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Some new existence and uniqueness results of solutions to semilinear impulsive fractional integro-differential equations

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Abstract

We consider the existence and uniqueness of solutions for the boundary value problem of semilinear impulsive integro-differential equations of fractional order $q \in (1, 2]$. Our results are based on the Altman fixed point theorem and a standard fixed point theorem. Two examples are presented to illustrate the main results.

MSC: 26A33; 34B15

Keywords: boundary value problem; impulsive fractional integro-differential equations; Caputo fractional derivative; existence of solutions; fixed point theorem

1 Introduction

Boundary value problems for nonlinear fractional differential equations have recently been addressed by several researchers. The interest in the study of differential equations of fractional order lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes [1–11]. For some recent development on the topic, see [12–25] and the references therein.

Daftardar-Gejji [12] considered the following system of fractional differential equations:

$$\begin{aligned} &D^{\alpha_i} u_i = f_i(t, u_1, u_2, \dots, u_n), \quad 0 < t < 1, 0 < \alpha_i < 1, 1 \le i \le n, \\ &u_i(0) = 0, \end{aligned}$$

where D^{α_i} denotes Riemann-Liouville derivative of order α_i. They obtained the existence of a solution by means of the Guo-Krasnosel'skii fixed point theorem and fixed point theorem in a cone.

Babakhani and Daftardar-Gejji [13] considered the following fractional initial value problem:

$$\left(D^{\alpha_n} - \sum_{j=1}^{n-1} p_j(x) D^{\alpha_{n-j}}\right) y = f(x, y), \qquad y(0) = 0, \quad 0 \le x \le \lambda, \lambda > 0,$$

where D^{α_j} is the standard Riemann-Liouville derivative of order α_i. They obtained the existence of positive solution by means of the Banach fixed point theorem.

Bai and Lü [14] studied the following two-point boundary value problem of a fractional differential equation:

$$\begin{aligned} &D^{\alpha}u(t)+f\bigl(t,u(t)\bigr)=0, \quad 0<t<1, 1<\alpha\leq 2, \\ &u(0)=u(1)=0, \end{aligned}$$

where D^{α} is the standard Riemann-Liouville fractional derivative. They obtained the existence of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem and the Leggett-Williams fixed point theorem.

Baleanu *et al.* [15] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned} &D^{\alpha}u(t)+f\bigl(t,u(t)\bigr)=0, \quad 0<\varepsilon<T, T\geq 1, 0<\alpha<1, t\in[\varepsilon,T], \\ &u(\eta)=u(T), \quad \eta\in(\varepsilon,t), \end{aligned}$$

where D^{α} is the standard Riemann-Liouville fractional derivative. They obtained the existence and uniqueness of positive solutions by means of the fixed point theorem on cones.

Baleanu *et al.* [16] studied the following initial value problem of a fractional differential equation:

$$\begin{aligned} &D^{\alpha}u(t)+D^{\beta}u(t)=f\bigl(t,u(t)\bigr), \quad 0<t<1, 0<\beta<\alpha<1, \\ &u(0)=0, \end{aligned}$$

where D^{α} is the standard Riemann-Liouville fractional derivative. They obtained the existence and multiplicity of positive solutions by means of the Guo-Krasnosel'skii fixed point theorem and Leggett-Williams fixed point theorem.

Baleanu *et al.* [17] studied the existence and uniqueness of a solution for the nonlinear fractional differential equation boundary value problem $D^{\alpha}u(t)=f(t,u(t))$ with a Riemann-Liouville fractional derivative via the different boundary value problems $u(0)=u(T)$, and the three-point boundary condition $u(0)=\beta_1 u(\eta)$ and $u(T)=\beta_2 u(\eta)$, where $T>0$, $t\in I=[0,T]$, $0<\alpha<1$, $0<\eta<T$, $0<\beta_1<\beta_2<1$.

By using a fixed point result on ordered metric spaces, Baleanu *et al.* [18] proved the existence and uniqueness of a solution of the nonlinear fractional differential equation $D^{\alpha}u(t)=f(t,u(t))$ ($t\in I=[0,T]$, $0<\alpha<1$) via the periodic boundary condition $u(0)=0$, where $T>0$ and $f: I\times R\to R$ is a continuous increasing function and ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α. Also, they solved it by using the anti-periodic boundary conditions $u(0)+u(T)=0$ with $u(0)\leq 0$ and $u(0)+\mu u(T)=0$ with $u(0)\leq 0$ and $\mu>0$ separately.

Zhou and Chu [19] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned} &{}^{c}D^{\alpha}u(t)=f\bigl(t,u(t),(Ku)(t),(Hu)(t)\bigr), \quad 0<t<1, 1<\alpha\leq 2, \\ &a_1u(0)-b_1u'(0)=d_1u(\xi_1), \qquad a_2u(1)+b_2u'(1)=d_2u(\xi_2), \end{aligned}$$

where ${}^cD^\alpha$ is the standard Caputo fractional derivative. They obtained the existence and uniqueness of solution by means of the contraction mapping principle and the Krasnoselskii fixed point theorem.

Zhou *et al.* [20] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned}
&{}^cD^\alpha u(t) = f\bigl(t,u(t)\bigr), \quad 0 \le t \le T, 1 < \alpha < 2, \\
&u(0) = \lambda_1 u(T) + \mu_1, \qquad u'(0) = \lambda_2 u'(T) + \mu_2, \quad \lambda_1 \neq 1, \lambda_2 \neq 1,
\end{aligned}$$

where ${}^cD^\alpha$ is the standard Caputo fractional derivative and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$. They obtained the existence of solution by means of the Mönch fixed point theorem combined with the technique of measures of weak noncompactness.

Zhou and Liu [21] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned}
&{}^cD^\alpha u(t) \in F\bigl(t,u(t)\bigr), \quad 0 \le t \le T, T > 0, 1 < \alpha \le 2, \\
&u(0) = \lambda_1 u(T) + \mu_1, \qquad u'(0) = \lambda_2 u'(T) + \mu_2,
\end{aligned}$$

where ${}^cD^\alpha$ is the standard Caputo fractional derivative and $\lambda_1 \neq 1$, $\lambda_2 \neq 1$, $\mu_1, \mu_2 \in \mathbb{R}$. They obtained the existence of solution by means of the Mönch fixed point theorem combined with the technique of measures of weak noncompactness.

Zhou *et al.* [22] presented some new multiplicity of positive solutions results for nonlinear semi-positone fractional boundary value problem $D^\alpha u(t) = p(t)f(t,u(t)) - q(t)$, $0 < t < 1$, $u(0) = u(1) = u'(1) = 0$, where $2 < \alpha \le 3$ is a real number and D^α is the standard Riemann-Liouville differentiation.

Zhou *et al.* [23] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned}
&D^\alpha u(t) + f\bigl(t,u(t)\bigr) = 0, \quad 0 < t < 1, 1 < \alpha \le 2, \\
&u(0) = 0, \qquad u(1) = aD^{\frac{\alpha-1}{2}} u(\xi),
\end{aligned}$$

where $\xi \in (0, \frac{1}{2}]$, $a \in (0, +\infty)$, $a\Gamma(\alpha)\xi^{\frac{\alpha-1}{2}} < \Gamma(\frac{\alpha+1}{2})$, and D^α is the standard Riemann-Liouville fractional derivative. They obtained the uniqueness of positive solution by means of the fixed point theorem in a partially ordered set.

Zhou and Liu [24] discussed the existence of weak solutions for a nonlinear boundary value problem of fractional q-difference equations in Banach space. They obtained the uniqueness of positive solution by means of the Mönch fixed point theorem combined with the technique of measures of weak noncompactness.

Zhou *et al.* [25] studied the following boundary value problem of a fractional differential equation:

$$\begin{aligned}
&D^\alpha u(t) = f\bigl(t,u(t)\bigr), \quad 0 < t < 1, 1 < \alpha \le 2, \\
&u(0) = u'(1) = 0,
\end{aligned}$$

where D^α is the standard Riemann-Liouville fractional derivative. They obtained the existence and multiplicity of positive solutions of the nonlinear fractional differential equation boundary value problem by means of the Leray-Schauder nonlinear alternative, a fixed point theorem on cones, and a mixed monotone method. In one word, the nonlinear term f of the above-mentioned boundary value problem does not include Ku, Hu (Ku, Hu are given functions satisfying some assumptions that will be specified later), and boundary conditions are particularly chosen.

Impulsive differential equations, which provide a natural description of observed evolution processes, are regarded as important mathematical tools for the better understanding of several real world problems in applied sciences. The theory of impulsive differential equations of integer order has found extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, we refer the reader to [26–29]. The impulsive differential equations of fractional order have also attracted considerable attention and a variety of results can be found in [30–39] and the references therein.

Ahmad and Sivasundaram [31] considered the following impulsive fractional differential equations:

$$\begin{cases} {}^cD^q x(t) = f(t,x(t)), \quad t \in J = [0,1] \setminus \{t_1, t_2, t_3, \dots, t_p\}, \\ \Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta x'(t_k) = J_k(x(t_k^-)), \quad t_k \in (0,1), k = 1, \dots, p, \\ x(0) + x'(0) = 0, \qquad x(1) + x'(1) = 0, \end{cases}$$

where ${}^cD^q$ is the Caputo fractional derivative. The results are based on the contraction mapping principle and the Krasnoselskii fixed point theorem.

Tian and Bai [33] considered the following impulsive fractional differential equations:

$$\begin{cases} {}^cD^q u(t) = f(t,u), \quad 0 < t < 1, t \neq t_k, k = 1, \dots, p, 1 < q \leq 2, \\ \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = \bar{I}_k(u(t_k^-)), \quad k = 1, \dots, p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(\xi) = 0, \quad \xi \in (0,1), \xi \neq t_k, k = 1, \dots, p. \end{cases}$$

The results are based on the contraction mapping principle and the Schauder fixed point theorem.

Zhang and Wang [38] considered the following impulsive fractional differential equations:

$$\begin{cases} {}^cD^q y(t) = f(t,y), \quad \forall t \in J = [0,T], t \neq t_k, k = 1, \dots, m, 1 < q \leq 2, \\ \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = \bar{I}_k(u(t_k^-)), \quad k = 1, \dots, p, \\ y(0) = -y(T), \qquad y'(0) = -y'(T). \end{cases}$$

The results are based on the Altman fixed point theorem and the Leray-Schauder fixed point theorem.

Zhou and Liu [39] considered the following nonlinear impulsive fractional differential equations:

$$\begin{cases} {}^cD^q u(t) = f(t,u(t)), \quad 1 < q \leq 2, t \in J' = J \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = \bar{I}_k(u(t_k^-)), \quad k = 1, \dots, m, \\ au(0) - bu'(0) = x_0, \qquad cu(1) + du'(1) = x_1. \end{cases}$$

The results are based on the nonlinear alternative of Leray-Schauder type and the Krasnoselskii fixed point theorem.

On the other hand, the impulsive boundary value problems for nonlinear fractional differential equations have not been addressed extensively and many aspects of these problems are yet to be explored. For example, we observed that in the above-mentioned work [30–39], the authors all demand that the nonlinear term f is bounded and continuous, and the impulse functions I_k and $\overline{I}_k$ are bounded, it is easy to see that these conditions are very strong restrictive and difficult to satisfy in applications. Motivated by the above-mentioned work [30–39], this article is mainly concerned with the existence and uniqueness of solution for the boundary value problems for the semilinear impulsive fractional integro-differential equations:

$$\begin{cases} {}^cD^q u(t) = \lambda u(t) + f(t, u(t), (Ku)(t), (Hu)(t)), & 1 < q \le 2, t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = \overline{I}_k(u(t_k^-)), & k = 1, \ldots, m, \\ au(0) - bu'(0) = x_0, \qquad cu(1) + du'(1) = x_1, \end{cases} \tag{1.1}$$

where ${}^cD^q$ is the Caputo fractional derivative, $\lambda \ge 0$, $a \ge 0$, $b > 0$, $c \ge 0$, $d > 0$, $\delta = ac + ad + bc \ne 0$, and $x_0, x_1 \in \mathbb{R}$. $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_k, \overline{I}_k \in C(\mathbb{R}, \mathbb{R})$, $J = [0,1]$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim_{h \to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$, $k = 1, \ldots, m$. $\Delta u'(t_k)$ has a similar meaning for $u'(t)$. Here

$$(Ku)(t) = \int_0^t k(t,s)u(s)\,ds, \quad t \in J,$$

$$(Hu)(t) = \int_0^1 h(t,s)u(s)\,ds, \quad t \in J,$$

are integral operators with integral kernels $k \in C[D, \mathbb{R}^+]$, $D = \{(t,s) : 0 \le s \le t \le 1\}$, and $h \in C[D_0, \mathbb{R}^+]$, $D_0 = \{(t,s) : 0 \le t, s \le 1\}$, $\mathbb{R}^+ = [0, +\infty)$, where $k^* = \sup_{t \in J} \int_0^t |k(t,s)|\,ds$, $h^* = \sup_{t \in J} \int_0^1 |h(t,s)|\,ds$.

Evidently, the problem (1.1) not only includes the boundary value problems mentioned above [37, 39] but also extends them to a much wider case. Our main tools are the Altman fixed point theorem and a standard fixed point theorem. Thus, our results are new and generalize some earlier ones.

The remainder of this article is organized as follows. In Section 2, we introduce some preliminary results, including basic definitions of fractional integrals and derivatives, some properties and a fixed point theorem. Section 3 will be devoted to the existence and uniqueness of solutions for boundary value problems of semilinear impulsive integro-differential equations of fractional order. In the last section, we give two examples to demonstrate the applications.

2 Preliminaries and lemmas

Let $J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, 1]$, and we introduce the spaces: $L^1(J, \mathbb{R})$ denotes the Banach space of measurable functions $u : J \mapsto \mathbb{R}$ which are Bochner integrable, equipped with the norm $\|u\|_{L^1} := \int_J \|u(t)\|\,dt$, $PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} : u \in C(J_k),$ $k = 0, 1, \ldots, m$, and $u(t_k^+)$ exists, $k = 1, \ldots, m\}$, is a Banach space with the norm $\|u\|_{PC} :=$

$\sup_{t\in J}\|u(t)\|$, and $PC^1(J,\mathbb{R}) = \{u : J \to \mathbb{R} : u \in C^1(J_k),\ k = 0,1,\dots,m$, and $u(t_k^+), u'(t_k^+)$ exist, $k = 1,\dots,m\}$ is a Banach space with the norm $\|u\|_{PC^1} := \max_{t\in J}\{\|u\|, \|u'\|\}$.

Definition 2.1 (see [1]) The Riemann-Liouville fractional integral of order r for a continuous function h is defined as

$$I^r h(t) = \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} h(s)\,ds, \quad r > 0,$$

provided the integral exists.

Definition 2.2 (see [1]) For an at least n-times continuously differentiable function $h : [0,\infty) \to R$, the Caputo derivative of fractional order r is defined as

$${}^cD^r h(t) = \frac{1}{\Gamma(n-r)} \int_0^t (t-s)^{n-r-1} h^{(n)}(s)\,ds, \quad n-1 < r < n, n = [r]+1,$$

where $[r]$ denotes the integer part of the real number r.

Lemma 2.1 (see [1]) *Let $r > 0$, $h \in C[0,1] \cap L(0,1)$. Then the differential equation ${}^cD^r h(t) = 0$ has solutions*

$$h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [r]+1$.

Lemma 2.2 (see [1]) *Assume that $h \in C[0,1] \cap L(0,1)$ with a derivative of order r that belongs to $C[0,1] \cap L(0,1)$. Then*

$$I_{0+}^r\, {}^cD_{0+}^r h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\dots,n-1$, $n = [r]+1$.

Lemma 2.3 (see [39]) *For a given $h \in C[0,1]$, a function u is a solution of the following impulsive boundary value problem*:

$$\begin{cases} {}^cD^q u(t) = h(t), \quad 1 < q \le 2, t \in J', \\ \Delta u(t_k) = I_k(u(t_k^-)), \qquad \Delta u'(t_k) = \bar{I}_k(u(t_k^-)), \quad k = 1,\dots,m, \\ au(0) - bu'(0) = x_0, \qquad cu(1) + du'(1) = x_1, \end{cases} \tag{2.1}$$

if and only if u is a solution of the impulsive fractional integral equation

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + C_1 + C_2 t, & \text{if } t \in J_0, \\ \frac{1}{\Gamma(q)} \int_{t_k}^t (t-s)^{q-1} h(s)\,ds + \frac{1}{\Gamma(q)} \sum_{i=1}^k \int_{t_{i-1}}^{t_i} (t_i-s)^{q-1} h(s)\,ds & \\ \quad + \frac{1}{\Gamma(q-1)} \sum_{i=1}^k (t-t_k) \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} h(s)\,ds & \\ \quad + \frac{1}{\Gamma(q-1)} \sum_{i=1}^{k-1} (t_k-t_i) \int_{t_{i-1}}^{t_i} (t_i-s)^{q-2} h(s)\,ds & \\ \quad + \sum_{i=1}^k I_i(y(t_i^-)) + \sum_{i=1}^k (t-t_k) \bar{I}_i(u(t_i^-)) & \\ \quad + \sum_{i=1}^{k-1} (t_k-t_i) \bar{I}_i(u(t_i^-)) + C_1 + C_2 t, & \text{if } t \in J_k, \end{cases} \tag{2.2}$$

where

$$\begin{aligned}
C_1 = -\Biggl\{ & \sum_{i=1}^{m+1} \frac{bc}{\delta\Gamma(q)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} h(s)\,ds + \sum_{i=1}^{m} \frac{bc(1-t_m)}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds \\
& + \sum_{i=1}^{m-1} \frac{bc(t_m - t_i)}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds + \sum_{i=1}^{m+1} \frac{bd}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds \\
& + \sum_{i=1}^{m} \frac{bc}{\delta} I_i\bigl(u(t_i^-)\bigr) + \sum_{i=1}^{m} \frac{bc(1-t_p)}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) + \sum_{i=1}^{m-1} \frac{bc(t_p - t_i)}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) \\
& + \sum_{i=1}^{m} \frac{bd}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) + \frac{(bc-\delta)x_0 - abx_1}{a\delta} \Biggr\},
\end{aligned}$$

$$\begin{aligned}
C_2 = -\Biggl\{ & \sum_{i=1}^{m+1} \frac{ac}{\delta\Gamma(q)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} h(s)\,ds + \sum_{i=1}^{m} \frac{ac(1-t_m)}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds \\
& + \sum_{i=1}^{m-1} \frac{ac(t_m - t_i)}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds + \sum_{i=1}^{m+1} \frac{ad}{\delta\Gamma(q-1)} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} h(s)\,ds \\
& + \sum_{i=1}^{m} \frac{ac}{\delta} I_i\bigl(u(t_i^-)\bigr) + \sum_{i=1}^{m} \frac{ac(1-t_m)}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) + \sum_{i=1}^{m-1} \frac{ac(t_m - t_i)}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) \\
& + \sum_{i=1}^{m} \frac{ad}{\delta} \overline{I}_i\bigl(u(t_i^-)\bigr) + \frac{cx_0 - ax_1}{a\delta} \Biggr\}.
\end{aligned}$$

We need the following well-known result to prove the existence of solutions for (1.1).

Lemma 2.4 (see [40]) *Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta \in \Omega$ and let $T : \overline{\Omega} \to E$ be a completely continuous operator such that*

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega.$$

Then T has a fixed point in $\overline{\Omega}$.

3 Main results

Define an operator $\mathcal{A} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ as

$$\begin{aligned}
(\mathcal{A}u)(t) = & \int_{t_k}^{t} \frac{(t-s)^{q-1}}{\Gamma(q)} \bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr]\,ds \\
& + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)} \bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr]\,ds \\
& + \sum_{i=1}^{k} (t - t_k) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr]\,ds \\
& + \sum_{i=1}^{k-1} (t_k - t_i) \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)} \bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr]\,ds
\end{aligned}$$

$$\begin{aligned}&+\sum_{i=1}^{k} I_i\bigl(u(t_i^-)\bigr)+\sum_{i=1}^{k}(t-t_k)\overline{I}_i\bigl(u(t_i^-)\bigr)\\&+\sum_{i=1}^{k-1}(t_k-t_i)\overline{I}_i\bigl(u(t_i^-)\bigr)+M_1+M_2t,\end{aligned}\tag{3.1}$$

where

$$\begin{aligned}M_1={}&-\Biggl\{\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{bc(t_i-s)^{q-1}}{\delta\Gamma(q)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{bc(1-t_m)(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{bc(t_m-t_i)(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{bd(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m}\frac{bc}{\delta}I_i\bigl(u(t_i^-)\bigr)+\sum_{i=1}^{m}\frac{bc(1-t_m)}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)\\&+\sum_{i=1}^{m}\frac{bd}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)+\frac{(bc-\delta)x_0-abx_1}{a\delta}\Biggr\},\end{aligned}\tag{3.2}$$

$$\begin{aligned}M_2={}&-\Biggl\{\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{ac(t_i-s)^{q-1}}{\delta\Gamma(q)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{ac(1-t_m)(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{ac(t_m-t_i)(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{ad(t_i-s)^{q-2}}{\delta\Gamma(q-1)}\bigl[f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr)+\lambda u(s)\bigr]\,ds\\&+\sum_{i=1}^{m}\frac{ac}{\delta}I_i\bigl(u(t_i^-)\bigr)+\sum_{i=1}^{m}\frac{ac(1-t_m)}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)+\sum_{i=1}^{m-1}\frac{bc(t_m-t_i)}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)\\&+\sum_{i=1}^{m}\frac{ad}{\delta}\overline{I}_i\bigl(u(t_i^-)\bigr)+\frac{cx_0-ax_1}{a\delta}\Biggr\}.\end{aligned}\tag{3.3}$$

Theorem 3.1 *Let*

$$\lim_{u\to 0}\frac{f(t,u,Ku,Hu)+\lambda u}{u}=0,\qquad\lim_{u\to 0}\frac{I_k(u)}{u}=0,\qquad\lim_{u\to 0}\frac{\overline{I}_k(u)}{u}=0.\tag{3.4}$$

Then the problem (1.1) *has at least one solution on* J.

Proof Consider the operator $\mathcal{A}$ defined by (3.1). Firstly, we show that the operator $\mathcal{A}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is continuous and completely continuous. Note that $\mathcal{A}$ is continuous in view of continuity of f, I_k, and $\bar{I}_k$. Let $\Omega \subset PC(J,\mathbb{R})$ be bounded. Then there exist positive constants $L_i > 0$ ($i = 0,1,2,3$) such that $|u| \le L_0$, $|f(t,u,Ku,Hu)| \le L_1$, $|I_k(u)| \le L_2$, $|\bar{I}_k(u)| \le L_3$, $\forall u \in \Omega$. Thus, for any $u \in \Omega$, we get

$$\begin{aligned}
|M_1| \le{}& \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bc(t_i-s)^{q-1}}{\delta\Gamma(q)} \bigl|f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr|\,ds \\
&+ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{bc(1-t_m)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} \bigl|f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr|\,ds \\
&+ \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{bc(t_m-t_i)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} \bigl|f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr|\,ds \\
&+ \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bd(t_i-s)^{q-2}}{\delta\Gamma(q-1)} \bigl|f\bigl(s,u(s),(Ku)(s),(Hu)(s)\bigr) + \lambda u(s)\bigr|\,ds \\
&+ \sum_{i=1}^{m} \frac{bc}{\delta}\bigl|I_i\bigl(u(t_i)\bigr)\bigr| + \sum_{i=1}^{m} \frac{bc(1-t_m)}{\delta}\bigl|\bar{I}_i\bigl(u(t_i)\bigr)\bigr| + \sum_{i=1}^{m-1} \frac{bc(t_m-t_i)}{\delta}\bigl|\bar{I}_i\bigl(u(t_i)\bigr)\bigr| \\
&+ \sum_{i=1}^{m} \frac{bd}{\delta}\bigl|\bar{I}_i\bigl(u(t_i)\bigr)\bigr| + \frac{(c+d)|x_0| + b|x_1|}{\delta} \\
\le{}& \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bc(t_i-s)^{q-1}}{\delta\Gamma(q)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m} \frac{bc}{\delta} L_2 \\
&+ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{bc(1-t_m)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m} \frac{bc(1-t_m)}{\delta} L_3 \\
&+ \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{bc(t_m-t_i)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m-1} \frac{bc(t_m-t_i)}{\delta} L_3 \\
&+ \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bd(t_i-s)^{q-2}}{\delta\Gamma(q-1)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m} \frac{bd}{\delta} L_3 + \frac{(c+d)|x_0| + b|x_1|}{\delta} \\
\le{}& \left[\frac{(m+1)bc}{\delta\Gamma(q+1)} + \frac{mbc}{\delta\Gamma(q)} + \frac{(m-1)bc}{\delta\Gamma(q)} + \frac{(m+1)bd}{\delta\Gamma(q)}\right](L_1 + \lambda L_0) + \frac{mbc}{\delta} L_2 \\
&+ \left[\frac{mbc}{\delta} + \frac{(m-1)bc}{\delta} + \frac{mbd}{\delta}\right] L_3 + \frac{(c+d)|x_0| + b|x_1|}{\delta}.
\end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned}
|M_2| \le{}& \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{ac(t_i-s)^{q-1}}{\delta\Gamma(q)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m} \frac{ac}{\delta} L_2 \\
&+ \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{ac(1-t_m)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m} \frac{ac(1-t_m)}{\delta} L_3 \\
&+ \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{ac(t_m-t_i)(t_i-s)^{q-2}}{\delta\Gamma(q-1)} (L_1 + \lambda L_0)\,ds + \sum_{i=1}^{m-1} \frac{ac(t_m-t_i)}{\delta} L_3
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m+1}\int_{t_{i-1}}^{t_i}\frac{ad(t_i-s)^{q-2}}{\delta\Gamma(q-1)}(L_1+\lambda L_0)\,ds+\sum_{i=1}^{m}\frac{ad}{\delta}L_3+\frac{c|x_0|+a|x_1|}{\delta} \\
\leq&\left[\frac{(m+1)ac}{\delta\Gamma(q+1)}+\frac{mac}{\delta\Gamma(q)}+\frac{(m-1)ac}{\delta\Gamma(q)}+\frac{(m+1)ad}{\delta\Gamma(q)}\right](L_1+\lambda L_0)+\frac{mac}{\delta}L_2 \\
&+\left[\frac{mac}{\delta}+\frac{(m-1)ac}{\delta}+\frac{mbd}{\delta}\right]L_3+\frac{c|x_0|+a|x_1|}{\delta}.
\end{aligned}
\tag{3.6}
$$

Therefore, we obtain

$$
\begin{aligned}
\left|Au(t)\right| \leq& \int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\left|f\left(s,u(s),(Ku)(s),(Hu)(s)\right)+\lambda u(s)\right|ds \\
&+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\left|f\left(s,u(s),(Ku)(s),(Hu)(s)\right)+\lambda u(s)\right|ds \\
&+\sum_{i=1}^{k}(t-t_k)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left|f\left(s,u(s),(Ku)(s),(Hu)(s)\right)+\lambda u(s)\right|ds \\
&+\sum_{i=1}^{k-1}(t_k-t_i)\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\left|f\left(s,u(s),(Ku)(s),(Hu)(s)\right)+\lambda u(s)\right|ds \\
&+\sum_{i=1}^{k}\left|I_i\left(u\left(t_i^-\right)\right)\right|+\sum_{i=1}^{k}(t-t_k)\left|\overline{I}_i\left(u\left(t_i^-\right)\right)\right| \\
&+\sum_{i=1}^{k-1}(t_k-t_i)\left|\overline{I}_i\left(u\left(t_i^-\right)\right)\right|+|M_1|+|M_2| \\
\leq&\left[\int_{t_k}^{t}\frac{(t-s)^{q-1}}{\Gamma(q)}\,ds+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-1}}{\Gamma(q)}\,ds\right. \\
&\left.+\sum_{i=1}^{m}\int_{t_{i-1}}^{t_i}\frac{(t-t_k)(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds+\sum_{i=1}^{m-1}\int_{t_{i-1}}^{t_i}\frac{(t_k-t_i)(t_i-s)^{q-2}}{\Gamma(q-1)}\,ds\right](L_1+\lambda L_0) \\
&+\sum_{i=1}^{m}L_2+\sum_{i=1}^{m}(t-t_k)L_3+\sum_{i=1}^{m-1}(t_k-t_i)L_3+|M_1|+|M_2| \\
\leq&\frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}(L_1+\lambda L_0) \\
&+\frac{(2m-1)[c(a+b)+\delta]+(m+1)(a+b)d}{\delta\Gamma(q)}(L_1+\lambda L_0) \\
&+\frac{m[c(a+b)+\delta]}{\delta}L_2+\frac{(2m-1)[c(a+b)+\delta]+md(a+b)}{\delta}L_3 \\
&+\frac{(2c+d)|x_0|+(a+b)|x_1|}{\delta},
\end{aligned}
\tag{3.7}
$$

which implies that

$$
\begin{aligned}
\|Au\| \leq& \frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}(L_1+\lambda L_0) \\
&+\frac{(2m-1)[c(a+b)+\delta]+(m+1)(a+b)d}{\delta\Gamma(q)}(L_1+\lambda L_0)
\end{aligned}
$$

$$+\frac{m[c(a+b)+\delta]}{\delta}L_2+\frac{(2m-1)[c(a+b)+\delta]+md(a+b)}{\delta}L_3$$
$$+\frac{(2c+d)|x_0|+(a+b)|x_1|}{\delta}=:L. \tag{3.8}$$

On the other hand, for any $t\in J_k$, $0\le k\le m$, we have

$$\begin{aligned}\big|(Au)'(t)\big| &\le \int_{t_k}^{t}\frac{(t-s)^{q-2}}{\Gamma(q-1)}\big|f\big(s,u(s),(Ku)(s),(Hu)(s)\big)+\lambda u(s)\big|\,ds\\
&\quad+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\frac{(t_i-s)^{q-2}}{\Gamma(q-1)}\big|f\big(s,u(s),(Ku)(s),(Hu)(s)\big)+\lambda u(s)\big|\,ds\\
&\quad+\sum_{i=1}^{k}\big|\bar{I}_i\big(u\big(t_i^-\big)\big)\big|+|M_2|\\
&\le\frac{(m+1)ac}{\delta\Gamma(q+1)}(L_1+\lambda L_0)+\frac{(2m-1)ac+(m+1)(ad+\delta)}{\delta\Gamma(q)}(L_1+\lambda L_0)\\
&\quad+\frac{mac}{\delta}L_2+\frac{(2m-1)ac+m(ad+\delta)}{\delta}L_3+\frac{c|x_0|+a|x_1|}{\delta}:=\hat{L}.\end{aligned}$$

Hence, for $t_1,t_2\in J_k$ with $t_1<t_2$, $0\le k\le m$, we have

$$\big|(Au)(t_2)-(Au)(t_1)\big|\le\int_{t_1}^{t_2}\big|(Au)'(s)\big|\,ds\le\hat{L}(t_2-t_1).$$

This implies that A is equicontinuous on all the subintervals J_k, $k=0,1,2,\dots,m$. Thus, by the Arzela-Ascoli theorem, it follows that $\mathcal{A}:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Now, in view of $\lim_{u\to0}\frac{f(t,u,Ku,Hu)+\lambda u}{u}=0$, $\lim_{u\to0}\frac{I_k(u)}{u}=0$, $\lim_{u\to0}\frac{\bar{I}_k(u)}{u}=0$, there exists a constant $r>0$ such that $|f(t,u,Ku,Hu)|+\lambda|u|\le\delta_1|u|$, $|I_k(u)|\le\delta_2|u|$, and $|\bar{I}_k(u)|\le\delta_3|u|$ for $0<|u|<r$, where $\delta_j>0$ ($j=1,2,3$) satisfy

$$\begin{aligned}&\frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}\delta_1+\frac{(2m-1)[c(a+b)+\delta]+(m+1)(a+b)d}{\delta\Gamma(q)}\delta_1+\frac{m[c(a+b)+\delta]}{\delta}\delta_2\\
&\quad+\frac{(2m-1)[c(a+b)+\delta]+md(a+b)}{\delta}\delta_3+\frac{(2c+d)|x_0|+(a+b)|x_1|}{\delta}\le1.\end{aligned}$$

Let $\Omega=\{u\in PC(J,\mathbb{R}):\|u\|<r\}$ and take $u\in PC(J,\mathbb{R})$ such that $\|u\|=r$ so that $u\in\partial\Omega$. Then, by the process used to obtain (3.7), we have

$$\begin{aligned}\big|Au(t)\big|\le\bigg\{&\frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}\delta_1+\frac{(2m-1)[c(a+b)+\delta]+(m+1)(a+b)d}{\delta\Gamma(q)}\delta_1\\
&+\frac{m[c(a+b)+\delta]}{\delta}\delta_2+\frac{(2m-1)[c(a+b)+\delta]+md(a+b)}{\delta}\delta_3\\
&+\frac{(2c+d)|x_0|+(a+b)|x_1|}{\delta}\bigg\}\|u\|,\end{aligned}$$

which implies that $\|Au\|\le\|u\|$, $u\in\partial\Omega$. Therefore, by Lemma 2.4, the operator A has at least one fixed point, which in turn implies that the problem (1.1) has at least one solution $u\in\overline{\Omega}$. This completes the proof. □

Theorem 3.2 *Assume that* (H1) *there exist nonnegative constants* γ_i $(i = 1, 2, 3, 4, 5)$ *such that* $|f(t,u,Ku,Hu) - f(t,\overline{u},K\overline{u},H\overline{u})| \le \gamma_1|u-\overline{u}| + \gamma_2|Ku - K\overline{u}| + \gamma_3|Hu - H\overline{u}|$, $t \in J$, $|I_k(u) - I_k(\overline{u})| \le \gamma_4|u-\overline{u}|$, $|\overline{I}_k(u) - \overline{I}_k(\overline{u})| \le \gamma_5|u-\overline{u}|$, *for* $t \in J$, $u, \overline{u} \in \mathbb{R}$ *and* $k = 1, 2, \ldots, m$. *Then the problem* (1.1) *has a unique solution if*

$$\begin{aligned}\Lambda = {} & \frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \\ & + \frac{(2m-1)[c(a+b)+\delta] + md(a+b)}{\delta}\gamma_5 + \frac{m[c(a+b)+\delta]}{\delta}\gamma_4 \\ & + \frac{(2m-1)[c(a+b)+\delta] + (m+1)(a+b)d}{\delta\Gamma(q)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) < 1. \end{aligned} \tag{3.9}$$

Proof For $u, \overline{u} \in C(J,\mathbb{R})$, it follows by the condition (H1) that

$$\begin{aligned}& \bigl|f\bigl(s,u(s),Ku(s),Hu(s)\bigr) - f\bigl(s,\overline{u}(s),K\overline{u}(s),H\overline{u}(s)\bigr)\bigr| \\ & \quad \le \gamma_1\bigl|u(s) - \overline{u}(s)\bigr| + \gamma_2\int_0^s k(s,\tau)\bigl|u(\tau) - \overline{u}(\tau)\bigr|\,d\tau \\ & \qquad + \gamma_3\int_0^1 h(s,\tau)\bigl|u(\tau) - \overline{u}(\tau)\bigr|\,d\tau \\ & \quad \le \bigl\{\gamma_1 + \gamma_2 k^* + \gamma_3 h^*\bigr\}\|u - \overline{u}\|. \end{aligned} \tag{3.10}$$

In view of the estimate (3.10), we obtain

$$\begin{aligned}& \bigl|Au(t) - A\overline{u}(t)\bigr| \\ & \quad \le \int_{t_k}^t \frac{(t-s)^{q-1}}{\Gamma(q)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^k \int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-1}}{\Gamma(q)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^k (t - t_k)\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^{k-1} (t_k - t_i)\int_{t_{i-1}}^{t_i} \frac{(t_i - s)^{q-2}}{\Gamma(q-1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^k \bigl|I_i\bigl(u\bigl(t_i^-\bigr)\bigr) - I_i\bigl(\overline{u}\bigl(t_i^-\bigr)\bigr)\bigr| + \sum_{i=1}^k (t - t_k)\bigl|\overline{I}_i\bigl(u\bigl(t_i^-\bigr)\bigr) - \overline{I}_i\bigl(\overline{u}\bigl(t_i^-\bigr)\bigr)\bigr| \\ & \qquad + \sum_{i=1}^{k-1} (t_k - t_i)\bigl|\overline{I}_i\bigl(u\bigl(t_i^-\bigr)\bigr) - \overline{I}_i\bigl(\overline{u}\bigl(t_i^-\bigr)\bigr)\bigr| \\ & \qquad + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bc(t_i - s)^{q-1}}{\delta\Gamma(q)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{bc(1 - t_m)(t_i - s)^{q-2}}{\delta\Gamma(q-1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \\ & \qquad + \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{bc(t_m - t_i)(t_i - s)^{q-2}}{\delta\Gamma(q-1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr)\|u - \overline{u}\|\,ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{bd(t_i - s)^{q-2}}{\delta\Gamma(q-1)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \|u - \overline{u}\| \, ds \\
&\quad + \sum_{i=1}^{m} \frac{bc}{\delta} \bigl| I_i\bigl(u(t_i)\bigr) - I_i\bigl(\overline{u}(t_i)\bigr)\bigr| + \sum_{i=1}^{m} \frac{bc(1-t_m)}{\delta} \bigl| \overline{I}_i\bigl(u(t_i)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i)\bigr)\bigr| \\
&\quad + \sum_{i=1}^{m-1} \frac{bc(t_m - t_i)}{\delta} \bigl| \overline{I}_i\bigl(u(t_i)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i)\bigr)\bigr| + \sum_{i=1}^{m} \frac{bd}{\delta} \bigl| \overline{I}_i\bigl(u(t_i)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i)\bigr)\bigr| \\
&\quad + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{ac(t_i - s)^{q-1}}{\delta\Gamma(q)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \|u - \overline{u}\| \, ds \\
&\quad + \sum_{i=1}^{m} \int_{t_{i-1}}^{t_i} \frac{ac(1-t_m)(t_i - s)^{q-2}}{\delta\Gamma(q-1)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \|u - \overline{u}\| \, ds \\
&\quad + \sum_{i=1}^{m-1} \int_{t_{i-1}}^{t_i} \frac{ac(t_m - t_i)(t_i - s)^{q-2}}{\delta\Gamma(q-1)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \|u - \overline{u}\| \, ds \\
&\quad + \sum_{i=1}^{m+1} \int_{t_{i-1}}^{t_i} \frac{ad(t_i - s)^{q-2}}{\delta\Gamma(q-1)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \|u - \overline{u}\| \, ds \\
&\quad + \sum_{i=1}^{m} \frac{ac}{\delta} \bigl| I_i\bigl(u(t_i^-)\bigr) - I_i\bigl(\overline{u}(t_i^-)\bigr)\bigr| + \sum_{i=1}^{m} \frac{ac(1-t_m)}{\delta} \bigl| \overline{I}_i\bigl(u(t_i^-)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i^-)\bigr)\bigr| \\
&\quad + \sum_{i=1}^{m-1} \frac{ac(t_m - t_i)}{\delta} \bigl| \overline{I}_i\bigl(u(t_i^-)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i^-)\bigr)\bigr| + \sum_{i=1}^{m} \frac{ad}{\delta} \bigl| \overline{I}_i\bigl(u(t_i^-)\bigr) - \overline{I}_i\bigl(\overline{u}(t_i^-)\bigr)\bigr| \\
&\le \biggl\{ \frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \\
&\quad + \frac{(2m-1)[c(a+b)+\delta] + md(a+b)}{\delta} \gamma_5 + \frac{m[c(a+b)+\delta]}{\delta} \gamma_4 \\
&\quad + \frac{(2m-1)[c(a+b)+\delta] + (m+1)(a+b)d}{\delta\Gamma(q)} \bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) \biggr\} \|u - \overline{u}\| \\
&\le \Lambda \|u - \overline{u}\|, \qquad (3.11)
\end{aligned}
$$

where Λ is given by (3.9). Thus, $\|Au - A\overline{u}\| \le \Lambda \|u - \overline{u}\|$. As $\Lambda < 1$, A is a contraction operator. Hence, by the contraction mapping principle, the problem (1.1) has a unique solution. The proof is complete. □

Remark 3.1 The problem (1.1) not only includes the boundary value problems mentioned above [37, 39], but it also extends them to a much wider case. Thus, our results are new and generalize some earlier ones.

In the sequel we present two examples which illustrate Theorem 3.1 and Theorem 3.2.

4 Examples

Example 4.1 Consider the following boundary value problem:

$$
\begin{cases}
{}^cD_{0+}^q u(t) = 2\ln(1+u^2) - \cos u + 1, & 0 < t < 1, t \neq t_1, 0 < t_1 < 1, \\
\Delta u(t_1) = e^{u^3(t_1)} - 1, \qquad \Delta u'(t_1) = (1 + u^2(t_1))^{\frac{1}{3}} - 1, \\
u(0) - u'(0) = 0.03, \qquad u(1) + u'(1) = 0.06,
\end{cases}
\tag{4.1}
$$

where

$$f(t,u) = 2\ln\bigl(1+u^2\bigr) - \cos u + 1,$$
$$I_1(u) = e^{u^3} - 1, \qquad \bar{I}_1(u) = \bigl(1+u^2\bigr)^{\frac{1}{3}} - 1.$$

Here $q = \frac{3}{2}$, $\lambda = 0$, $a = 1$, $b = 1$, $c = 1$, $d = 1$, $m = 1$, $\delta = ac + ad + bc = 3$, $k(t,s) = 0$, $h(t,s) = 0$, $x_0 = 0.03$, $x_1 = 0.06$. Clearly

$$\lim_{u\to 0}\frac{f(t,u)}{u} = \lim_{u\to 0}\frac{2\ln(1+u^2) - \cos u + 1}{u} = \lim_{u\to 0}\frac{2\ln(1+u^2)}{u} + \lim_{u\to 0}\frac{-\cos u + 1}{u} = 0,$$
$$\lim_{u\to 0}\frac{I_1(u)}{u} = \lim_{u\to 0}\frac{e^{u^3}-1}{u} = \lim_{u\to 0} 3u^2 = 0,$$
$$\lim_{u\to 0}\frac{\bar{I}_1(u)}{u} = \lim_{u\to 0}\frac{(1+u^2)^{\frac{1}{3}}-1}{u} = \lim_{u\to 0}\frac{u}{3} = 0,$$

Furthermore, in this case, δ_i $(i = 1,2,3)$ satisfy the inequality

$$\frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}\delta_1 + \frac{(2m-1)[c(a+b)+\delta] + (m+1)(a+b)d}{\delta\Gamma(q)}\delta_1 + \frac{m[c(a+b)+\delta]}{\delta}\delta_2 + \frac{(2m-1)[c(a+b)+\delta] + md(a+b)}{\delta}\delta_3 + \frac{(2c+d)|x_0| + (a+b)|x_1|}{\delta} \le 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. Hence, by the conclusion of Theorem 3.1, the impulsive fractional boundary value problem (4.1) has at least one solution.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} {}^{c}D^{q}_{0+}u(t) = \frac{1}{40}u(t) + \frac{\sin t}{(t+4)^2}\frac{|u(t)|}{1+|u(t)|} + \frac{1}{25}\int_0^t \frac{e^{-(s-t)}}{5}u(s)\,ds \\ \qquad\qquad + \frac{1}{25}\int_0^1 \frac{e^{-(s-t)/2}}{5}u(s)\,ds, \quad 0 < t < 1, t \neq \frac{3}{5}, \\ \Delta u(\frac{3}{5}) = \frac{1}{80}\frac{|u(\frac{3}{5})|}{16+|u(\frac{3}{5})|}, \qquad \Delta u'(\frac{3}{5}) = \frac{1}{300}\frac{|u(\frac{3}{5})|}{25+|u(\frac{3}{5})|}, \\ u(0) - u'(0) = 0.03, \qquad u(1) + u'(1) = 0.06, \end{cases} \tag{4.2}$$

where

$$f\bigl(t,u(t),Ku(t),Hu(t)\bigr) = \frac{\sin t}{(t+4)^2}\frac{|u(t)|}{1+|u(t)|} + \frac{1}{25}\int_0^t \frac{e^{-(s-t)}}{5}u(s)\,ds + \frac{1}{25}\int_0^1 \frac{e^{-(s-t)/2}}{5}u(s)\,ds,$$

$q = \frac{3}{2}$, $\lambda = \frac{1}{40}$, $a = 1$, $b = 1$, $c = 1$, $d = 1$, $m = 1$, $\delta = ac + ad + bc = 3$, $k(t,s) = \frac{e^{-(s-t)}}{5}$, $h(t,s) = \frac{e^{-(s-t)/2}}{5}$, $x_0 = 0.03$, $x_1 = 0.06$. Clearly $\gamma_1 = \frac{1}{16}$, $\gamma_2 = \frac{1}{25}$, $\gamma_3 = \frac{1}{25}$, $\gamma_4 = \frac{1}{80}$, $\gamma_5 = \frac{1}{300}$, $k^* = \frac{e-1}{5}$, $h^* = \frac{2(\sqrt{e}-1)}{5}$.

Moreover, we have

$$\Lambda = \frac{(m+1)[c(a+b)+\delta]}{\delta\Gamma(q+1)}\bigl(\gamma_1 + \gamma_2 k^* + \gamma_3 h^* + \lambda\bigr) + \frac{(2m-1)[c(a+b)+\delta] + md(a+b)}{\delta}\gamma_5 + \frac{m[c(a+b)+\delta]}{\delta}\gamma_4$$

$$
\begin{aligned}
&+\frac{(2m-1)[c(a+b)+\delta]+(m+1)(a+b)d}{\delta\Gamma(q)}\left(\gamma_1+\gamma_2 k^*+\gamma_3 h^*+\lambda\right)\\
&\approx 0.68638575<1.
\end{aligned}
$$

Thus, all the assumptions of Theorem 3.2 are satisfied. Hence, by the conclusion of Theorem 3.2, the impulsive fractional boundary value problem (4.2) has a unique solution on J.

5 Conclusions

In this paper, the boundary value problem of semilinear impulsive integro-differential equations of fractional order $q \in (1,2]$ have been investigated. Based on the Altman fixed point theorem and a standard fixed point theorem, the existence and uniqueness of solutions for the boundary value problem of semilinear impulsive integro-differential equations of fractional order $q \in (1,2]$ are presented. Two examples are presented to demonstrate the effectiveness and feasibility of the proposed scheme.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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