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Solving second-order fuzzy differential equations by the fuzzy Laplace transform method

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Abstract
In this paper, we study the fuzzy Laplace transforms introduced by the authors in (Allahviranloo and Ahmadi in Soft Comput. 14:235-243, 2010) to solve only first-order fuzzy linear differential equations. We extend and use this method to solve second-order fuzzy linear differential equations under generalized Hukuhara differentiability.

Keywords: fuzzy differential equation; fuzzy second-order differential equation; fuzzy Laplace transform; generalized Hukuhara differentiability

1 Introduction

In recent years, the theory of FDEs has attracted widespread attention and has been rapidly growing. It was massively studied by several authors (see [1–4]). Concerning the solutions of this kind of equations, some numerical methods and algorithms have been developed by other researchers (see [5–9]).

Allahviranloo *et al.* proposed in [7] a novel method for solving fuzzy linear differential equations, which its construction based on the equivalent integral forms of original problems under the assumption of strongly generalized differentiability. But their method was limited to solving only fuzzy linear differential equations with crisp constant coefficients, and its main result was formal and lacks proof.

Motivated by their work, we have developed and extended in 2015 (see [9]) this operator method to solve some first-order fuzzy linear differential equations, with variable coefficients. Moreover, we gave the general formula's solution with necessary proofs.

Before, in 2010, Allahviranloo and Ahmadi introduced in [10] the fuzzy Laplace transform, which they used under the strongly generalized differentiability, in an analytic solution method for some first-order fuzzy differential equations (FDEs).

In their main result the authors established the relation between the fuzzy Laplace transforms of a fuzzy function and its first derivative.

They gave two numerical examples to illustrate the efficiency of the method, but these two examples are all first-order FDEs.

The aim of this work is to develop their method and to extend their main result by establishing the relationship between the fuzzy Laplace transforms of a function and its second derivative, with the purpose of solving second-order fuzzy linear differential equations under strongly generalized differentiability.

The remainder of this paper is organized as follows:

In Section 2, which is reserved for some preliminaries, we collect some useful results on fuzzy derivation and integration. In Section 3, we first introduce the fuzzy Laplace transform, we recall its basic properties. Then we announce and prove our main result. In Section 4, we propose the procedure for solving second-order FDEs by the fuzzy Laplace transform. For illustration, we give some numerical examples in Section 5. In the last section, we present our conclusion and a further research topic.

2 Preliminaries

By $P_K(\mathbb{R})$ we denote the family of all nonempty, compact, and convex subsets of $\mathbb{R}$ and define the addition and scalar multiplication in $P_K(\mathbb{R})$ as usual. Denote

$$E = \{u : \mathbb{R} \to [0,1] | u \text{ satisfies (i)-(iv) below}\},$$

where

(i) u is normal, *i.e.* $\exists x_0 \in \mathbb{R}$ for which $u(x_0) = 1$,

(ii) u is fuzzy convex, *i.e.*

$$u(\lambda x + (1 - \lambda y)) \geq \min(u(x), u(y)) \quad \text{for any } x, y \in \mathbb{R}, \text{ and } \lambda \in [0,1],$$

(iii) u is upper semi-continuous,

(iv) $\operatorname{supp} u = \{x \in \mathbb{R} | u(x) > 0\}$ is the support of the u, and its closure cl ($\operatorname{supp} u$) is compact.

For $0 < \alpha \leq 1$, denote

$$[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}.$$

Then, from (i)-(iv), it follows that the α-level set $[u]^\alpha \in P_K(\mathbb{R})$ for all $0 \leq \alpha \leq 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy-number space E as usual.

It is well known that the following properties are true for all levels:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha.$$

Let $D : E \times E \to [0, \infty)$ be a function which is defined by the equation

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_K(\mathbb{R})$. Then it is easy to see that D is a metric in E and has the following properties [11]:

(1) (E, D) is a complete metric space;

(2) $D(u + w, v + w) = D(u, v)$ for all $u, v, w \in E$;

(3) $D(ku, kv) = |k|D(u, v)$ for all $u, v \in E$ and $k \in \mathbb{R}$;

(4) $D(u + w, v + t) \leq D(u, v) + D(w, t)$ for all $u, v, w, t \in E$.

Definition 2.1 A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$, $\overline{u}(r)$, $0 \leq r \leq 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded non-decreasing left continuous function in $(0,1]$, and right continuous at 0;

(2) $\overline{u}(r)$ is a bounded non-increasing left continuous function in $(0,1]$, and right continuous at 0;

(3) $\underline{u}(r) \leq \overline{u}(r)$ for all $0 \leq r \leq 1$.

A crisp number k is simply represented by $\underline{u}(r) = \overline{u}(r) = k$, $0 \leq r \leq 1$.

Let $T = [c,d] \subset \mathbb{R}$ be a compact interval.

Definition 2.2 A mapping $F : T \to E$ is strongly measurable if for all $\alpha \in [0,1]$ the set-valued function $F_\alpha : T \to \mathcal{P}_K(\mathbb{R})$ defined by $F_\alpha(t) = [F(t)]^\alpha$ is Lebesgue measurable.

A mapping $F : T \to E$ is called integrably bounded if there exists an integrable function k such that $\|x\| \leq k(t)$ for all $x \in F_0(t)$.

Definition 2.3 Let $F : T \to E$, then the integral of F over T, denoted by $\int_T F(t)\,dt$ or $\int_c^d F(t)\,dt$, is defined by the equation

$$\left[\int_T F(t)\,dt\right]^\alpha = \int_T F_\alpha(t)\,dt; \quad \alpha \in]0,1]$$

i.e.

$$\left[\int_T F(t)\,dt\right]^\alpha = \left\{\int_T f(t)\,dt | f : T \to \mathbb{R} \text{ is a measurable selection for } F_\alpha\right\}.$$

Also, a strongly measurable and integrably bounded mapping $F : T \to E$ is said to be integrable over T if $\int_T F(t)\,dt \in E$.

Proposition 2.4 (Aumann [12]) *If $F : T \to E$ is strongly measurable and integrably bounded, then F is integrable.*

For more measurability, integrability properties for fuzzy set-valued mappings see [2, 8, 13].

Theorem 2.5 (see [14, 15]) *Let $f(x)$ be a fuzzy valued-function on $[a,\infty[$ which is represented by $(\underline{f}(x,r), \overline{f}(x,r))$. For any fixed $r \in [0,1]$, assume $\underline{f}(x,r)$, $\overline{f}(x,r)$ are Riemann integrable on $[a,b]$ for every $b \geq a$, and assume there are two positive constants $\underline{M}(r)$ and $\overline{M}(r)$ such that $\int_a^b |\underline{f}(x,r)|\,dx \leq \underline{M}(r)$ and $\int_a^b |\overline{f}(x,r)|\,dx \leq \overline{M}(r)$ for every $b \geq a$. Then $f(x)$ is improper fuzzy Riemann integrable on $[a,\infty[$ and the improper fuzzy Riemann integral is a fuzzy number. Furthermore, we have*

$$\int_a^\infty f(x)\,dx = \left(\int_a^\infty \underline{f}(x,r)\,dx, \int_a^\infty \overline{f}(x,r)\,dx\right).$$

Proposition 2.6 (see [14]) *If each of $f(x)$ and $g(x)$ is a fuzzy valued function and fuzzy Riemann integrable on $[a,\infty[$ then $f(x) + g(x)$ is fuzzy Riemann integrable on $[a,\infty[$. Moreover,* *we have*

$$\int_a^\infty \big(f(x)+g(x)\big)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx.$$

For $u, v \in E$, if there exists $w \in E$ such that $u = v + w$, then w is the Hukuhara difference of u and v denoted by $u \ominus v$.

Definition 2.7 We say that a mapping $f : (a,b) \to E$ is strongly generalized differentiable at $x_0 \in (a,b)$ if there exists an element $f'(x_0) \in E$ such that

(i) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$, and the limits

$$\lim_{h\to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0)$$

or

(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$, $f(x_0-h) \ominus f(x_0)$, and the limits

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0)$$

or

(iii) for all $h > 0$ sufficiently small, there exist $f(x_0+h) \ominus f(x_0)$, $f(x_0-h) \ominus f(x_0)$, and the limits

$$\lim_{h\to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(x_0-h) \ominus f(x_0)}{(-h)} = f'(x_0)$$

or

(iv) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0+h)$, $f(x_0) \ominus f(x_0-h)$, and the limits

$$\lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0+h)}{(-h)} = \lim_{h\to 0^+} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0).$$

The following theorem (see [13]) allows us to consider case (i) or (ii) of the previous definition almost everywhere in the domain of the functions under discussion.

Theorem 2.8 *Let $f : (a,b) \to E$ be strongly generalized differentiable on each point $x \in (a,b)$ in the sense of Definition* 2.3, (iii) *or* (iv). *Then $f'(x) \in \mathbb{R}$ for all $x \in (a,b)$.*

Theorem 2.9 (see *e.g.* [16]) *Let $f : \mathbb{R} \to E$ be a function and denote $f(t) = (\underline{f}(t,r), \overline{f}(t,r))$, for each $r \in [0,1]$. Then*

(1) *If f is* (i)*-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\underline{f}'(t,r), \overline{f}'(t,r))$.*

(2) *If f is* (ii)*-differentiable, then $\underline{f}(t,r)$ and $\overline{f}(t,r)$ are differentiable functions and $f'(t) = (\overline{f}'(t,r), \underline{f}'(t,r))$.*

Definition 2.10 We say that a mapping $f : (a,b) \to E$ is strongly generalized differentiable of the second-order at $x_0 \in (a,b)$; if there exists an element $f''(x_0) \in E$ such that

(i) for all $h > 0$ sufficiently small, there exist $f'(x_0 + h) \ominus f'(x_0)$, $f'(x_0) \ominus f'(x_0 - h)$, and the limits

$$\lim_{h \to 0^+} \frac{f'(x_0 + h) \ominus f'(x_0)}{h} = \lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 - h)}{h} = f''(x_0)$$

or

(ii) for all $h > 0$ sufficiently small, there exist $f'(x_0) \ominus f'(x_0 + h)$, $f'(x_0 - h) \ominus f'(x_0)$, and the limits

$$\lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f'(x_0 - h) \ominus f'(x_0)}{(-h)} = f''(x_0)$$

or

(iii) for all $h > 0$ sufficiently small, there exist $f'(x_0 + h) \ominus f'(x_0)$, $f'(x_0 - h) \ominus f'(x_0)$, and the limits

$$\lim_{h \to 0^+} \frac{f'(x_0 + h) \ominus f'(x_0)}{h} = \lim_{h \to 0^+} \frac{f'(x_0 - h) \ominus f'(x_0)}{(-h)} = f''(x_0)$$

or

(iv) for all $h > 0$ sufficiently small, there exist $f'(x_0) \ominus f'(x_0 + h)$, $f'(x_0) \ominus f'(x_0 - h)$, and the limits

$$\lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{f'(x_0) \ominus f'(x_0 - h)}{h} = f''(x_0).$$

All the limits are taken in the metric space (E, D), and at the end points of (a, b) and we consider only one-sided derivatives.

3 Fuzzy Laplace transform

Definition 3.1 (see [10]) Let $f(x)$ be a continuous fuzzy-valued function. Suppose that $e^{-px}f(x)$ is improper fuzzy Riemann integrable on $[0, \infty[$, then $\int_0^\infty e^{-px} f(x)\,dx$ is called the fuzzy Laplace transform of f and is denoted

$$\mathbf{L}\bigl[f(x)\bigr] = \int_0^\infty e^{-px} f(x)\,dx, \quad p > 0.$$

Denote by $\mathcal{L}(g(x))$ the classical Laplace transform of a crisp function $g(x)$.
Since $\int_0^\infty e^{-px} f(x)\,dx = (\int_0^\infty e^{-px} \underline{f}(x, r)\,dx, \int_0^\infty e^{-px} \overline{f}(x, r)\,dx)$, then

$$\mathbf{L}\bigl[f(x)\bigr] = \bigl(\mathcal{L}\bigl(\underline{f}(x, r)\bigr), \mathcal{L}\bigl(\underline{f}(x, r)\bigr)\bigr).$$

Theorem 3.2 (see [10]) *Let $f'(x)$ be an integrable fuzzy-valued function and $f(x)$ the primitive of $f'(x)$ on $[0, \infty[$. Then*

(a) *if f is* (i)*-differentiable*:

$$\mathbf{L}\bigl[f'(x)\bigr] = p\mathbf{L}\bigl[f(x)\bigr] \ominus f(0) \tag{1}$$

(b) *or if f is* (ii)*-differentiable*:

$$\mathbf{L}\bigl[f'(x)\bigr] = \bigl(-f(0)\bigr) \ominus (-p)\mathbf{L}\bigl[f(x)\bigr]. \tag{2}$$

Theorem 3.3 (see [10]) *Let $f(x), g(x)$ be continuous fuzzy-valued functions and c_1, c_2 two real constants, then*

$$\mathbf{L}\big[c_1 f(x) + c_2 g(x)\big] = c_1 \mathbf{L}\big[f(x)\big] + c_2 \mathbf{L}\big[g(x)\big].$$

Now, we present our main result about the relation between the Laplace transforms of a function and its second derivative.

Theorem 3.4 *Let $f(x)$ be a continuous fuzzy-valued function such that $e^{-px}f(x)$, $e^{-px}f'(x)$, and $e^{-px}f''(x)$ exist, are continuous, and Riemann integrable on $[0,\infty[$. We distinguish the following cases*:

(a) *If $f(x)$ and $f'(x)$ are* (i)*-differentiable, then*

$$\mathbf{L}\big[f''(x)\big] = \big\{p^2\mathbf{L}\big[f(x)\big] \ominus pf(0)\big\} \ominus f'(0). \tag{3}$$

(b) *If $f(x)$ is* (i)*-differentiable and $f'(x)$ is* (ii)*-differentiable, then*

$$\mathbf{L}\big[f''(x)\big] = \big(-f'(0)\big) \ominus \big\{-p^2\mathbf{L}\big[f(x)\big] \ominus \big(-pf(0)\big)\big\}. \tag{4}$$

(c) *If $f(x)$ is* (ii)*-differentiable and $f'(x)$ is* (i)*-differentiable, then*

$$\mathbf{L}\big[f''(x)\big] = \big\{\big(-pf(0)\big) \ominus \big(-p^2\mathbf{L}\big[f(x)\big]\big)\big\} \ominus f'(0). \tag{5}$$

(d) *If $f(x)$ is* (ii)*-differentiable and $f'(x)$ is* (ii)*-differentiable, then*

$$\mathbf{L}\big[f''(x)\big] = \big(-f'(0)\big) \ominus \big\{pf(0) \ominus p^2\mathbf{L}\big[f(x)\big]\big\}. \tag{6}$$

Proof

(a) Assume that $f(x)$ and $f'(x)$ are (i)-differentiable, then applying (1) to $f(x)$ and $f'(x)$, respectively, we get

$$\mathbf{L}\big[f'(x)\big] = p\mathbf{L}\big[f(x)\big] \ominus f(0) \quad \text{and} \quad \mathbf{L}\big[f''(x)\big] = p\mathbf{L}\big[f'(x)\big] \ominus f'(0).$$

Combining these identities yields

$$\begin{aligned}\mathbf{L}\big[f''(x)\big] &= p\big\{p\mathbf{L}\big[f(x)\big] \ominus f(0)\big\} \ominus f'(0)\\ &= \big\{p^2\mathbf{L}\big[f(x)\big] \ominus pf(0)\big\} \ominus f'(0).\end{aligned}$$

(b) Assume that $f(x)$ is (i)-differentiable and $f'(x)$ is (ii)-differentiable, then applying (1) and (2) to $f(x)$ and $f'(x)$, respectively, we get

$$\mathbf{L}\big[f'(x)\big] = p\mathbf{L}\big[f(x)\big] \ominus f(0) \quad \text{and} \quad \mathbf{L}\big[f''(x)\big] = \big(-f'(0)\big) \ominus (-p)\mathbf{L}\big[f'(x)\big].$$

By combination of these identities we get

$$\begin{aligned}\mathbf{L}\big[f''(x)\big] &= \big(-f'(0)\big) \ominus (-p)\big\{p\mathbf{L}\big[f(x)\big] \ominus f(0)\big\}\\ &= \big(-f'(0)\big) \ominus \big\{-p^2\mathbf{L}\big[f(x)\big] \ominus \big(-pf(0)\big)\big\}.\end{aligned}$$

(c) If $f(x)$ is (ii)-differentiable and $f'(x)$ is (i)-differentiable, then

$$\mathbf{L}\big[f'(x)\big] = \big(-f(0)\big) \ominus (-p)\mathbf{L}\big[f(x)\big] \quad \text{and} \quad \mathbf{L}\big[f''(x)\big] = p\mathbf{L}\big[f'(x)\big] \ominus f'(0).$$

By combination of these identities we get

$$\begin{aligned}\mathbf{L}\big[f''(x)\big] &= p\big\{\big(-f(0)\big) \ominus (-p)\mathbf{L}\big[f(x)\big]\big\} \ominus f'(0)\\ &= \big\{\big(-pf(0)\big) \ominus \big(-p^2\big)\mathbf{L}\big[f(x)\big]\big\} \ominus f'(0).\end{aligned}$$

(d) Assume that $f(x)$ and $f'(x)$ are (ii)-differentiable, then

$$\mathbf{L}\big[f'(x)\big] = \big(-f(0)\big) \ominus (-p)\mathbf{L}\big[f(x)\big] \quad \text{and} \quad \mathbf{L}\big[f''(x)\big] = \big(-f'(0)\big) \ominus (-p)\mathbf{L}\big[f'(x)\big].$$

Combining these identities yields

$$\begin{aligned}\mathbf{L}\big[f''(x)\big] &= \big(-f'(0)\big) \ominus (-p)\big\{\big(-f(0)\big) \ominus (-p)\mathbf{L}\big[f(x)\big]\big\}\\ &= \big(-f'(0)\big) \ominus \big\{pf(0) \ominus p^2\mathbf{L}\big[f(x)\big]\big\}.\end{aligned}$$

□

4 Fuzzy Laplace transform algorithm for second-order fuzzy differential equations

Our aim now is to solve the following second-order fuzzy differential equation, using the fuzzy Laplace transform method under strongly generalized differentiability:

$$\begin{cases} y''(t) = f(t, y(t), y'(t)),\\ y(0) = y_0 = (\underline{y}_0, \overline{y}_0) \in E,\\ y'(0) = z_0 = (\underline{z}_0, \overline{z}_0) \in E, \end{cases} \tag{7}$$

where $y(t) = (\underline{y}(t,\alpha), \overline{y}(t,\alpha))$ is a fuzzy function of $t \geq 0$ and $f(t, y(t), y'(t))$ is a fuzzy-valued function, which is linear with respect to $(y(t), y'(t))$.

By using the fuzzy Laplace transform, we obtain

$$\mathbf{L}\big[y''(t)\big] = \mathbf{L}\big[f\big(t, y(t), y'(t)\big)\big]. \tag{8}$$

Then we have the following alternatives for solving (8):

(a) Case I: If y and y' are (i)-differentiable: $y'(t) = (\underline{y}'(t,\alpha), \overline{y}'(t,\alpha))$ and $y''(t) = (\underline{y}''(t,\alpha), \overline{y}''(t,\alpha))$ and

$$\mathbf{L}\big[y''(t)\big] = \big\{p^2\mathbf{L}\big[y(t)\big] \ominus py(0)\big\} \ominus y'(0).$$

Therefore

$$\mathbf{L}\big[f\big(t, y(t), y'(t)\big)\big] = \big\{p^2\mathbf{L}\big[y(t)\big] \ominus py_0\big\} \ominus z_0.$$

Hence

$$\begin{cases} \mathcal{L}[\underline{f}(t, y(t), y'(t), \alpha)] = p^2\mathcal{L}[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha),\\ \mathcal{L}[\overline{f}(t, y(t), y'(t), \alpha)] = p^2\mathcal{L}[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha), \end{cases} \tag{9}$$

where $\underline{f}(t,y(t),y'(t),\alpha) = \min\{f(t,u,v)/u \in (\underline{y}(t,\alpha),\overline{y}(t,\alpha)); v \in (\underline{y}'(t,\alpha),\overline{y}'(t,\alpha))\}$ and $\overline{f}(t,y(t),y'(t),\alpha) = \max\{f(t,u,v)/u \in (\underline{y}(t,\alpha),\overline{y}(t,\alpha)); v \in (\underline{y}'(t,\alpha),\overline{y}'(t,\alpha))\}$. Assume that this leads to

$$\begin{cases} \mathcal{L}[\underline{y}(t,\alpha)] = H_1(p,\alpha), \\ \mathcal{L}[\overline{y}(t,\alpha)] = K_1(p,\alpha), \end{cases}$$

where the couple $(H_1(p,\alpha),K_1(p,\alpha))$ is a solution of the system (9).

By using the inverse Laplace transform we get

$$\begin{cases} \underline{y}(t,\alpha) = \mathcal{L}^{-1}[H_1(p,\alpha)], \\ \overline{y}(t,\alpha) = \mathcal{L}^{-1}[K_1(p,\alpha)]. \end{cases}$$

(b) Case II: If y is (i)-differentiable and y' is (ii)-differentiable: $y'(t) = (\underline{y}'(t,\alpha),\overline{y}'(t,\alpha))$ and $y''(t) = (\overline{y}''(t,\alpha),\underline{y}''(t,\alpha))$ and

$$\mathbf{L}\bigl[y''(t)\bigr] = \bigl(-y'(0)\bigr) \ominus \bigl\{-p^2\mathbf{L}\bigl[y(t)\bigr] \ominus \bigl(-py(0)\bigr)\bigr\}.$$

Therefore

$$\mathbf{L}\bigl[f\bigl(t,y(t),y'(t)\bigr)\bigr] = (-z_0) \ominus \bigl\{-p^2\mathbf{L}\bigl[y(t)\bigr] \ominus (-py_0)\bigr\}.$$

Hence

$$\begin{cases} \mathcal{L}[\overline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha), \\ \mathcal{L}[\underline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha). \end{cases} \tag{10}$$

Assume that this implies

$$\begin{cases} \mathcal{L}[\underline{y}(t,\alpha)] = H_2(p,\alpha), \\ \mathcal{L}[\overline{y}(t,\alpha)] = K_2(p,\alpha), \end{cases}$$

where $(H_2(p,\alpha),K_2(p,\alpha))$ is a solution of the system (10).

By using the inverse Laplace transform we get

$$\begin{cases} \underline{y}(t,\alpha) = \mathcal{L}^{-1}[H_2(p,\alpha)], \\ \overline{y}(t,\alpha) = \mathcal{L}^{-1}[K_2(p,\alpha)]. \end{cases}$$

(c) Case III: If y is (ii)-differentiable and y' is (i)-differentiable: $y'(t) = (\overline{y}'(t,\alpha),\underline{y}'(t,\alpha))$ and $y''(t) = (\overline{y}''(t,\alpha),\underline{y}''(t,\alpha))$ and

$$\mathbf{L}\bigl[y''(t)\bigr] = \bigl\{\bigl(-py(0)\bigr) \ominus \bigl(-p^2\mathbf{L}\bigl[y(t)\bigr]\bigr)\bigr\} \ominus y'(0).$$

Therefore

$$\mathbf{L}\bigl[f\bigl(t,y(t),y'(t)\bigr)\bigr] = \bigl\{\bigl(-py(0)\bigr) \ominus \bigl(-p^2\mathbf{L}\bigl[y(t)\bigr]\bigr)\bigr\} \ominus y'(0).$$

Hence

$$\begin{cases} \mathcal{L}[\overline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha), \\ \mathcal{L}[\underline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha). \end{cases} \tag{11}$$

Assume that this leads to

$$\begin{cases} \mathcal{L}[\underline{y}(t,\alpha)] = H_3(p,\alpha), \\ \mathcal{L}[\overline{y}(t,\alpha)] = K_3(p,\alpha), \end{cases}$$

where $(H_3(p,\alpha), K_3(p,\alpha))$ is a solution of the system (11).

By using the inverse Laplace transform we get

$$\begin{cases} \underline{y}(t,\alpha) = \mathcal{L}^{-1}[H_3(p,\alpha)], \\ \overline{y}(t,\alpha) = \mathcal{L}^{-1}[K_3(p,\alpha)]. \end{cases}$$

(d) Case IV: If y and y' are (ii)-differentiable: $y'(t) = (\overline{y}'(t,\alpha), \underline{y}'(t,\alpha))$ and $y''(t) = (\underline{y}''(t,\alpha), \overline{y}''(t,\alpha))$ and

$$\mathbf{L}\big[y''(t)\big] = \big(-y'(0)\big) \ominus \big\{py(0) \ominus p^2\mathbf{L}\big[y(t)\big]\big\}.$$

Therefore

$$\mathbf{L}\big[f\big(t,y(t),y'(t)\big)\big] = \big(-y'(0)\big) \ominus \big\{py(0) \ominus p^2\mathbf{L}\big[y(t)\big]\big\}.$$

Hence

$$\begin{cases} \mathcal{L}[\underline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \overline{z}_0(\alpha), \\ \mathcal{L}[\overline{f}(t,y(t),y'(t),\alpha)] = p^2\mathcal{L}[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \underline{z}_0(\alpha). \end{cases} \tag{12}$$

Assume that this implies

$$\begin{cases} \mathcal{L}[\underline{y}(t,\alpha)] = H_4(p,\alpha), \\ \mathcal{L}[\overline{y}(t,\alpha)] = K_4(p,\alpha), \end{cases}$$

where $(H_4(p,\alpha), K_4(p,\alpha))$ is a solution of the system (12).

By using the inverse Laplace transform we get

$$\begin{cases} \underline{y}(t,\alpha) = \mathcal{L}^{-1}[H_4(p,\alpha)], \\ \overline{y}(t,\alpha) = \mathcal{L}^{-1}[K_4(p,\alpha)]. \end{cases}$$

Remark 4.1 To show that $H_1(p,\alpha)$ and $K_1(p,\alpha)$ can easily be calculated, we suppose that the fuzzy linear function f is given by $f(t,y(t),y'(t)) = ay(t) + by'(t) + c(t)$, where a, b are real constants and $c(t)$ is a crisp mapping.

We have to discuss the following cases:

(1) Case (I.1): If $a \geq 0$ and $b \geq 0$, then the system (9) is equivalent to

$$\begin{cases} p^2\mathcal{L}[\underline{y}(t,\alpha)] - p\underline{y}_0(\alpha) - \underline{z}_0(\alpha) = (a+bp)\mathcal{L}[\underline{y}(t,\alpha)] - b\underline{y}_0(\alpha) + \mathcal{L}[c(t)], \\ p^2\mathcal{L}[\overline{y}(t,\alpha)] - p\overline{y}_0(\alpha) - \overline{z}_0(\alpha) = (a+bp)\mathcal{L}[\overline{y}(t,\alpha)] - b\overline{y}_0(\alpha) + \mathcal{L}[c(t)]. \end{cases}$$

By consequence

$$\begin{cases} H_1(p,\alpha) = \mathcal{L}[\underline{y}(t,\alpha)] = \frac{(p-b)\underline{y}_0(\alpha)+\underline{z}_0(\alpha)+\mathcal{L}[c(t)]}{p^2-bp-a}, \\ K_1(p,\alpha) = \mathcal{L}[\overline{y}(t,\alpha)] = \frac{(p-b)\overline{y}_0(\alpha)+\overline{z}_0(\alpha)+\mathcal{L}[c(t)]}{p^2-bp-a}. \end{cases}$$

(2) Case (I.2): If $a \geq 0$ and $b < 0$, then (9) is equivalent to the following system:

$$\begin{cases} (p^2-a)\mathcal{L}[\underline{y}(t,\alpha)] - bp\mathcal{L}[\overline{y}(t,\alpha)] = p\underline{y}_0(\alpha) + \underline{z}_0(\alpha) - b\overline{y}_0(\alpha) + \mathcal{L}[c(t)], \\ bp\mathcal{L}[\underline{y}(t,\alpha)] + (p^2-a)\mathcal{L}[\overline{y}(t,\alpha)] = p\overline{y}_0(\alpha) + \overline{z}_0(\alpha) - b\underline{y}_0(\alpha) + \mathcal{L}[c(t)]. \end{cases}$$

Denote

$$\begin{cases} B(p,\alpha) = p\underline{y}_0(\alpha) + \underline{z}_0(\alpha) - b\overline{y}_0(\alpha) + \mathcal{L}[c(t)], \\ C(p,\alpha) = p\overline{y}_0(\alpha) + \overline{z}_0(\alpha) - b\underline{y}_0(\alpha) + \mathcal{L}[c(t)]. \end{cases} \tag{13}$$

Hence

$$\begin{cases} H_1(p,\alpha) = \mathcal{L}[\underline{y}(t,\alpha)] = \frac{(p^2-a)B(p,\alpha)+bpC(p,\alpha)}{(p^2-a)^2+(bp)^2}, \\ K_1(p,\alpha) = \mathcal{L}[\overline{y}(t,\alpha)] = \frac{(p^2-a)C(p,\alpha)-bpB(p,\alpha)}{(p^2-a)^2+(bp)^2}. \end{cases}$$

(3) Case (I.3): If $a < 0$ and $b \geq 0$, then (9) is equivalent to the following system:

$$\begin{cases} (p^2-bp)\mathcal{L}[\underline{y}(t,\alpha)] - a\mathcal{L}[\overline{y}(t,\alpha)] = p\underline{y}_0(\alpha) + \underline{z}_0(\alpha) - b\overline{y}_0(\alpha) + \mathcal{L}[c(t)], \\ -a\mathcal{L}[\underline{y}(t,\alpha)] + (p^2-bp)\mathcal{L}[\overline{y}(t,\alpha)] = p\overline{y}_0(\alpha) + \overline{z}_0(\alpha) - b\underline{y}_0(\alpha) + \mathcal{L}[c(t)]. \end{cases}$$

Therefore

$$\begin{cases} H_1(p,\alpha) = \mathcal{L}[\underline{y}(t,\alpha)] = \frac{(p^2-bp)B(p,\alpha)+aC(p,\alpha)}{(p^2-bp)^2+a^2}, \\ K_1(p,\alpha) = \mathcal{L}[\overline{y}(t,\alpha)] = \frac{(p^2-a)C(p,\alpha)+aB(p,\alpha)}{(p^2-bp)^2+a^2}. \end{cases}$$

(4) Case (I.4): If $a < 0$ and $b < 0$, then (9) is equivalent to the following system:

$$\begin{cases} p^2\mathcal{L}[\underline{y}(t,\alpha)] - (a+bp)\mathcal{L}[\overline{y}(t,\alpha)] = p\underline{y}_0(\alpha) + \underline{z}_0(\alpha) - b\overline{y}_0(\alpha) + \mathcal{L}[c(t)], \\ -(a+bp)\mathcal{L}[\underline{y}(t,\alpha)] + p^2\mathcal{L}[\overline{y}(t,\alpha)] = p\overline{y}_0(\alpha) + \overline{z}_0(\alpha) - b\underline{y}_0(\alpha) + \mathcal{L}[c(t)]. \end{cases}$$

Therefore

$$\begin{cases} H_1(p,\alpha) = \mathcal{L}[\underline{y}(t,\alpha)] = \frac{p^2B(p,\alpha)+(a+bp)C(p,\alpha)}{p^4+(a+bp)^2}, \\ K_1(p,\alpha) = \mathcal{L}[\overline{y}(t,\alpha)] = \frac{p^2C(p,\alpha)+(a+bp)B(p,\alpha)}{p^4+(a+bp)^2}. \end{cases}$$

Here $B(p,\alpha)$ and $C(p,\alpha)$ are given by (13).

Similarly, the respective expressions of $H_2(p,\alpha)$, $K_2(p,\alpha)$, $H_3(p,\alpha)$, $K_3(p,\alpha)$, $H_4(p,\alpha)$, $K_4(p,\alpha)$ can be computed.

5 Numerical examples

The following first example was studied in [8] using the fuzzy double integral method:

Example 1

$$\begin{cases} y''(x) + y(x) = \sigma_0, \quad \sigma_0 = (\alpha, 2-\alpha), \\ y(0,\alpha) = (\alpha-1, 1-\alpha), \\ y'(0,\alpha) = (\alpha-1, 1-\alpha). \end{cases} \tag{14}$$

- Case I: If $y(x)$ and $y'(x)$ are (i)-differentiable, then

$$\begin{cases} \underline{y}''(x,\alpha) + \underline{y}(x,\alpha) = \alpha, \\ \overline{y}''(x,\alpha) + \overline{y}(x,\alpha) = 2-\alpha. \end{cases}$$

  Therefore

$$\begin{cases} \mathcal{L}[\underline{y}''(x,\alpha)] + \mathcal{L}[\underline{y}(x,\alpha)] = \frac{\alpha}{p}, \\ \mathcal{L}[\overline{y}''(x,\alpha)] + \mathcal{L}[\overline{y}(x,\alpha)] = \frac{2-\alpha}{p}. \end{cases}$$

  Using Theorem 3.4, we get

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = (\alpha-1)\frac{p+1}{p^2+1} + \alpha(\frac{1}{p} - \frac{p}{p^2+1}), \\ \mathcal{L}[\overline{y}(x,\alpha)] = (1-\alpha)\frac{p+1}{p^2+1} + (2-\alpha)(\frac{1}{p} - \frac{p}{p^2+1}). \end{cases}$$

  By the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha(1+\sin(x)) - \sin(x) - \cos(x), \\ \overline{y}(x,\alpha) = (2-\alpha)(1+\sin(x)) - \sin(x) - \cos(x). \end{cases}$$

  In this case, no solution exists, since $y'(x)$ is not an (i)-differentiable fuzzy-valued function (see [8]).

- Case II: If $y(x)$ is (i)-differentiable and $y'(x)$ is (ii)-differentiable, then

$$\begin{cases} \mathcal{L}[\overline{y}''(x,\alpha)] + \mathcal{L}[\underline{y}(x,\alpha)] = \frac{\alpha}{p}, \\ \mathcal{L}[\underline{y}''(x,\alpha)] + \mathcal{L}[\overline{y}(x,\alpha)] = \frac{2-\alpha}{p}. \end{cases}$$

  Using Theorem 3.4, we get

$$\begin{cases} p^2\mathcal{L}[\overline{y}(x,\alpha)] + \mathcal{L}[\underline{y}(x,\alpha)] = (1-\alpha)(p+1) + \frac{\alpha}{p}, \\ \mathcal{L}[\overline{y}(x,\alpha)] + p^2\mathcal{L}[\underline{y}(x,\alpha)] = (\alpha-1)(p+1) + \frac{2-\alpha}{p}. \end{cases}$$

Thus

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = \alpha(\frac{1}{2(p-1)} - \frac{1}{2(p+1)} + \frac{1}{p}) + \frac{1}{2(p+1)} - \frac{1}{2(p-1)} - \frac{p}{p^2+1}, \\ \mathcal{L}[\overline{y}(x,\alpha)] = \alpha(\frac{1}{2(p+1)} - \frac{1}{2(p-1)} - \frac{1}{p}) + \frac{2}{p} + \frac{1}{2(p-1)} - \frac{1}{2(p+1)} - \frac{p}{p^2+1}. \end{cases}$$

By the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha(1 + \sinh(x)) - \sinh(x) - \cos(x), \\ \overline{y}(x,\alpha) = (2-\alpha)(1 + \sinh(x)) - \sinh(x) - \cos(x). \end{cases}$$

As in case I, no solution exists (see [8]).

- Case III: If $y(x)$ is (ii)-differentiable and $y'(x)$ is (i)-differentiable, then

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = \alpha(\frac{1}{2(p+1)} - \frac{1}{2(p-1)} + \frac{1}{p}) + \frac{1}{2(p-1)} - \frac{1}{2(p+1)} - \frac{p}{p^2+1}, \\ \mathcal{L}[\overline{y}(x,\alpha)] = \alpha(\frac{1}{2(p-1)} - \frac{1}{2(p+1)} - \frac{1}{p}) + \frac{2}{p} + \frac{1}{2(p+1)} - \frac{1}{2(p-1)} - \frac{p}{p^2+1}. \end{cases}$$

By the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha(1 - \sinh(x)) + \sinh(x) - \cos(x), \\ \overline{y}(x,\alpha) = (2-\alpha)(1 - \sinh(x)) + \sinh(x) - \cos(x). \end{cases}$$

In this case, since $y(x)$ is (ii)-differentiable and $y'(x)$ is (i)-differentiable, the solution is acceptable for $x \in (0, \ln(1+\sqrt{2}))$ (see [8]).

- Case IV: If $y(x)$ and $y'(x)$ are (ii)-differentiable, then

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = (\alpha-1)(\frac{p}{p^2+1} - \frac{1}{p^2+1}) + \alpha(\frac{1}{p} - \frac{p}{p^2+1}), \\ \mathcal{L}[\overline{y}(x,\alpha)] = (1-\alpha)(\frac{p}{p^2+1} - \frac{1}{p^2+1}) + (2-\alpha)(\frac{1}{p} - \frac{p}{p^2+1}). \end{cases}$$

By the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha(1 - \sin(x)) + \sin(x) - \cos(x), \\ \overline{y}(x,\alpha) = (2-\alpha)(1 - \sin(x)) + \sin(x) - \cos(x). \end{cases}$$

In this case, the solution is acceptable for $x \in (0, \frac{\pi}{2})$ (see [8]).

Example 2 We consider the following fuzzy differential equation:

$$\begin{cases} y''(x) = y'(x) + x + 1, \\ y(0,\alpha) = (\alpha - 2, 1 - 2\alpha), \\ y'(0,\alpha) = (\alpha - 2, 1 - 2\alpha). \end{cases} \tag{15}$$

- Case I: If $y(x)$ and $y'(x)$ are (i)-differentiable, then

$$\begin{cases} \mathcal{L}[\underline{y}''(x,\alpha)] = \mathcal{L}[\underline{y}'(x)] + \mathcal{L}[t+1], \\ \mathcal{L}[\overline{y}''(x,\alpha)] = \mathcal{L}[\overline{y}'(x)] + \mathcal{L}[t+1]. \end{cases}$$

Using Theorems 3.2 and 3.4, we get

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = \frac{\alpha-2}{p-1} + \frac{p+1}{p^3(p-1)}, \\ \mathcal{L}[\overline{y}(x,\alpha)] = \frac{1-2\alpha}{p-1} + \frac{p+1}{p^3(p-1)}. \end{cases}$$

By the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha e^x - \frac{x^2}{2} - 2x - 2, \\ \overline{y}(x,\alpha) = e^x(3-2\alpha) - \frac{x^2}{2} - 2x - 2. \end{cases}$$

In this case, the solution is valid for all $x \in \mathbb{R}$.

- Case II: If $y(x)$ is (i)-differentiable and $y'(x)$ is (ii)-differentiable, then by Theorems 3.2 and 3.4 we get

$$\begin{cases} -p\mathcal{L}[\underline{y}(x,\alpha)] + p^2\mathcal{L}[\overline{y}(x,\alpha)] = (1-2\alpha)p + 3 - 3\alpha + \frac{p+1}{p^2}, \\ p^2\mathcal{L}[\underline{y}(x,\alpha)] - p\mathcal{L}[\overline{y}(x,\alpha)] = (\alpha-2)p + 3\alpha - 3 + \frac{p+1}{p^2}. \end{cases}$$

Therefore

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = \frac{\alpha-2}{p-1} + \frac{4-3\alpha}{p(p^2-1)} + \frac{2p+1}{p^3(p^2-1)}, \\ \mathcal{L}[\overline{y}(x,\alpha)] = \frac{1-2\alpha}{p-1} + \frac{3\alpha-2}{p(p^2-1)} + \frac{2p+1}{p^3(p^2-1)}. \end{cases}$$

Using the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = \alpha e^x + 3(1-\alpha)\cosh(x) - \frac{x^2}{2} - 2x - 5 + 3\alpha, \\ \overline{y}(x,\alpha) = (3-2\alpha)e^x + 3(\alpha-1)\cosh(x) - \frac{x^2}{2} - 2x + 1 - 3\alpha. \end{cases}$$

This solution is not acceptable since $\overline{y}''(x,\alpha) \neq \underline{y}'(x,\alpha) + x + 1$.

- Case III: If $y(x)$ is (ii)-differentiable and $y'(x)$ is (i)-differentiable, then by Theorems 3.2 and 3.4 we get

$$\begin{cases} \mathcal{L}[\underline{y}(x,\alpha)] = \frac{1-2\alpha}{p-1} + \frac{\alpha-2}{p(p-1)} + \frac{p+1}{p^3(p-1)}, \\ \mathcal{L}[\overline{y}(x,\alpha)] = \frac{\alpha-2}{p-1} + \frac{1-2\alpha}{p(p-1)} + \frac{p+1}{p^3(p-1)}. \end{cases}$$

Using the inverse Laplace transform we deduce

$$\begin{cases} \underline{y}(x,\alpha) = (3-2\alpha)e^x - \frac{x^2}{2} - 2x - 5 + 3\alpha, \\ \overline{y}(x,\alpha) = \alpha e^x - \frac{x^2}{2} - 2x + 1 - 3\alpha. \end{cases}$$

In this case, the solution is valid for all $x \in \mathbb{R}$.

- Case IV: If $y(x)$ and $y'(x)$ are (ii)-differentiable, then by Theorems 3.2 and 3.4 we get

$$\begin{cases} p^2\mathcal{L}[\underline{y}(x,\alpha)] - p\mathcal{L}[\overline{y}(x,\alpha)] = (\alpha-2)p + \frac{p+1}{p^2}, \\ -p\mathcal{L}[\underline{y}(x,\alpha)] + p^2\mathcal{L}[\overline{y}(x,\alpha)] = (1-2\alpha)p + \frac{p+1}{p^2}. \end{cases}$$

By solving this linear system and using the inverse Laplace transform, we get

$$\begin{cases} \underline{y}(x,\alpha) = (\alpha-1)\cosh(x) + (1-2\alpha)\sinh(x) + 2e^x - \frac{x^2}{2} - 2x - 3, \\ \overline{y}(x,\alpha) = 2(1-\alpha)\cosh(x) + (\alpha-2)\sinh(x) + 2e^x - \frac{x^2}{2} - 2x - 3. \end{cases}$$

As in case II, no solution exists.

6 Conclusion

In the present work, the relation between the fuzzy Laplace transforms of a fuzzy function and its second derivative is established and proved. The main purpose of the paper is to solve fuzzy linear second-order differential equations (FDEs) using the fuzzy Laplace transform method, under generalized differentiability. The efficiency of the proposed algorithm is illustrated by giving two numerical examples.

For future research, we will investigate the relationship between the fuzzy Laplace transform of a fuzzy function and that of its kth derivative, with $k \geq 1$, then we will apply the Laplace method to solve a large class of FDEs.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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