solution $y(t)$ of Eq. (4.2) is oscillatory.

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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