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On Poly-Bernoulli polynomials of the second kind with umbral calculus viewpoint

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Abstract

Poly-Bernoulli polynomials of the second kind were introduced in Kim *et al.* (Adv. Differ. Equ. 2014:219, 2014) as a generalization of the Bernoulli polynomial of the second kind. Here we investigate those polynomials and derive further results about them by using umbral calculus.

Keywords: Bernoulli polynomials; poly-Bernoulli polynomials; Stirling numbers; umbral calculus

1 Introduction

Following Kaneko [1], the poly-Bernoulli polynomials have been studied by many researchers in recent decades. Poly-Bernoulli polynomials $B_n^{(k)}(x)$ were defined as $\frac{Li_k(1-e^{-t})}{1-e^{-t}}e^{xt} = \sum_{n\geq 0} B_n^{(k)}(x)\frac{t^n}{n!}$, where $Li_k(x) = \sum_{r\geq 1}\frac{x^r}{r^k}$ is the classical polylogarithm function, which satisfies $\frac{d}{dx}Li_k(x) = \frac{1}{x}Li_{k-1}(x)$. The poly-Bernoulli polynomials have wide-ranging applications in mathematics and applied mathematics (see [2–4]). For $k \in \mathbb{Z}$, the poly-Bernoulli polynomials $b_n^{(k)}(x)$ of the second kind are given by the generating function

$$\frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^x = \sum_{n\geq 0} b_n^{(k)}(x)\frac{t^n}{n!}. \tag{1.1}$$

When $x = 0$, $b_n^{(k)} = b_n^{(k)}(0)$ are called the poly-Bernoulli numbers of the second kind. When $k = 1$, $b_n(x) = b_n^{(1)}(x)$ are called the Bernoulli polynomial of the second kind (see [5–11]). Poly-Bernoulli polynomials of the second kind were introduced as a generalization of the Bernoulli polynomial of the second kind (see [12]). The aim of this paper is to use umbral calculus to obtain several new and interesting explicit formulas, recurrence relations and identities of poly-Bernoulli polynomials of the second kind. Umbral calculus has been used in numerous problems of mathematics. Umbral techniques have been of use in different areas of physics; for example it is used in group theory and quantum mechanics by Biedenharn *et al.* (see [13–15]).

Let Π be the algebra of polynomials in a single variable x over $\mathbb{C}$ and let Π^* be the vector space of all linear functionals on Π. We denote the action of a linear functional L on a polynomial $p(x)$ by $\langle L|p(x)\rangle$. Define the vector space structure on Π^* by $\langle cL + c'L'|p(x)\rangle = c\langle L|p(x)\rangle + c'\langle L'|p(x)\rangle$, where $c, c' \in \mathbb{C}$ (see [16–19]). We define the algebra of a formal power series in a single variable t to be

$$\mathcal{H} = \left\{ f(t) = \sum_{k\geq 0} a_k \frac{t^k}{k!} \middle| a_k \in \mathbb{C} \right\}. \tag{1.2}$$

The formal power series in the variable t defines a linear functional on Π by setting $\langle f(t)|x^n\rangle = a_n$, for all $n \geq 0$ (see [16–19]). Thus

$$\langle t^k|x^n\rangle = n!\delta_{n,k}, \quad \text{for all } n,k \geq 0 \text{ (see [16–19])}, \tag{1.3}$$

where $\delta_{n,k}$ is the Kronecker symbol. Let $f_L(t) = \sum_{n\geq 0}\langle L|x^n\rangle \frac{t^n}{n!}$. By (1.3), we have $\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle$. Thus, the map $L \mapsto f_L(t)$ is a vector space isomorphism from Π^* onto $\mathcal{H}$. Therefore, $\mathcal{H}$ is thought of as a set of both formal power series and linear functionals. We call $\mathcal{H}$ the *umbral algebra*. The *umbral calculus* is the study of the umbral algebra.

Let $f(t)$ be a non-zero power series, the *order* $O(f(t))$ is the smallest integer k for which the coefficient of t^k does not vanish. If $O(f(t)) = 1$ (respectively, $O(f(t)) = 0$), then $f(t)$ is called a *delta* (respectively, an *invertable*) series. Suppose that $f(t)$ is a delta series and $g(t)$ is an invertable series, then there exists a unique sequence $s_n(x)$ of polynomials such that $\langle g(t)(f(t))^k|s_n(x)\rangle = n!\delta_{n,k}$, where $n,k \geq 0$. The sequence $s_n(x)$ is called the *Sheffer* sequence for $(g(t),f(t))$ which is denoted by $s_n(x) \sim (g(t),f(t))$ (see [18, 19]). For $f(t) \in \mathcal{H}$ and $p(x) \in \Pi$, we have $\langle e^{yt}|p(x)\rangle = p(y)$, $\langle f(t)g(t)|p(x)\rangle = \langle g(t)|f(t)p(x)\rangle$, and $f(t) = \sum_{n\geq 0}\langle f(t)|x^n\rangle \frac{t^n}{n!}$ and $p(x) = \sum_{n\geq 0}\langle t^n|p(x)\rangle \frac{x^n}{n!}$ (see [18, 19]). Thus, we obtain $\langle t^k|p(x)\rangle = p^{(k)}(0)$ and $\langle 1|p^{(k)}(x)\rangle = p^{(k)}(0)$, where $p^{(k)}(0)$ denotes the kth derivative of $p(x)$ with respect to x at $x = 0$. Therefore, we get $t^k p(x) = p^{(k)}(x) = \frac{d^k}{dx^k}p(x)$, for all $k \geq 0$ (see [18, 19]). Thus, for $s_n(x) \sim (g(t),f(t))$, we have

$$\frac{1}{g(\bar{f}(t))} e^{y\bar{f}(t)} = \sum_{n\geq 0} s_n(y)\frac{t^n}{n!}, \tag{1.4}$$

for all $y \in \mathbb{C}$, where $\bar{f}(t)$ is the compositional inverse of $f(t)$ (see [18, 19]). For $s_n(x) \sim (g(t),f(t))$ and $r_n(x) \sim (h(t),\ell(t))$, let $s_n(x) = \sum_{k=0}^{n} c_{n,k} r_k(x)$, then we have

$$c_{n,k} = \frac{1}{k!}\left\langle \frac{h(\bar{f}(t))}{g(\bar{f}(t))}\left(\ell(\bar{f}(t))\right)^k \middle| x^n \right\rangle \tag{1.5}$$

(see [18, 19]).

It is immediate from (1.1) and (1.4) to see that $b_n^{(k)}(x)$ is the Sheffer polynomial for the pair $g(x) = \frac{t}{Li_k(1-e^{1-e^t})}$ and $f(t) = e^t - 1$, that is,

$$b_n^{(k)}(x) \sim \left(\frac{t}{Li_k(1-e^{1-e^t})}, e^t - 1\right). \tag{1.6}$$

The aim of the present paper is to present several new identities for the poly-Bernoulli polynomials by the use of umbral calculus.

2 Explicit expressions

Before proceeding, we observe that

$$\begin{aligned}
Li_k\left(1-e^{-t}\right) &= \sum_{r\geq 1}\frac{1}{r^k}\left(1-e^{-t}\right)^r = \sum_{r\geq 1}\frac{(-1)^r}{r^k}\left(e^{-t}-1\right)^r \\
&= \sum_{r\geq 1}\frac{(-1)^r r!}{r^k}\sum_{\ell\geq r}S_2(\ell,r)\frac{(-t)^\ell}{\ell!} = \sum_{r\geq 1}\sum_{\ell\geq r}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)\frac{t^\ell}{\ell!} \\
&= \sum_{\ell\geq 1}\sum_{r=1}^{\ell}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)\frac{t^\ell}{\ell!},
\end{aligned}\tag{2.1}$$

where $S_2(n,k)$ is the Stirling number of the second kind, which is defined by the identity $x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k$ with $(x)_0 = 1$ and $(x)_k = x(x-1)\cdots(x-k+1)$. This shows

$$\frac{1}{t}Li_k\left(1-e^{-t}\right) = \sum_{\ell\geq 0}\sum_{r=1}^{\ell+1}\frac{(-1)^{r+\ell+1}r!}{r^k}\frac{S_2(\ell+1,r)}{\ell+1}\frac{t^\ell}{\ell!}.\tag{2.2}$$

Thus,

$$\begin{aligned}
Li_k\left(1-e^{1-e^t}\right) &= \sum_{r\geq 1}\sum_{\ell\geq r}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)\frac{(e^t-1)^\ell}{\ell!} \\
&= \sum_{r\geq 1}\sum_{\ell\geq r}\sum_{m\geq \ell}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)S_2(m,\ell)\frac{t^m}{m!} \\
&= \sum_{m\geq 1}\sum_{r=1}^{m}\sum_{\ell=r}^{m}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)S_2(m,\ell)\frac{t^m}{m!},
\end{aligned}$$

which implies that

$$\frac{1}{t}Li_k\left(1-e^{1-e^t}\right) = \sum_{m\geq 0}\sum_{r=1}^{m+1}\sum_{\ell=r}^{m+1}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)S_2(m+1,\ell)\frac{t^m}{(m+1)!}.\tag{2.3}$$

Now, we are ready to present several formulas for the nth poly-Bernoulli polynomials of the second kind.

Theorem 2.1 *For all* $n\geq 1$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{r=1}^{j-m+1}\sum_{\ell=r}^{j-m+1}\frac{(-1)^{r+\ell}}{j-m+1}\frac{r!}{r^k}\binom{n-1}{j-1}\binom{j}{m}S_2(\ell,r)S_2(j-m+1,\ell)B_{n-j}^{(n)}x^m.$$

Proof Since $x^n \sim (1,t)$ and $\frac{t}{Li_k(1-e^{1-e^t})}b_n^{(k)}(x) \sim (1,e^t-1)$ (see (1.6)), we obtain

$$\begin{aligned}
\frac{t}{Li_k(1-e^{1-e^t})}b_n^{(k)}(x) &= x\left(\frac{t}{e^t-1}\right)^n x^{n-1} = x\left(\sum_{j\geq 0}B_j^{(n)}\frac{t^j}{j!}\right)x^{n-1} \\
&= x\sum_{j=0}^{n-1}\binom{n-1}{j}B_j^{(n)}x^{n-1-j} = \sum_{j=1}^{n}\binom{n-1}{j-1}B_{n-j}^{(n)}x^j.
\end{aligned}$$

Thus, by (2.3) we have

$$\begin{aligned}
b_n^{(k)}(x) &= \sum_{j=1}^{n}\binom{n-1}{j-1}B_{n-j}^{(n)}\frac{Li_k(1-e^{1-e^t})}{t}x^j \\
&= \sum_{j=1}^{n}\binom{n-1}{j-1}B_{n-j}^{(n)}\left(\sum_{m=0}^{j}\sum_{r=1}^{m+1}\sum_{\ell=r}^{m+1}\frac{(-1)^{r+\ell}r!}{r^k}S_2(\ell,r)S_2(m+1,\ell)\frac{t^m}{(m+1)!}\right)x^j \\
&= \sum_{j=1}^{n}\sum_{m=0}^{j}\sum_{r=1}^{m+1}\sum_{\ell=r}^{m+1}\frac{(-1)^{r+\ell}}{m+1}\frac{r!}{r^k}\binom{n-1}{j-1}\binom{j}{m}S_2(\ell,r)S_2(m+1,\ell)B_{n-j}^{(n)}x^{j-m} \\
&= \sum_{j=1}^{n}\sum_{m=0}^{j}\sum_{r=1}^{j-m+1}\sum_{\ell=r}^{j-m+1}\frac{(-1)^{r+\ell}}{j-m+1}\frac{r!}{r^k}\binom{n-1}{j-1}\binom{j}{m}S_2(\ell,r)S_2(j-m+1,\ell)B_{n-j}^{(n)}x^m,
\end{aligned}$$

which completes the proof. □

Let $S_1(n,k)$ be the Stirling number of the first kind, which is defined by the identity $(x)_n = \sum_{j=0}^{n} S_1(n,k)x^k$. Now, we are ready to present our second explicit formula.

Theorem 2.2 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n}\sum_{j=m}^{n}\sum_{r=1}^{j-m+1}\sum_{\ell=r}^{j-m+1}\frac{(-1)^{r+\ell}}{j-m+1}\frac{r!}{r^k}\binom{j}{m}S_1(n,j)S_2(\ell,r)S_2(j-m+1,\ell)x^m.$$

Proof Note that $(x)_n = \sum_{j=0}^{n} S_1(n,j)x^j \sim (1, e^t - 1)$. So, by (1.6) we have $\frac{t}{Li_k(1-e^{1-e^t})}b_n^{(k)}(x) \sim (1, e^t - 1)$, which implies that

$$b_n^{(k)}(x) = \sum_{j=0}^{n} S_1(n,j)\frac{Li_k(1-e^{1-e^t})}{t}x^j. \tag{2.4}$$

Thus, by (2.3) and using the arguments in the proof of Theorem 2.1, we obtain the required formula. □

For the next explicit formula, we use the conjugation representation, namely (1.5).

Theorem 2.3 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = b_n^{(k)} + \sum_{j=1}^{n}\frac{1}{j}\left(\sum_{m=j-1}^{n-1}\sum_{r=1}^{n-m}\frac{(-1)^{r+n-m}r!}{r^k}\binom{n}{m}S_1(m,j-1)S_2(n-m,r)\right)x^j.$$

Proof By (1.5) and (1.6), we have $b_n^{(k)}(x) = \sum_{j=0}^{n} c_{n,j}x^j$, where

$$j!c_{n,j} = \langle \left(g(\bar{f}(t))\right)^{-1}\bar{f}^j(t)|x^n\rangle = \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)}\left(\log(1+t)\right)^j \middle| x^n\right\rangle.$$

If $j = 0$, then $c_{n,0} = b_n^{(k)}$. Thus, assume now that $1 \leq j \leq n$. So

$$\begin{aligned} j!c_{n,j} &= \langle Li_k(1-e^{-t})(\log(1+t))^{j-1} | x^n \rangle \\ &= \left\langle Li_k(1-e^{-t}) \Big| (j-1)! \sum_{m \geq j-1} S_1(m,j-1)\frac{t^m}{m!}x^n \right\rangle \\ &= (j-1)! \sum_{m=j-1}^{n} \binom{n}{m} S_1(m,j-1) \langle Li_k(1-e^{-t}) | x^{n-m} \rangle, \end{aligned}$$

which, by (2.1), implies that

$$\begin{aligned} j!c_{n,j} &= (j-1)! \sum_{m=j-1}^{n} \binom{n}{m} S_1(m,j-1) \left\langle \sum_{\ell \geq 1} \sum_{r=1}^{\ell} \frac{(-1)^{r+\ell} r!}{r^k} S_2(\ell,r)\frac{t^\ell}{\ell!} \Big| x^{n-m} \right\rangle \\ &= (j-1)! \sum_{m=j-1}^{n} \binom{n}{m} S_1(m,j-1) \left(\sum_{r=1}^{n-m} \frac{(-1)^{r+n-m} r!}{r^k} S_2(n-m,r) \right), \end{aligned}$$

which completes the proof. □

In order to state our next formula, we recall that $b_n(x) = b_n^{(1)}(x)$ is the Bernoulli polynomial of the second kind, which is given by the generating function $\frac{t}{\log(1+t)}(1+t)^x = \sum_{n \geq 0} b_n(x)\frac{t^n}{n!}$.

Theorem 2.4 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \left(B_{n+1-j}^{(k)} - B_{n+1-j}^{(k)}(-1) \right) b_\ell(x),$$

where $B_n^{(k)}(x)$ *is the nth poly-Bernoulli polynomial.*

Proof From the definitions, we have

$$\begin{aligned} b_n^{(k)}(y) &= \left\langle \sum_{\ell \geq 0} b_\ell^{(k)}(y)\frac{t^\ell}{\ell!} \Big| x^n \right\rangle = \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^y \Big| x^n \right\rangle \\ &= \left\langle \frac{e^{-t}-1}{-t} \frac{Li_k(1-e^{-t})}{1-e^{-t}} \frac{t}{\log(1+t)}(1+t)^y \Big| x^n \right\rangle \\ &= \left\langle \frac{e^{-t}-1}{-t} \frac{Li_k(1-e^{-t})}{1-e^{-t}} \Big| \frac{t}{\log(1+t)}(1+t)^y x^n \right\rangle \\ &= \left\langle \frac{e^{-t}-1}{-t} \frac{Li_k(1-e^{-t})}{1-e^{-t}} \Big| \sum_{\ell \geq 0} b_\ell(y)\frac{t^\ell}{\ell!}x^n \right\rangle. \end{aligned}$$

Since $B_n^{(k)}(x)$ is the poly-Bernoulli polynomial given by the generating function $\frac{Li_k(1-e^{-t})}{1-e^{-t}} \times e^{xt} = \sum_{n \geq 0} B_n^{(k)}(x)\frac{t^n}{n!}$, we have $\frac{Li_k(1-e^{-t})}{1-e^{-t}}x^n = B_n^{(k)}(x)$ and $\frac{d}{dx}B_n^{(k)}(x) = nB_{n-1}^{(k)}(x)$. Thus $b_n^{(k)}(y) = \sum_{\ell=0}^{n} \binom{n}{\ell} b_\ell(y) \langle \frac{e^{-t}-1}{-t} | B_{n-\ell}^{(k)}(x) \rangle$. By the fact that $\langle f(at)|p(x)\rangle = \langle f(t)|p(ax)\rangle$ for constant a (see

Proposition 2.1.11 in [19]), we obtain

$$b_n^{(k)}(y) = \sum_{\ell=0}^{n} \binom{n}{\ell} b_\ell(y) \left\langle \frac{e^t - 1}{t} \middle| B_{n-\ell}^{(k)}(-x) \right\rangle.$$

Note that $\langle \frac{e^t-1}{t} | B_{n-\ell}^{(k)}(-x) \rangle = \int_0^1 B_{n-\ell}^{(k)}(-u)\,du = \frac{1}{n+1-\ell}(B_{n+1-\ell}^{(k)} - B_{n+1-\ell}^{(k)}(-1))$, which leads to

$$\begin{aligned} b_n^{(k)}(y) &= \sum_{\ell=0}^{n} \binom{n}{\ell} b_\ell(y) \frac{1}{n+1-\ell} \big(B_{n+1-\ell}^{(k)} - B_{n+1-\ell}^{(k)}(-1)\big) \\ &= \frac{1}{n+1} \sum_{j=0}^{n} \binom{n+1}{j} \big(B_{n+1-j}^{(k)} - B_{n+1-j}^{(k)}(-1)\big) b_\ell(y), \end{aligned}$$

which completes the proof. □

Theorem 2.5 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} \left[\sum_{r=1}^{m+1} (-1)^{r+m+1} \frac{r! S_2(m+1, r)}{r^k (m+1)} \right] b_{n-m}(x).$$

Proof By using a similar argument as in the proof of Theorem 2.4, we obtain

$$\begin{aligned} b_n^{(k)}(y) &= \left\langle \frac{Li_k(1-e^{-t})}{t} \frac{t}{\log(1+t)} (1+t)^y \middle| x^n \right\rangle = \left\langle \frac{Li_k(1-e^{-t})}{t} \middle| \sum_{m \geq 0} b_m(y) \frac{t^m}{m!} x^n \right\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} b_m(y) \left\langle \frac{Li_k(1-e^{-t})}{t} \middle| x^{n-m} \right\rangle, \end{aligned}$$

which, by (2.2), gives

$$\begin{aligned} b_n^{(k)}(y) &= \sum_{m=0}^{n} \binom{n}{m} b_m(y) \left\langle \sum_{\ell \geq 0} \sum_{r=1}^{\ell+1} \frac{(-1)^{r+\ell+1} r!}{r^k} \frac{S_2(\ell+1, r)}{\ell+1} \frac{t^\ell}{\ell!} \middle| x^{n-m} \right\rangle \\ &= \sum_{m=0}^{n} \binom{n}{m} b_m(y) \left(\sum_{r=1}^{n-m+1} \frac{(-1)^{r+n-m+1} r!}{r^k} \frac{S_2(n-m+1, r)}{n-m+1} \right) \\ &= \sum_{m=0}^{n} \binom{n}{m} \left[\sum_{r=1}^{m+1} (-1)^{r+m+1} \frac{r! S_2(m+1, r)}{r^k (m+1)} \right] b_{n-m}(y), \end{aligned}$$

as required. □

Note that the statement of Theorem 2.5 has been obtained in Theorem 2.2 of [12].

3 Recurrence relations

By (1.6) we have $b_n^{(k)}(x) \sim (\frac{t}{Li_k(1-e^{1-e^t})}, e^t - 1)$ with $P_n(x) = \frac{t}{Li_k(1-e^{1-e^t})} b_n^{(k)}(x) = (x)_n = x(x-1)\cdots(x-n+1) \sim (1, e^t - 1)$. Thus,

$$b_n^{(k)}(x+y) = \sum_{j=0}^{n} \binom{n}{j} b_j^{(k)}(x) (y)_{n-j}.$$

The aim of this section is to derive recurrence relations for the poly-Bernoulli polynomials of the second kind. As first trivial recurrence, by using the fact that if $S_n(x) \sim (g(t), f(t))$ then $f(t)S_n(x) = nS_{n-1}(x)$, we derive that $(e^t - 1)b_n^{(k)}(x) = nb_{n-1}^{(k)}(x)$, and hence $b_n^{(k)}(x+1) = b_n^{(k)}(x) + nb_{n-1}^{(k)}(x)$. Our next results establish other types of recurrence relations.

Theorem 3.1 *For all* $n \geq 0$,

$$\begin{aligned} b_{n+1}^{(k)}(x) &= xb_n^{(k)}(x-1) \\ &\quad + \sum_{j=0}^{n} \sum_{\ell=0}^{j+1} \sum_{m=0}^{j+1-\ell} \frac{1}{m} \binom{j}{m-1} S_1(n,j) S_2(j+1-m,\ell) \big(B_\ell^{(k-1)}(-1)x^m - b_\ell^{(k)}(x-1)^m \big). \end{aligned}$$

Proof It is well known that if $S_n(x) \sim (g(t), f(t))$ then $S_{n+1}(x) = (x - \frac{g'(t)}{g(t)}) \frac{1}{f'(t)} S_n(x)$. Hence, by (1.6), we have

$$b_{n+1}^{(k)}(x) = xb_n^{(k)}(x-1) - e^{-t} \frac{g'(t)}{g(t)} b_n^{(k)}(x)$$

with

$$\frac{g'(t)}{g(t)} = \big(\log\big(g(t)\big)\big)' = \big(\log t - \log Li_k\big(1 - e^{1-e^t}\big)\big)' = \frac{1}{t} \left(1 - \frac{te^t e^{1-e^t} Li_{k-1}(1 - e^{1-e^t})}{(1 - e^{1-e^t}) Li_k(1 - e^{1-e^t})} \right),$$

where $1 - \frac{te^t e^{1-e^t} Li_{k-1}(1-e^{1-e^t})}{(1-e^{1-e^t}) Li_k(1-e^{1-e^t})}$ has order at least one. Thus, by (2.4), we get

$$\begin{aligned} -e^{-t} \frac{g'(t)}{g(t)} b_n^{(k)}(x) &= \frac{-e^{-t}}{t} \left(1 - \frac{te^t e^{1-e^t} Li_{k-1}(1 - e^{1-e^t})}{(1 - e^{1-e^t}) Li_k(1 - e^{1-e^t})} \right) \sum_{j=0}^{n} S_1(n,j) \frac{Li_k(1 - e^{1-e^t})}{t} x^j \\ &= -\sum_{j=0}^{n} \frac{S_1(n,j)}{j+1} \left(\frac{e^{-t} Li_k(1 - e^{1-e^t})}{\log(1 + e^t - 1)} - \frac{e^{1-e^t} Li_{k-1}(1 - e^{1-e^t})}{(1 - e^{1-e^t})} \right) x^{j+1} \\ &= -\sum_{j=0}^{n} \frac{S_1(n,j)}{j+1} \left(e^{-t} \sum_{\ell \geq 0} b_\ell^{(k)} \frac{(e^t - 1)^\ell}{\ell!} - \sum_{\ell \geq 0} B_\ell^{(k-1)}(-1) \frac{(e^t - 1)^\ell}{\ell!} \right) x^{j+1}, \end{aligned}$$

where

$$\begin{aligned} & e^{-t} \sum_{\ell \geq 0} b_\ell^{(k)} \frac{(e^t - 1)^\ell}{\ell!} x^{j+1} \\ &= e^{-t} \sum_{\ell=0}^{j+1} b_\ell^{(k)} \sum_{m=\ell}^{j+1} S_2(m,\ell) \frac{t^m}{m!} x^{j+1} = e^{-t} \sum_{\ell=0}^{j+1} \sum_{m=\ell}^{j+1} \binom{j+1}{m} b_\ell^{(k)} S_2(m,\ell) x^{j+1-m} \\ &= e^{-t} \sum_{\ell=0}^{j+1} \sum_{m=0}^{j+1-\ell} \binom{j+1}{m} b_\ell^{(k)} S_2(j+1-m,\ell) x^m \\ &= \sum_{\ell=0}^{j+1} \sum_{m=0}^{j+1-\ell} \binom{j+1}{m} b_\ell^{(k)} S_2(j+1-m,\ell)(x-1)^m \end{aligned}$$

and

$$\begin{aligned}\sum_{\ell\geq 0} B_\ell^{(k-1)}(-1)\frac{(e^t-1)^\ell}{\ell!}x^{j+1} &= \sum_{\ell=0}^{j+1} B_\ell^{(k-1)}(-1)\sum_{m=\ell}^{j+1} S_2(m,\ell)\frac{t^m}{m!}x^{j+1}\\ &= \sum_{\ell=0}^{j+1}\sum_{m=\ell}^{j+1}\binom{j+1}{m}S_2(m,\ell)B_\ell^{(k-1)}(-1)x^{j+1-m}\\ &= \sum_{\ell=0}^{j+1}\sum_{m=0}^{j+1-\ell}\binom{j+1}{m}S_2(j+1-m,\ell)B_\ell^{(k-1)}(-1)x^m.\end{aligned}$$

Thus,

$$\begin{aligned} b_{n+1}^{(k)}(x) &= xb_n^{(k)}(x-1)\\ &\quad+\sum_{j=0}^{n}\frac{S_1(n,j)}{j+1}\sum_{\ell=0}^{j+1}\sum_{m=0}^{j+1-\ell}\binom{j+1}{m}S_2(j+1-m,\ell)\bigl(B_\ell^{(k-1)}(-1)x^m-b_\ell^{(k)}(x-1)^m\bigr)\\ &= xb_n^{(k)}(x-1)\\ &\quad+\sum_{j=0}^{n}\sum_{\ell=0}^{j+1}\sum_{m=0}^{j+1-\ell}\frac{1}{m}\binom{j}{m-1}S_1(n,j)S_2(j+1-m,\ell)\bigl(B_\ell^{(k-1)}(-1)x^m-b_\ell^{(k)}(x-1)^m\bigr),\end{aligned}$$

which completes the proof. □

Theorem 3.2 *For all* $n\geq 0$, $\frac{d}{dx}b_n^{(k)}(x)=n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-1-\ell}}{\ell!(n-\ell)}b_\ell^{(k)}(x)$.

Proof We proceed in the proof by using the fact that if $S_n(x)\sim(g(t),f(t))$ then

$$\frac{d}{dx}S_n(x)=\sum_{\ell=0}^{n-1}\binom{n}{\ell}\langle\bar{f}(t)|x^{n-\ell}\rangle S_\ell(x).$$

By (1.6), we have $\langle\bar{f}(t)|x^{n-\ell}\rangle=\langle\log(1+t)|x^{n-\ell}\rangle$, which leads to

$$\langle\bar{f}(t)|x^{n-\ell}\rangle=\left\langle\sum_{m\geq 1}(-1)^{m-1}(m-1)!\frac{t^m}{m!}\middle|x^{n-\ell}\right\rangle=(-1)^{n-1-\ell}(n-1-\ell)!.$$

Thus $\frac{d}{dx}b_n^{(k)}(x)=n!\sum_{\ell=0}^{n-1}\frac{(-1)^{n-1-\ell}}{\ell!(n-\ell)}b_\ell^{(k)}(x)$, as required. □

Theorem 3.3 *For all* $n\geq 1$,

$$b_n^{(k)}(x)=xb_{n-1}^{(k)}(x-1)+\frac{1}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\bigl(B_\ell^{(k-1)}(-1)b_{n-\ell}(x)-b_\ell^{(k)}b_{n-\ell}(x-1)\bigr).$$

Proof Let $n\geq 1$. Then (1.6), we have

$$\begin{aligned} b_n^{(k)}(y) &= \left\langle\frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^y\middle|x^n\right\rangle=\left\langle\frac{d}{dt}\left[\frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^y\right]\middle|x^{n-1}\right\rangle\\ &= \left\langle\frac{Li_k(1-e^{-t})}{\log(1+t)}\frac{d}{dt}\bigl[(1+t)^y\bigr]\middle|x^{n-1}\right\rangle+\left\langle\frac{d}{dt}\left[\frac{Li_k(1-e^{-t})}{\log(1+t)}\right](1+t)^y\middle|x^{n-1}\right\rangle. \end{aligned}\tag{3.1}$$

The first term in (3.1) is given by

$$\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \frac{d}{dt}\left[(1+t)^y\right] \middle| x^{n-1} \right\rangle = y\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)}(1+t)^{y-1} \middle| x^{n-1} \right\rangle = yb_{n-1}^{(k)}(y-1). \tag{3.2}$$

For the second term in (3.1), we note that

$$\begin{aligned} \frac{d}{dt}\left[\frac{Li_k(1-e^{-t})}{\log(1+t)}\right](1+t)^y &= \frac{1}{t}\frac{t}{\log(1+t)}\left(\frac{Li_{k-1}(1-e^{-t})}{1-e^{-t}}e^{-t} - \frac{Li_k(1-e^{-t})}{\log(1+t)}\frac{1}{1+t}\right)(1+t)^y \\ &= \frac{1}{t}\left(\frac{t(1+t)^y}{\log(1+t)}\frac{Li_{k-1}(1-e^{-t})}{1-e^{-t}}e^{-t} - \frac{t(1+t)^{y-1}}{\log(1+t)}\frac{Li_k(1-e^{-t})}{\log(1+t)}\right), \end{aligned}$$

which has order at least zero. So, the second term in (3.1) is given by

$$\begin{aligned} &\left\langle \frac{d}{dt}\left[\frac{Li_k(1-e^{-t})}{\log(1+t)}\right](1+t)^y \middle| x^{n-1} \right\rangle \\ &\quad = \frac{1}{n}\left(\left\langle \frac{t}{\log(1+t)}(1+t)^y \middle| \frac{Li_{k-1}(1-e^{-t})}{1-e^{-t}}e^{-t}x^n \right\rangle\right. \\ &\quad\quad \left. - \left\langle \frac{t}{\log(1+t)}(1+t)^{y-1} \middle| \frac{Li_k(1-e^{-t})}{\log(1+t)}x^n \right\rangle\right) \\ &\quad = \frac{1}{n}\left(\left\langle \frac{t}{\log(1+t)}(1+t)^y \middle| \sum_{\ell\geq 0} B_\ell^{(k-1)}(-1)\frac{t^\ell}{\ell!}x^n \right\rangle\right. \\ &\quad\quad \left. - \left\langle \frac{t}{\log(1+t)}(1+t)^{y-1} \middle| \sum_{\ell\geq 0} b_\ell^{(k)}\frac{t^\ell}{\ell!}x^n \right\rangle\right) \\ &\quad = \frac{1}{n}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}B_\ell^{(k-1)}(-1)\left\langle \frac{t}{\log(1+t)}(1+t)^y \middle| x^{n-\ell} \right\rangle\right. \\ &\quad\quad \left. - \sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(k)}\left\langle \frac{t}{\log(1+t)}(1+t)^{y-1} \middle| x^{n-\ell} \right\rangle\right) \\ &\quad = \frac{1}{n}\left(\sum_{\ell=0}^{n}\binom{n}{\ell}B_\ell^{(k-1)}(-1)b_{n-\ell}(y) - \sum_{\ell=0}^{n}\binom{n}{\ell}b_\ell^{(k)}b_{n-\ell}(y-1)\right) \\ &\quad = \frac{1}{n}\sum_{\ell=0}^{n}\binom{n}{\ell}\left(B_\ell^{(k-1)}(-1)b_{n-\ell}(y) - b_\ell^{(k)}b_{n-\ell}(y-1)\right). \end{aligned} \tag{3.3}$$

By substituting (3.2) and (3.3) into (3.1), we complete the proof. □

4 Identities

In this section we present some identities related to poly-Bernoulli numbers of the second kind.

Theorem 4.1 *For all* $n \geq 0$,

$$\sum_{\ell=0}^{n}(-1)^{n-\ell}(n-\ell)!\binom{n+1}{\ell}b_\ell^{(k)} = \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}B_m^{(k-1)}.$$

Proof We compute $A = \langle Li_k(1-e^{-t}) | x^{n+1}\rangle$ in two different ways. On the one hand, by (1.6), it is

$$\begin{aligned} A &= \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \,\middle|\, \log(1+t)x^{n+1} \right\rangle = \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \,\middle|\, \sum_{\ell\geq 1} \frac{(-1)^{\ell-1}t^\ell}{\ell} x^{n+1} \right\rangle \\ &= \sum_{\ell=0}^{n} (-1)^{n-\ell}(n-\ell)! \binom{n+1}{\ell} \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \,\middle|\, x^\ell \right\rangle \\ &= \sum_{\ell=0}^{n} (-1)^{n-\ell}(n-\ell)! \binom{n+1}{\ell} b_\ell^{(k)}. \end{aligned} \tag{4.1}$$

On the other hand, by (1.6), it is

$$\begin{aligned} A &= \langle Li_k(1-e^{-t}) | x^{n+1}\rangle = \left\langle \int_0^t \frac{d}{ds} Li_k(1-e^{-s})\, ds \,\middle|\, x^{n+1} \right\rangle = \left\langle \int_0^t e^{-s} \frac{Li_{k-1}(1-e^{-s})}{1-e^{-s}}\, ds \,\middle|\, x^{n+1} \right\rangle \\ &= \left\langle \int_0^t \sum_{a\geq 0} \frac{(-s)^a}{a!} \sum_{m\geq 0} B_m^{(k-1)} \frac{s^m}{m!}\, ds \,\middle|\, x^{n+1} \right\rangle = \left\langle \sum_{\ell\geq 0} \sum_{m=0}^{\ell} (-1)^{\ell-m} \binom{\ell}{m} B_m^{(k-1)} \frac{t^{\ell+1}}{(\ell+1)!} \,\middle|\, x^{n+1} \right\rangle \\ &= \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} B_m^{(k-1)}. \end{aligned} \tag{4.2}$$

By comparing (4.1) and (4.2), we obtain the required identity. □

By using similar techniques as in the proof of Theorem 4.1 with computing

$$\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \big(\log(1+t)\big)^m \,\middle|\, x^n \right\rangle$$

in two different ways, we obtain the following result (we leave the proof as an exercise to the interested reader).

Theorem 4.2 *For all* $n-1 \geq m \geq 1$,

$$\begin{aligned} &\sum_{\ell=0}^{n-m} \binom{n}{\ell} S_1(n-\ell, m) b_\ell^{(k)} \\ &\quad = \sum_{\ell=0}^{n-m} \binom{n-1}{\ell} S_1(n-1-\ell, m-1) b_\ell^{(k)}(-1) \\ &\qquad + \frac{1}{n} \sum_{\ell=0}^{n-1-m} \sum_{j=0}^{\ell+1} \binom{n}{\ell+1} \binom{\ell+1}{j} S_1(n-1-\ell, m) \big(b_{\ell+1-j} B_j^{(k-1)}(-1) - b_{\ell+1-j}(-1) b_j^{(k)} \big). \end{aligned}$$

Let $b_n^{(k)}(x) = \sum_{m=0}^{n} c_{n,m}(x)_m$. By (1.5), (1.6) and the fact that $(x)_m \sim (1, e^t-1)$, we obtain

$$c_{n,m} = \frac{1}{m!} \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \,\middle|\, t^m x^n \right\rangle = \binom{n}{m} \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \,\middle|\, x^{n-m} \right\rangle = \binom{n}{m} b_{n-m}^{(k)},$$

which leads to the following identity.

Theorem 4.3 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n} \binom{n}{m} b_{n-m}^{(k)}(x)_m.$$

Let $\mathbb{B}_n^{(s)}(x)$ be the nth Bernoulli polynomial of order s. Then $\mathbb{B}_n^{(s)}(x) \sim (((e^t-1)/t)^s, t)$. Also, the Bernoulli numbers of the second kind of order s are given by $\frac{t^s}{\log^s(1+t)} = \sum_{j\geq 0} \mathbf{b}_j^{(s)} \frac{t^j}{j!}$ and let $b_n^{(k)}(x) = \sum_{m=0}^{n} c_{n,m}\mathbb{B}_m^{(s)}(x)$. By (1.5) and (1.6), we obtain

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!}\left\langle \frac{\frac{t^s}{\log^s(1+t)}}{\frac{\log(1+t)}{Li_k(1-e^{-t})}} \log^m(1+t) \middle| x^n \right\rangle = \frac{1}{m!}\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \frac{t^s}{\log^s(1+t)} \middle| \log^m(1+t)x^n \right\rangle \\
&= \frac{1}{m!}\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \frac{t^s}{\log^s(1+t)} \middle| m! \sum_{\ell\geq m} S_1(\ell,m)\frac{t^\ell}{\ell!} x^n \right\rangle \\
&= \sum_{\ell=m}^{n} \binom{n}{\ell} S_1(\ell,m) \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \frac{t^s}{\log^s(1+t)} \middle| x^{n-\ell} \right\rangle \\
&= \sum_{\ell=0}^{n-m} \binom{n}{\ell} S_1(n-\ell,m) \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| \frac{t^s}{\log^s(1+t)} x^\ell \right\rangle \\
&= \sum_{\ell=0}^{n-m} \binom{n}{\ell} S_1(n-\ell,m) \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| \sum_{j\geq 0} \mathbf{b}_j^{(s)} \frac{t^j}{j!} x^\ell \right\rangle \\
&= \sum_{\ell=0}^{n-m} \sum_{j=0}^{\ell} \binom{n}{\ell} \binom{\ell}{j} S_1(n-\ell,m) \mathbf{b}_j^{(s)} \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| x^{\ell-j} \right\rangle \\
&= \sum_{\ell=0}^{n-m} \sum_{j=0}^{\ell} \binom{n}{\ell} \binom{\ell}{j} S_1(n-\ell,m) \mathbf{b}_j^{(s)} b_{\ell-j}^{(k)},
\end{aligned}$$

which gives the following identity.

Theorem 4.4 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n} \left(\sum_{\ell=0}^{n-m} \sum_{j=0}^{\ell} \binom{n}{\ell} \binom{\ell}{j} S_1(n-\ell,m) \mathbf{b}_j^{(s)} b_{\ell-j}^{(k)} \right) \mathbb{B}_m^{(s)}(x).$$

Define $H_n^{(s)}(\lambda,x)$ to be the nth Frobenius-Euler polynomials of order s. Note that these polynomial satisfy $H_n^{(s)}(\lambda,x) \sim (((e^t-\lambda)/(1-\lambda))^s, t)$. Let $b_n^{(k)}(x) = \sum_{m=0}^{n} c_{n,m} H_m^{(s)}(\lambda,x)$. By (1.5) and (1.6), we obtain

$$\begin{aligned}
c_{n,m} &= \frac{1}{m!}\left\langle \frac{\frac{(1+t-\lambda)^s}{(1-\lambda)^s}}{\frac{\log(1+t)}{Li_k(1-e^{-t})}} \log^m(1+t) \middle| x^n \right\rangle = \frac{1}{m!(1-\lambda)^s}\left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \log^m(1+t) \middle| (1-\lambda+t)^s x^n \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{j=0}^{n-m} \binom{s}{j} (1-\lambda)^{s-j} (n)_j \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| \log^m(1+t) x^{n-j} \right\rangle \\
&= \frac{1}{m!(1-\lambda)^s} \sum_{j=0}^{n-m} \binom{s}{j} (1-\lambda)^{s-j} (n)_j \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| m! \sum_{\ell\geq m} S_1(\ell,m) \frac{t^\ell}{\ell!} x^{n-j} \right\rangle
\end{aligned}$$

$$= \sum_{j=0}^{n-m} \sum_{\ell=m}^{n-j} \binom{s}{j} \binom{n-j}{\ell} S_1(\ell, m)(1-\lambda)^{-j}(n)_j \left\langle \frac{Li_k(1-e^{-t})}{\log(1+t)} \middle| x^{n-j-\ell} \right\rangle$$

$$= \sum_{j=0}^{n-m} \sum_{\ell=m}^{n-j} \binom{s}{j} \binom{n-j}{\ell} S_1(\ell, m)(1-\lambda)^{-j}(n)_j b_{n-j-\ell}^{(k)}$$

$$= \sum_{j=0}^{n-m} \sum_{\ell=0}^{n-m-j} \binom{s}{j} \binom{n-j}{\ell} S_1(n-j-\ell, m)(1-\lambda)^{-j}(n)_j b_{\ell}^{(k)},$$

which gives the following identity.

Theorem 4.5 *For all* $n \geq 0$,

$$b_n^{(k)}(x) = \sum_{m=0}^{n} \left(\sum_{j=0}^{n-m} \sum_{\ell=0}^{n-m-j} \binom{s}{j} \binom{n-j}{\ell} S_1(n-j-\ell, m)(1-\lambda)^{-j}(n)_j b_{\ell}^{(k)} \right) H_m^{(s)}(\lambda, x).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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