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Boundedness and stability of discrete Volterra equations

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Abstract

In this paper, using the fixed point theorem, we investigate the boundedness and asymptotic stability of the zero solution of the discrete Volterra equation

$$x(n + 1) = a(n) + b(n)x(n) + \sum_{i=n_0}^{n} K(n,i)x(i).$$

Necessary conditions for the existence of a periodic solution of the above equation are also obtained.

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Keywords: Volterra difference equation; stability; bounded solution; periodic solution; contraction mapping

1 Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of nonnegative integers, the set of all integers and the set of real numbers, respectively, and $\mathbb{N}_0 = \{n_0, n_0 + 1, \dots\}$, where $n_0 \in \mathbb{N}$.

In this paper we consider the linear discrete Volterra equation of non-convolution type of the form

$$x(n + 1) = a(n) + b(n)x(n) + \sum_{i=n_0}^{n} K(n,i)x(i), \quad n \geq n_0, \tag{1}$$

where $a, b \colon \mathbb{N}_0 \to \mathbb{R}$, $b(n) \neq 0$ for $n \geq n_0$, $K \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$. By a solution of equation (1) we mean a sequence $x \colon \mathbb{N}_0 \to \mathbb{R}$ satisfying equation (1) for all $n \geq n_0$.

We introduce some notation and definitions that will be used in the paper. Hereafter, we denote the solution of equation (1) with the initial condition $x(n_0) = x_0$ by $x(n, n_0, x_0)$.

Definition 1 Solutions of equation (1) are equi-bounded if for each constant $M_1 > 0$, there is $M_2 > 0$ such that $|x_0| \leq M_1$ and $n \geq n_0$ implies that $|x(n, n_0, x_0)| \leq M_2$.

If $a(n) \equiv 0$ in equation (1), then (1) takes the following form:

$$x(n + 1) = b(n)x(n) + \sum_{i=n_0}^{n} K(n,i)x(i), \quad n \geq n_0. \tag{2}$$

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Definition 2 The zero solution of equation (2) is stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0| \leq \delta$ implies $|x(n, n_0, x_0)| \leq \varepsilon$ for $n \geq n_0$.

Definition 3 The zero solution of equation (2) is asymptotically stable if there exists $\mu > 0$ such that $|x_0| \leq \mu$ implies $\lim_{n \to \infty} x(n) = 0$.

Denote the Banach space of all bounded real sequences $x \colon \mathbb{N}_0 \to \mathbb{R}$ equipped with the supremum norm $\|x\| = \sup_{n \geq n_0} |x(n)|$ by BS.

We will also adopt the customary notations

$$\sum_{i=k+s}^{k} u(i) = 0, \qquad \prod_{i=k+s}^{k} u(i) = 1,$$

where k is an integer, s is a positive integer and u is a sequence of real numbers.

Volterra difference equations are widely used in modeling of processes in many branches of natural sciences, economics and engineering. Some important properties of their solutions are boundedness, stability and periodicity. Boundedness of solutions of linear discrete Volterra equations has been studied by many authors. For example, Kolmanovskii and Shaikhet [1] have established a necessary and sufficient condition for boundedness of all solutions of the linear Volterra equation

$$\Delta x(n) = \sum_{i=0}^{n} A(n, i) x(i), \quad n \geq 0,$$

where $A(j, i) \geq 0, j \geq i \geq 0$.

Appleby *et al.* [2] found sufficient conditions in terms of the kernel H under which every solution of the system of linear Volterra equations

$$x(n + 1) = h(n) + \sum_{i=0}^{n} H(n, i) x(i), \quad n \geq 0,$$

is convergent. The necessary and sufficient conditions for boundedness of all solutions of the above equation are given by Győri and Reynolds in [3].

In [4], Khandaker and Raffoul studied the stability properties of the zero solution of the nonlinear Volterra discrete system

$$x(n + 1) = A(n) x(n) + \sum_{s=0}^{n} B(n, s) x(s) + g(n, x(n))$$

by expressing the solution in terms of the resolvent matrix. Here, the authors assumed that $|g(n, x(n))| \leq \lambda(n) |x(n)|$, where $\lambda(n)$ is such that $0 \leq \lambda(n) \leq N < \infty$ for some constant N. A scalar case of the above equation is also considered in Section 4 of Raffoul [5].

Yankson, in [6], studied the asymptotic stability of the zero solution of the Volterra difference delay equation

$$x(n + 1) = a(n) x(n) + c(n) \Delta x(n - g(n)) + \sum_{s=n-g(n)}^{n-1} k(n, s) h(x(s)).$$

Since the condition $0 \leq g(n) \leq g_0$ is assumed, then equation (1) cannot be obtained as a special case of this equation.

Boundedness of solutions of linear and nonlinear discrete Volterra equations was also studied by Crisci *et al.* [7], Diblík and Schmeidel [8], Gronek and Schmeidel [9], Győri and Awwad [10], Győri and Horváth [11], Kolmanovskii and Shaikhet [1], Medina [12], Migda and Migda [13] or Migda and Morchało [14]. A survey of the fundamental results on the stability of linear Volterra difference equations, of both convolution and non-convolution type, can be found in Elaydi [15], see also Crisci *et al.* [16] and [17]. The problem of finding periodic and asymptotically periodic solutions of linear discrete Volterra equations of type (1) was investigated, for example, by Diblik *et al.* [8, 18–21], Elaydi [22], Gajda *et al.* [23], Győri and Reynolds [24], Migda and Migda [13] or Song and Baker [25].

Motivated by the results obtained in the papers by Islam and Yankson [26] and Raffoul [27], in this paper, we derive explicit sufficient conditions for the equi-boundedness of solutions of equation (1) and the asymptotic stability of the zero solution of equation (2).

We prove our main results using the variation of constants formula and the contraction mapping principle. We study necessary conditions for the existence of periodic solutions. Our results generalize certain results obtained in [9] and [1]. Moreover, they can be applied to equations for which the results obtained in some of the above mentioned papers could not be used (see Examples 1-3).

2 Boundedness and stability

Let us note that by the variation of constants formula, the solution of equation (1) satisfies the following equation:

$$
x(n) = x_0 \prod_{j=n_0}^{n-1} b(j) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) \sum_{i=n_0}^{k} K(k,i) x(i)
$$
$$
+ \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) a(k) \quad \text{for } n \geq n_0, \tag{3}
$$

where $x(n_0) = x_0$. Using this fact we prove the following theorems.

Theorem 1 *Suppose that*

$$
\sum_{n=n_0}^{\infty} \left| a(n) \right| < \infty \tag{4}
$$

and there exists a nonnegative real constant B such that

$$
\prod_{i=n_0}^{n-1} \left| b(i) \right| \leq B, \quad n \geq n_0. \tag{5}
$$

Assume also that there exists $\alpha \in (0,1)$ such that

$$
\sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} \left| b(j) \right| \sum_{i=n_0}^{k} \left| K(k,i) \right| \leq \alpha \quad \text{for } n \geq n_0. \tag{6}
$$

Then the solutions of equation (1) *are equi-bounded.*

Proof By (4), there exists a nonnegative real constant A such that $\sum_{n=n_0}^{\infty} |a(n)| = A$. Let $M_1 > 0$ be given. Choose a real number M_2 such that $\frac{(A+M_1)B}{1-\alpha} \leq M_2$. Let u_0 be a real number such that $|u_0| \leq M_1$. We define a subset S of BS by

$$S = \{x \in BS \colon x_0 = u_0 \text{ and } \|x\| \leq M_2\}.$$

Then S is a closed subset of BS. Now, we define the mapping $T \colon S \to BS$ as follows:

$$(Tx)(n) = u_0 \prod_{j=n_0}^{n-1} b(j) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j)\right) \sum_{i=n_0}^{k} K(k,i)x(i)$$

$$+ \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j)\right) a(k) \quad \text{for } n \geq n_0. \tag{7}$$

We will prove that T has a fixed point in S.

Firstly, we show that $T(S) \subset S$. Indeed, if $x \in S$, then by (7), (6), (4) and (5), we get

$$\bigl|(Tx)(n)\bigr| \leq M_1 \prod_{j=n_0}^{n-1} |b(j)|$$

$$+ M_2 \sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| + \prod_{j=n_0}^{n-1} |b(j)| \sum_{k=n_0}^{n-1} |a(k)|$$

$$\leq BM_1 + \alpha M_2 + BA \leq M_2 \quad \text{for } n \geq n_0.$$

Moreover, we have $(Tx)(n_0) = u_0$. Hence, $T(x) \in S$ and $T(S) \subset S$.

Next, we prove that T is a contraction. Let $y, z \in S$. Then, using (6) and (7), we get

$$\bigl|(Ty)(n) - (Tz)(n)\bigr| \leq \sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| \,|y(i) - z(i)|$$

$$\leq \sup_{n \geq n_0} |y(n) - z(n)| \left(\sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| \right)$$

$$\leq \alpha \sup_{n \geq n_0} |y(n) - z(n)| \quad \text{for } n \geq n_0.$$

Thus

$$\|Ty - Tz\| = \sup_{n \geq n_0} \bigl|(Ty)(n) - (Tz)(n)\bigr| \leq \alpha \|y - z\|.$$

Hence, T is a contraction. By the contraction mapping principle, T has a unique fixed point x^* in S. It means that $(Tx^*)(n) = x^*(n)$. So, from (7) we have

$$x^*(n) = u_0 \prod_{j=n_0}^{n-1} b(j) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j)\right) \sum_{i=n_0}^{k} K(k,i)x^*(i) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j)\right) a(k).$$

By comparing the above equation and (3), we see that the unique fixed point x^* is also a solution of equation (1). Hence, by the definition of the set S, the solutions of equation (1) are equi-bounded. This completes the proof. □

Theorem 1 extends Theorem 2.1 in [1] and Theorem 1 in [9].

The following example illustrates the result presented in Theorem 1.

Example 1 Let us consider the linear Volterra difference equation

$$x(n+1) = \frac{1}{n(n+1)} + x(n) + \sum_{i=1}^{n} \frac{i}{n^2(n+1)^2} x(i), \quad n \geq 1. \tag{8}$$

Here

$$a(n) = \frac{1}{n(n+1)}, \qquad b(n) \equiv 1 \quad \text{and} \quad K(n,i) = \frac{i}{n^2(n+1)^2}.$$

Hence $A = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$, $B = \prod_{j=1}^{\infty} b(j) = 1$,

$$\sum_{k=1}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=1}^{k} |K(k,i)| = \sum_{k=1}^{n-1} \frac{1}{k^2(k+1)^2} \sum_{i=1}^{k} i = \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{k(k+1)} \leq \frac{1}{2} = \alpha.$$

So, by Theorem 1 the solutions of (1) are equi-bounded. For example, for $M_1 = 1$ we have $M_2 \geq \frac{(A+M_1)B}{1-\alpha} = 4$. Hence, all solutions with the initial condition $|x(1)| \leq 1$ have the property $\|x\| \leq 4$.

Note that for equation (8), Theorem 1 in [28] cannot be applied since the assumption given below of this theorem

$$|b(n)| + |K(n,n)| + \varphi(0) \leq 1 - \alpha,$$

where $\alpha \in (0,1)$, $\varphi(n) \geq 0$, $n \geq 1$, is not satisfied.

In the next example we present a Volterra difference equation for which condition (24) from [2] does not hold. Hence, Theorem 5.1 [2] is not applicable for this example whereas our Theorem 1 is.

Example 2 Let us consider the linear Volterra difference equation

$$x(n+1) = \left\lfloor \frac{n+1}{2} \right\rfloor^{(-1)^n} x(n) + \sum_{i=1}^{n} \frac{1}{2^{k+i}} x(i), \quad n \geq 1. \tag{9}$$

Here

$$b(n) = \left\lfloor \frac{n+1}{2} \right\rfloor^{(-1)^n} = \left(1, 1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \dots \right) \quad \text{and} \quad K(n,i) = \frac{1}{2^{n+i}}.$$

It is easy to verify that conditions (5) and (6) are satisfied. So, by Theorem 1 the solutions of (9) are equi-bounded.

Remark 1 From Theorem 3.1 in [7] it follows that the boundedness of equation (2) is equivalent to the stability of its zero solution. Hence, and by Theorem 1, conditions (5) and (6) ensure the stability of the zero solution of equation (2).

Now, we provide conditions for the asymptotic stability of the zero solution of equation (2).

Theorem 2 *If condition* (6) *holds and*

$$\lim_{n \to \infty} \prod_{i=n_0}^{n-1} b(i) = 0, \tag{10}$$

then the zero solution of equation (2) *is asymptotically stable.*

Proof By (10) there exists a constant B_1 such that $|\prod_{i=n_0}^{n-1} b(i)| \le B_1$ for $n \ge n_0$. Let $M > 0$ be given. Choose $\mu > 0$ such that

$$\mu \le \frac{(1-\alpha)M}{B_1}. \tag{11}$$

Let u_0 be a real number such that $|u_0| \le \mu$. We define a subset S_1 of BS by

$$S_1 = \left\{ x \in BS : x_0 = u_0, \|x\| \le M \text{ and } \lim_{n \to \infty} x(n) = 0 \right\}.$$

Then S_1 is a closed subset of BS. We define the mapping T_1 for $x \in S_1$ as follows:

$$(T_1 x)(n) = u_0 \prod_{j=n_0}^{n-1} b(j) + \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) \sum_{i=n_0}^{k} K(k,i) x(i) \quad \text{for } n \ge n_0. \tag{12}$$

We show that $T_1(S_1) \subset S_1$. Indeed, if $x \in S_1$, then by (6) and (11) we have

$$\left| (T_1 x)(n) \right| \le \mu \prod_{j=n_0}^{n-1} |b(j)| + M \sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)|$$

$$\le \mu B_1 + M\alpha \le M \quad \text{for } n \ge n_0.$$

Moreover, from (10) we immediately get that the first term on the right-hand side of (12) tends to zero. We show that

$$\left| \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) \sum_{i=n_0}^{k} K(k,i) x(i) \right| \to 0 \quad \text{as } n \to \infty.$$

Let $\varepsilon > 0$. Since $\lim_{n \to \infty} x(n) = 0$, there exists $n_1 \ge n_0$ such that

$$\sup_{n \ge n_1} |x(n)| \le \frac{\varepsilon}{2\alpha}. \tag{13}$$

By (10) there exists $n_2 \geq n_1 \geq n_0$ such that

$$\prod_{j=n_2}^{n-1} |b(j)| \leq \frac{\varepsilon}{2\alpha M} \quad \text{for } n \geq n_2. \tag{14}$$

Then, for $n \geq n_2$, we have

$$\left| \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) \sum_{i=n_0}^{k} K(k,i)x(i) \right|$$

$$\leq \sum_{k=n_0}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| |x(i)|$$

$$\leq \sum_{k=n_0}^{n_2-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| |x(i)|$$

$$+ \sum_{k=n_2}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)| |x(i)|$$

$$\leq \sup_{n \geq n_0} |x(n)| \sum_{k=n_0}^{n_2-1} \prod_{j=k+1}^{n_2-1} |b(j)| \prod_{j=n_2}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)|$$

$$+ \sup_{n \geq n_2} |x(n)| \sum_{k=n_2}^{n-1} \prod_{j=k+1}^{n-1} |b(j)| \sum_{i=n_0}^{k} |K(k,i)|.$$

From the above, the definition of the set S_1, (6), (13) and (14), we obtain

$$\left| \sum_{k=n_0}^{n-1} \left(\prod_{j=k+1}^{n-1} b(j) \right) \sum_{i=n_0}^{k} K(k,i)x(i) \right| \leq M \prod_{j=n_2}^{n-1} |b(j)| \alpha + \frac{\varepsilon}{2\alpha} \alpha \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, the second term of formula (12) tends to zero, too. So, we have $\lim_{n\to\infty}(Tx)(n) = 0$. Hence, $T_1(x) \in S_1$ and $T_1(S_1) \subset S_1$.

Similarly as in the proof of Theorem 1, we get that T has a fixed point. This fixed point solves equation (1) and tends to zero. Hence, the zero solution of equation (2) is asymptotically stable. $\qquad\square$

Example 3 Let us consider the linear Volterra difference equation

$$x(n+1) = \frac{1}{n}x(n) + \sum_{i=1}^{n} \frac{1}{2n}x(i), \quad n \geq 1. \tag{15}$$

Here

$$b(n) = \frac{1}{n} \quad \text{and} \quad K(n,i) = \frac{1}{2n}.$$

It is easy to verify that conditions (5) and (6) are satisfied. So, by Theorem 1 the solutions of (15) are equi-bounded. Moreover, since condition (10) is also satisfied, by Theorem 2 the zero solution of equation (15) is asymptotically stable.

Note that the result about asymptotic stability of the zero solution obtained in the section Scalar Equation (see Example 3.1) of [4] is not applicable here because assumption vi) $\sum_{n=1}^{\infty} |K(n,i)| \leq C$, where C is a positive constant, for (15) is not satisfied. Similarly, Theorem 2.7 of [7] could not be applied for (15) since the kernel $K(n,i) = \frac{1}{2n}$ does not satisfy condition (2.28).

In Proposition 3.2 of [3], the necessary and sufficient conditions for boundedness of all solutions of equations of type (2) are given. It is easy to see that for (15), assumption (17) of [3] does not hold. But, as it was shown above, each solution of this equation is bounded. Corollary 3.7 of [3] is not applicable here, too.

3 Existence of periodic solutions

In this section we give necessary conditions for the existence of periodic solutions of equation (1) assuming the periodicity conditions

$$a(n + q) = a(n), \qquad b(n + q) = b(n), \quad n \in \mathbb{N}_0, \tag{16}$$

and

$$K(n + q, i) = K(n, i) = K(n, i + q), \quad n, i \in \mathbb{N}_0, \tag{17}$$

for some integer $q \geq 1$. If condition (17) is satisfied, then the sequence K is said to be doubly q-periodic sequence.

Denote $P_q = \{x \in BS : x(n + q) = x(n)\}$. Let us consider the system of linear equations

$$\begin{cases} a_{11}x(n_0) + a_{12}x(n_0 + 1) + \cdots + a_{1q}x(n_0 + q - 1) = 0, \\ a_{21}x(n_0) + a_{22}x(n_0 + 1) + \cdots + a_{2q}x(n_0 + q - 1) = 0, \\ \vdots \\ a_{q1}x(n_0) + a_{q2}x(n_0 + 1) + \cdots + a_{qq}x(n_0 + q - 1) = 0, \end{cases} \tag{18}$$

where

$$a_{ij} = K(n_0 + i - 1, n_0 + j - 1) \quad \text{for } i,j \in \{1, 2, \ldots, q\}$$

and $x(n_0), x(n_0 + 1), \ldots, x(n_0 + q - 1)$ are the unknowns. Denote the fundamental matrix of system (18) by A_q, i.e.,

$$A_q = \begin{pmatrix} K(n_0, n_0) & K(n_0, n_0 + 1) & \ldots & K(n_0, n_0 + q - 1) \\ K(n_0 + 1, n_0) & K(n_0 + 1, n_0 + 1) & \ldots & K(n_0 + 1, n_0 + q - 1) \\ \vdots & \vdots & \ddots & \vdots \\ K(n_0 + q - 2, n_0) & K(n_0 + q - 2, n_0 + 1) & \ldots & K(n_0 + q - 2, n_0 + q - 1) \\ K(n_0 + q - 1, n_0) & K(n_0 + q - 1, n_0 + 1) & \ldots & K(n_0 + q - 1, n_0 + q - 1) \end{pmatrix}. \tag{19}$$

In the next theorem we give necessary conditions for the existence of a periodic solution of equation (1).

Theorem 3 *Let conditions* (16) *and* (17) *hold, and let A_q be the matrix defined by* (19). *If there exists a q-periodic solution of equation* (1), *then* $\det A_q = 0$.

Proof Let $x \in P_q$ be a nontrivial solution of equation (1). Then, by (1), we have

$$x(n + 1) = x(n + q + 1) = a(n + q) + b(n + q)x(n + q) + \sum_{i=n_0}^{n+q} K(n + q, i)x(i).$$

Hence, using (16) and (17), we get

$$x(n + 1) = a(n) + b(n)x(n) + \sum_{i=n_0}^{n} K(n, i)x(i) + \sum_{i=n+1}^{n+q} K(n, i)x(i)$$

$$= x(n + 1) + \sum_{i=n+1}^{n+q} K(n, i)x(i).$$

Thus, by the q-periodicity of x and K, we get

$$\sum_{i=r+1}^{r+q} K(r, i)x(i) = 0, \quad \text{for } r = n, n + 1, \ldots, n + q - 1. \tag{20}$$

This means that the above system has a nontrivial solution for any $n \geq n_0$. By the periodicity of the sequences x and K, system (20) is equivalent to system (18). Hence, system (18) has a nontrivial solution, too. Thus $\det A_q = 0$. This completes the proof. □

Remark 2 From the proof of Theorem 3 it follows that if x is a q-periodic solution of (1), then x has the following form

$$x(n) = (c_1, c_2, \ldots, c_q, c_1, c_2, \ldots, c_q, \ldots),$$

where $(c_1, c_2, \ldots, c_q)$ is a certain solution of system (18).

Example 4 Let us consider the linear Volterra difference equation of the form (1), where sequences a, b, K are 3-periodic, $n_0 = 1$ and

$$a(3k + 1) = 1, \qquad a(3k + 2) = -3, \qquad a(3k + 3) = 2, \quad k = 0, 1, \ldots,$$

$$b(3k + 1) = -2, \qquad b(3k + 2) = 2, \qquad b(3k + 2) = 1, \quad k = 0, 1, \ldots,$$

and

$$A_3 = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$$

The considered equation has a 3-periodic solution x of the form

$$x(3k + 1) = 1, \qquad x(3k + 2) = 0, \qquad x(3k + 3) = -1.$$

It is easy to check that $\det A_3 = 0$ and the triple $(1, 0, -1)$ is a solution of system (18).

The next example shows that the condition $\det A_q = 0$ is not sufficient for the existence of a q-periodic solution of (1).

Example 5 Let us consider the linear Volterra difference equation of the form (1), where sequences a, b, K are 2-periodic, $n_0 = 1$ and

$$a(2k + 1) = -1, \qquad a(2k + 2) = 1, \quad k = 0, 1, \ldots,$$

$$b(2k + 1) = 2, \qquad b(2k + 2) = 1, \quad k = 0, 1, \ldots,$$

$$A_2 = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

Then $\det A_2 = 0$, but it is easy to check that the considered equation does not have a 2-periodic solution.

For some periodic difference equations and systems, see also, for example, [29, 30] or [31].

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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