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# A note on stronger forms of sensitivity for inverse limit dynamical systems

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**Abstract**

In this paper we study stronger forms of sensitivity for inverse limit dynamical system which is induced from dynamical system on a compact metric space. We give the implication of stronger forms of sensitivity between inverse limit dynamical systems and original systems. More precisely, the inverse limit system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive) if and only if original system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive). Also, we prove that the inverse limit system is syndetically transitive if and only if original system is syndetically transitive.

**MSC:** 54H20; 37B20

**Keywords:** inverse limit dynamical system; syndetically sensitive; cofinitely sensitive; ergodically sensitive; multi-sensitive

## 1 Introduction

Throughout this paper a topological dynamical system we mean a pair $(X, f)$, where $X$ is a compact space and $f : X \to X$ is a surjective continuous map. Let $N^+$ denotes the set of all positive integers and let $N = N^+ \cup \{0\}$. When $X$ is finite, it is a discrete space and there is no non-trivial convergence. Hence, we assume that $X$ contains infinitely many points.

It is well known that sensitive dependence on initial conditions characterizes the unpredictability of chaotic phenomenon (see [1–12]). Sensitive dependence on initial conditions, or sensitivity for short, is the essential component of various definitions of chaos. Roughly speaking, a dynamical system $(X, f)$ is sensitive if for any open region $U$ of the phase space, there exist two points in $U$ and an integer $n \in N$ such that the $n$th iterates of the two points under the map $f$ are significantly separated. The largeness of the set of all $n \in N$ where this significant separation or sensitivity happens can be thought of as a measure of how sensitive the dynamical system is. In particular, if this set is quite thin with arbitrarily large gaps between consecutive entries, then one has some excuse for treating the dynamical system as practically non-sensitive!

For continuous self-maps of compact metric spaces, Moothathu [8] initiated a preliminary study of stronger forms of sensitivity formulated in terms of large subsets of $N$. He considered syndetic sensitivity and cofinite sensitivity. Moreover, he constructed a transitive, sensitive map which is not syndetically sensitive and established the following. (1) Any syndetically transitive, non-minimal map is syndetically sensitive (this improves the result that sensitivity is redundant in Devaney's definition of chaos). (2) Any sensitive map of

$[0,1]$ is cofinitely sensitive. (3) Any sensitive subshift of finite type is cofinitely sensitive. (4) Any syndetically transitive, infinite subshift is syndetically sensitive. (5) No Sturmian subshift is cofinitely sensitive.

More recently, Sharma and Nagar [13] studied the relations between the various forms of sensitivity of the systems $(X,f)$ and it induced hyperspace dynamical systems $(\kappa(X),\bar{f})$. They proved that all forms of sensitivity of $(\kappa(X),\bar{f})$ partly imply the same for $(X,f)$, and the converse holds in some cases. Li *et al.* [14–18] introduced the notion of ergodic sensitivity which is a stronger form of sensitivity, and presented some sufficient conditions for a dynamical system $(X,f)$ to be ergodically sensitive. Also, it is shown that $(\kappa(X),\bar{f})$ is syndetically sensitive (resp. multi-sensitive) if and only if $(X,f)$ is syndetically sensitive (resp. multi-sensitive).

Along with the deep research on the properties of topological dynamical systems, many people also considered dynamical properties in some induced dynamical systems such as inverse limit dynamical systems. Li [19] studied Devaney chaos of inverse limit dynamical systems and proved that an inverse limit dynamical system is Devaney chaos if and only if its original system is Devaney chaos. Chen and Li [20] discussed shadowing property for inverse limit spaces, Ye [21] studied topological entropy of inverse limit dynamical system, Block *et al.* [22], Bruin [23] and Raines and Stimac [24] discussed the properties of inverse limit spaces of tent maps. Liu and Zhao [25] investigated Martelli chaos of inverse limit dynamical systems and proved that inverse limit dynamical systems were Martelli chaos implied that original systems was Martelli chaos.

In this paper we discuss stronger forms of sensitivity for inverse limit dynamical systems on the basis of [8]. Our purpose is to discuss implication of stronger forms of sensitivity between inverse limit systems and original systems. It is shown that the inverse limit system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive) if and only if original system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive). Also, we prove that the inverse limit system is syndetically transitive if and only if original system is syndetically transitive.

## 2 Preliminaries

Let $(X,d)$ be a compact metric space and let $f: X \to X$ be a continuous map. The inverse limit space of $f$ is a metric space defined by the sequence

$$X \overset{f}{\leftarrow} X \overset{f}{\leftarrow} X \overset{f}{\leftarrow} \cdots$$

whose elements $\underline{x} = (x_0, x_1, x_2, \ldots)$ satisfy $f(x_{i+1}) = x_i$, $i = 0,1,2,\ldots$, and the metric is defined by

$$\underline{d}(\underline{x},\underline{y}) = \sum_{i=0}^{\infty} \frac{d(x_i,y_i)}{2^i}.$$

The inverse limit space of $(X,f)$ is denoted by $\lim_{\leftarrow}(X,f)$.

The inverse limit space $\lim_{\leftarrow}(X,f)$ is a compact subspace of product space $\prod_{i=1}^{\infty} X_i$ $(X_i = X, i = 1,2,\ldots)$, the shift map $\sigma_f : \lim_{\leftarrow}(X,f) \to \lim_{\leftarrow}(X,f)$ is defined by $\sigma_f(x_0,x_1,\ldots) = (f(x_0),x_0,x_1,\ldots)$. Furthermore, $\sigma_f^k(x_0,x_1,\ldots) = (f^k(x_0),f^k(x_1),\ldots)$, where $k \in N$. $\sigma_f$ is a homeomorphism and $\sigma_f^{-1}(x_0,x_1,x_2,\ldots) = (x_1,x_2,\ldots)$. The inverse limit dynamical system is denoted by $(\lim_{\leftarrow}(X,f),\sigma_f)$.

The projection map $\pi_i : \lim_{\leftarrow}(X,f) \to X$ is defined by $\pi_i(x_0, x_1, \ldots x_i, \ldots) = x_i$ for $i = 0, 1, \ldots$. Clearly, $\pi_i$ is a continuous mapping, and $f \circ \pi_i = \pi_i \circ \sigma_f$ for $i = 0, 1, \ldots$. If $f$ is a surjective map, then $\pi_i$ is an open surjective mapping for $i = 0, 1, \ldots$. The metric $\underline{d}$ induces the inverse limit topology. This topology has a basis

$$\mathcal{B} = \{V : V = \pi_i^{-1}(U) \text{ for some } i \geq 0 \text{ and some open set } U \text{ in } X\}.$$

Let $(X,f)$ be a dynamical system, $\mathrm{orb}(x,f)$ be the orbit of $x$ under $f$ for some $x \in X$, *i.e.*, $\mathrm{orb}(x,f) = \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ where $f^n = f \circ f^{n-1}$ and $f^0$ be the identity map on $X$. For any two nonempty sets $U, V \subset X$, we write $N_f(U,V) = \{n \in N^+ : U \cap f^{-n}(V) \neq \emptyset\}$.

A map $f : X \to X$ is topologically transitive if $N_f(U,V)$ is nonempty, for any nonempty open sets $U, V \subset X$.

A subset $S \subset N^+$ is thick if $S$ contains arbitrarily large blocks of consecutive numbers. A subset $S \subset N^+$ is syndetic if $N^+ \setminus S$ is not thick.

A map $f : X \to X$ is syndetically transitive if $N_f(U,V)$ is syndetic, for any nonempty open sets $U, V \subset X$.

We shall use $\mathrm{card}A$ to denote the cardinality of $A$.

An upper density of a set $A \subset N$ is the number

$$d^*(A) = \lim_{k\to\infty} \sup \frac{1}{k+1} \mathrm{card}\{0 \leq j \leq k : j \in A\}.$$

A lower density of a set $A \subset N$ is the number

$$d_*(A) = \lim_{k\to\infty} \inf \frac{1}{k+1} \mathrm{card}\{0 \leq j \leq k : j \in A\}.$$

$f$ is topologically ergodic if for every pair of nonempty open sets $U, V \subset X$, the set $N_f(U,V)$ has positive upper density.

Let $(X,f)$ be a dynamical system. According to the classical definition, $f$ has sensitive dependence if there is a $\delta > 0$ such that for any $x \in X$ and any open neighborhood $V_x$ of $x$, there is an $n \in N$ such that $\sup\{d(f^n(x), f^n(y)) : y \in V_x\} > \delta$. We can write this in a slightly different way. For $U \subset X$ and $\delta > 0$, let $N_f(U,\delta) = \{n \in N : \text{there exist } x, y \in V \text{ with } d(f^n(x), f^n(y)) > \delta\}$. Now, we say:

(1) $f$ is sensitive if there exists a $\delta > 0$ such that for any nonempty open set $U \subset X$, $N_f(U,\delta)$ is nonempty.

(2) $f$ is syndetically sensitive if there exists a $\delta > 0$ such that for every nonempty open subset $U \subset X$, $N_f(U,\delta)$ is syndetic.

(3) $f$ is cofinitely sensitive if there exists a $\delta > 0$ such that for every nonempty open subset $U \subset X$, $N_f(U,\delta)$ is cofinite, that is, $N \setminus N_f(U,\delta)$ is finite.

(4) $f$ is ergodically sensitive if there exists a $\delta > 0$ such that for every nonempty open subset $U \subset X$, $N_f(U,\delta)$ has positive upper density.

(5) $f$ is multi-sensitive if there exists $\delta > 0$ such that for every integer $k > 0$ and for any nonempty open subsets $U_1, U_2, \ldots, U_k \subset X$, $\bigcap_{i=1}^{k} N_f(U_i,\delta) \neq \emptyset$.

Here $\delta > 0$ will be referred as a constant of sensitivity. Clearly, syndetic sensitivity implies ergodic sensitivity. It is well known from the definition of the ergodic sensitive and Theorem 7 in [8] that ergodic sensitivity implies sensitivity and the converse does not hold.

By Theorem 5 and Corollary 3 in [8], one can conclude that both syndetic sensitivity and ergodic sensitivity are weaker than cofinite sensitivity. It is easy to show that:

(1) Cofinite sensitivity $\Rightarrow$ multi-sensitivity.

(2) If $f \times f$ is topologically transitive (this is known as topologically weak mixing), then $f$ is multi-sensitivity.

Corollary 3 and Theorem 5 from [8] show that every Sturmian subshift is syndetically sensitive, and that no Sturmian subshift is cofinitely sensitive. In addition, Theorem 7 in [8] shows that there exists a transitive, sensitive subshift which is not syndetically sensitive. Consequently, there are sensitive transformations that are not syndetically sensitive, and syndetically sensitive maps that are not cofinitely sensitive.

**Definition 2.1** Let $(X,f)$ and $(Y,g)$ be two dynamical systems. Then $f$ and $g$ are said to be topologically conjugate if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$. The homeomorphism $h$ is called a conjugate map.

Also, $f$ and $g$ are said to be topologically semiconjugate (or $g$ is a factor of $f$) if $h : X \to Y$ is a continuous surjection such that $h \circ f = g \circ h$.

## 3 Main results

In this section, we shall discuss in the inverse limit spaces and find that the inverse limit dynamical system $(\lim_{\leftarrow}(X,f), \sigma_f)$ has stronger forms of sensitivity if and only if $(X,f)$ has stronger forms of sensitivity, *i.e.*, the inverse limit system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive) if and only if original system is syndetically sensitive (resp. cofinitely sensitive, ergodically sensitive, multi-sensitive).

**Theorem 3.1** *Let* $(\lim_{\leftarrow}(X,f), \sigma_f)$ *be an inverse limit dynamical system. Then* $f$ *is syndetically transitive if and only if so is* $\sigma_f$.

*Proof* Necessity. Suppose that $f$ is syndetically transitive. We shall prove that $N_{\sigma_f}(\widetilde{U}, \widetilde{V})$ is a syndetic set for any nonempty open subsets $\widetilde{U}$ and $\widetilde{V}$ in $\lim_{\leftarrow}(X,f)$.

Let $\widetilde{U}$ and $\widetilde{V}$ be any nonempty open subsets $\lim_{\leftarrow}(X,f)$. Take $\underline{y} \in \widetilde{V}$ and $\delta > 0$ satisfying $B(\underline{y}, \delta) \subset \widetilde{V}$, where $B(\underline{y}, \delta)$ is a $\delta$-neighborhood of $\underline{y}$. Denote $M = \operatorname{diam} X = \sup\{d(x,y) : x, y \in X\}$. When $n$ is large enough, $\frac{M}{2^n} < \frac{\delta}{2}$. Since $\pi_n$ is an open map for the above enough large $n$, $\pi_n(\widetilde{U})$ and $\pi_n(B(\underline{y}, \delta))$ are two nonempty subsets in $X$. Moreover, $f$ is syndetically transitive, then $N_f(\pi_n(\widetilde{U}), \pi_n(B(\underline{y}, \delta)))$ is syndetic. Furthermore, for any $k \in N_f(\pi_n(\widetilde{U}), \pi_n(B(\underline{y}, \delta)))$, we have $f^k(\pi_n(\widetilde{U})) \cap \pi_n(B(\underline{y}, \delta)) \neq \emptyset$. Take $\underline{x} = (x_0, x_1, x_2, \ldots) \in \widetilde{U}$ and $\underline{z} = (z_0, z_1, z_2, \ldots) \in B(\underline{y}, \delta)$ such that $f^k(x_n) = z_n$. Hence, $f^k(x_j) = z_j$, $j = 1, 2, \ldots, n$. Since

$$\begin{aligned} \underline{d}\big(\sigma_f^k(\underline{x}), \underline{y}\big) &\le \underline{d}\big(\sigma_f^k(\underline{x}), \underline{z}\big) + \underline{d}(\underline{z}, \underline{y}) \\ &\le \sum_{j=0}^{n} \frac{d(f^k(x_j), z_j)}{2^j} + \sum_{j=n+1}^{\infty} \frac{d(f^k(x_j), z_j)}{2^j} + \frac{\delta}{2} \\ &\le 0 + \frac{M}{2^n} + \frac{\delta}{2} < \delta, \end{aligned}$$

we have $\sigma_f^k(\underline{x}) \in B(\underline{y}, \delta) \subset \widetilde{V}$, i.v., $\sigma_f^k(\widetilde{U}) \cap \widetilde{V} \neq \emptyset$. Therefore, $k \in N_{\sigma_f}(\widetilde{U}, \widetilde{V})$, furthermore, $N_f(\pi_n(\widetilde{U}), \pi_n(B(\underline{y}, \delta))) \subset N_{\sigma_f}(\widetilde{U}, \widetilde{V})$. This shows that $N_{\sigma_f}(\widetilde{U}, \widetilde{V})$ is syndetic, which implies that $\sigma_f$ is syndetically transitive.

Sufficiency. Suppose that $\sigma_f$ is syndetically transitive. We shall prove that $N_f(U,V)$ is a syndetic set for any nonempty open subsets $U$ and $V$ in $(X,f)$.

Let $U$ and $V$ be any nonempty subsets in $X$. Then $\pi_0^{-1}(U)$ and $\pi_0^{-1}(V)$ are two nonempty subsets in $\lim_{\leftarrow}(X,f)$ because $\pi_0 : \lim_{\leftarrow}(X,f) \to X$ is a continuous map. Since $\sigma_f$ is syndetically transitive, we have $N_{\sigma_f}(\pi_0^{-1}(U),\pi_0^{-1}(V))$ is syndetic. For every $k \in N_{\sigma_f}(\pi_0^{-1}(U),\pi_0^{-1}(V))$, we have $\pi_0^{-1}(U) \cap \sigma_f^{-k}(\pi_0^{-1}(V)) \neq \emptyset$. Since $f^k \circ \pi_0 = \pi_0 \circ \sigma_f^k$, we have $\pi_0^{-1}(U) \cap \pi_0^{-1}(f^{-k}(V)) \neq \emptyset$, furthermore, $\pi_0^{-1}(U \cap f^{-k}(V)) \neq \emptyset$, which implies $U \cap f^{-k}(V) \neq \emptyset$. Therefore, we have $k \in N_f(U,V)$ and $N_{\sigma_f}(\pi_0^{-1}(U),\pi_0^{-1}(V)) \subset N_f(U,V)$. This shows that $N_f(U,V)$ is syndetic, *i.e.*, $f$ is syndetically transitive. □

**Theorem 3.2** *Let $(X,f)$ be a dynamical system and $f : X \to X$ be a surjective map. Then $f$ is syndetically sensitive if and only if so is $\sigma_f$.*

*Proof* Necessity. Suppose that $f$ is syndetically sensitive with sensitive constant $\delta > 0$. We shall prove that $N_{\sigma_f}(\widetilde{U},\delta)$ is syndetic for any nonempty open subset $\widetilde{U}$ in $\lim_{\leftarrow}(X,f)$.

Let $\widetilde{U}$ be any nonempty open subset in $\lim_{\leftarrow}(X,f)$. Then $\pi_0(\widetilde{U})$ is a nonempty open subset in $X$ because $\pi_0$ is an open map. Since $f$ is syndetically sensitive with sensitive constant $\delta > 0$, $N_f(\pi_0(\widetilde{U}),\delta)$ is syndetic. For any $k \in N_f(\pi_0(\widetilde{U}),\delta)$, there exist $x_0, y_0 \in \pi_0(\widetilde{U})$ such that $d(f^k(x_0),f^k(y_0)) > \delta$. Let $\underline{x} = (x_0,x_1,\ldots) \in \pi_0^{-1}(x_0) \cap \widetilde{U}$, $\underline{y} = (y_0,y_1,\ldots) \in \pi_0^{-1}(y_0) \cap \widetilde{U}$. Then

$$\underline{d}\bigl(\sigma_f^k(\underline{x}),\sigma_f^k(\underline{y})\bigr) = \sum_{i=0}^{\infty} \frac{d(f^k(x_i),f^k(y_i))}{2^i} \geq d\bigl(f^k(x_0),f^k(y_0)\bigr) > \delta.$$

Hence, $k \in N_{\sigma_f}(\widetilde{U},\delta)$ and $N_f(\pi_0(\widetilde{U}),\delta) \subset N_{\sigma_f}(\widetilde{U},\delta)$. This shows $N_{\sigma_f}(\widetilde{U},\delta)$ is syndetic, *i.e.*, $\sigma_f$ is syndetically sensitive.

Sufficiency. Suppose that $\sigma_f$ is syndetically sensitive with sensitive constant $\delta > 0$. We shall prove that $N_f(U,\frac{\delta}{2})$ is syndetic for any nonempty subset $U$ in $X$.

Let $U$ be a nonempty subset in $X$. Then $\pi_0^{-1}(U)$ is a nonempty subset in $\lim_{\leftarrow}(X,f)$ because $\pi_0$ is a continuous map. Take $\underline{x} \in \pi_0^{-1}(U)$, then there exists $m > 8$ such that $B(\underline{x},\frac{\delta}{m}) \subset \pi_0^{-1}(U)$. Since $\sigma_f$ is syndetically sensitive with sensitive constant $\delta > 0$, $N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta)$ is syndetic. For any $k \in N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta)$, there exist $\underline{x}^*, \underline{y}^* \in B(\underline{x},\frac{\delta}{m})$ such that $\underline{d}(\sigma_f^k(\underline{x}^*), \sigma_f^k(\underline{y}^*)) > \delta$. Since $\sigma_f^{k-1}$ is continuous for $\underline{x}$, there exists $\frac{\delta}{m} < \delta' < \frac{\delta}{8}$, when $\underline{x}' \in B(\underline{x},\delta')$, we have $\underline{d}(\sigma_f^{k-1}(\underline{x}'),\sigma_f^{k-1}(\underline{x})) < \frac{\delta}{8}$. By the triangular inequality, $\underline{d}(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)) < \frac{\delta}{4}$.

Let $\underline{x}^* = (x_0^*,x_1^*,\ldots)$ and $\underline{y}^* = (y_0^*,y_1^*,\ldots)$. Then $x_0^* = \pi_0(\underline{x}^*) \in U$ and $y_0^* = \pi_0(\underline{y}^*) \in U$. Since

$$\begin{aligned}
\underline{d}\bigl(\sigma_f^k(\underline{x}^*),\sigma_f^k(\underline{y}^*)\bigr) &= \sum_{i=0}^{\infty} \frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\bigl(f^k(x_0^*),f^k(y_0^*)\bigr) + \sum_{i=1}^{\infty} \frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\bigl(f^k(x_0^*),f^k(y_0^*)\bigr) + \frac{1}{2}\sum_{i=0}^{\infty} \frac{d(f^{k-1}(x_i^*),f^{k-1}(y_i^*))}{2^i} \\
&= d\bigl(f^k(x_0^*),f^k(y_0^*)\bigr) + \frac{1}{2}\underline{d}\bigl(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)\bigr) \\
&\leq d\bigl(f^k(x_0^*),f^k(y_0^*)\bigr) + \frac{1}{8}\delta,
\end{aligned}$$

we have $d(f^k(x_0^*),f^k(y_0^*)) > \frac{7}{8}\delta > \frac{1}{2}\delta$, which implies that $k \in N_f(U,\frac{\delta}{2})$. Furthermore, $N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta) \subset N_f(U,\frac{\delta}{2})$. This shows that $N_f(U,\frac{\delta}{2})$ is syndetic, *i.e.*, $f$ is syndetically sensitive. □

**Theorem 3.3** *Let $(X,f)$ be a dynamical system and $f: X \to X$ be a surjective map. Then $f$ is cofinitely sensitive if and only if so is $\sigma_f$.*

*Proof* Necessity. Suppose that $f$ is cofinitely sensitive with sensitive constant $\delta > 0$. We shall prove that $N_{\sigma_f}(\widetilde{U},\delta)$ is cofinite for any nonempty open subset $\widetilde{U}$ in $\lim_{\leftarrow}(X,f)$.

Let $\widetilde{U}$ be any nonempty open subset in $\lim_{\leftarrow}(X,f)$. Then $\pi_0(\widetilde{U})$ is a nonempty open subset in $X$ because $\pi_0$ is an open map. Since $f$ is cofinitely sensitive with sensitive constant $\delta > 0$, then $N_f(\pi_0(\widetilde{U}),\delta)$ is cofinite, *i.e.*, $N \setminus N_f(\pi_0(\widetilde{U}),\delta)$ is finite. For any $k \in N_f(\pi_0(\widetilde{U}),\delta)$, there exist $x_0, y_0 \in \pi_0(\widetilde{U})$ such that $d(f^k(x_0),f^k(y_0)) > \delta$. Let $\underline{x} = (x_0,x_1,\ldots) \in \pi_0^{-1}(x_0) \cap \widetilde{U}$ and $\underline{y} = (y_0,y_1,\ldots) \in \pi_0^{-1}(y_0) \cap \widetilde{U}$. Then

$$\underline{d}\big(\sigma_f^k(\underline{x}),\sigma_f^k(\underline{y})\big) = \sum_{i=0}^{\infty} \frac{d(f^k(x_i),f^k(y_i))}{2^i} \geq d\big(f^k(x_0),f^k(y_0)\big) > \delta.$$

Hence, $k \in N_{\sigma_f}(\widetilde{U},\delta)$ and $N_f(\pi_0(\widetilde{U}),\delta) \subset N_{\sigma_f}(\widetilde{U},\delta)$. Furthermore, $N \setminus N_{\sigma_f}(\widetilde{U},\delta)$ is finite. This shows $N_{\sigma_f}(\widetilde{U},\delta)$ is cofinite, *i.e.*, $\sigma_f$ is cofinitely sensitive.

Sufficiency. Suppose that $\sigma_f$ is cofinitely sensitive with sensitive constant $\delta > 0$. We shall prove that $N_f(U,\frac{\delta}{2})$ is cofinite for any nonempty subset $U$ in $X$.

Let $U$ be a nonempty subset in $X$. Then $\pi_0^{-1}(U)$ is a nonempty subset in $\lim_{\leftarrow}(X,f)$ because $\pi_0$ is a continuous map. Take $\underline{x} \in \pi_0^{-1}(U)$, then there exists $m > 8$ such that $B(\underline{x},\frac{\delta}{m}) \subset \pi_0^{-1}(U)$. Since $\sigma_f$ is cofinitely sensitive with sensitive constant $\delta > 0$, then $N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta)$ is cofinite. For any $k \in N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta)$, there exist $\underline{x}^*,\underline{y}^* \in B(\underline{x},\frac{\delta}{m})$ such that $\underline{d}(\sigma_f^k(\underline{x}^*),\sigma_f^k(\underline{y}^*)) > \delta$. Since $\sigma_f^{k-1}$ is continuous for $\underline{x}$, then there exists $\frac{\delta}{m} < \delta' < \frac{\delta}{8}$, when $\underline{x}' \in B(\underline{x},\delta')$, we have $\underline{d}(\sigma_f^{k-1}(\underline{x}'),\sigma_f^{k-1}(\underline{x})) < \frac{\delta}{8}$. By the triangular inequality, $\underline{d}(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)) < \frac{\delta}{4}$.

Let $\underline{x}^* = (x_0^*,x_1^*,\ldots)$ and $\underline{y}^* = (y_0^*,y_1^*,\ldots)$. Then $x_0^* = \pi_0(\underline{x}^*) \in U$ and $y_0^* = \pi_0(\underline{y}^*) \in U$. Since

$$\begin{aligned}
\underline{d}\big(\sigma_f^k(\underline{x}^*),\sigma_f^k(\underline{y}^*)\big) &= \sum_{i=0}^{\infty} \frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \sum_{i=1}^{\infty} \frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{2}\sum_{i=0}^{\infty} \frac{d(f^{k-1}(x_i^*),f^{k-1}(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{2}\underline{d}\big(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)\big) \\
&\leq d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{8}\delta,
\end{aligned}$$

we have $d(f^k(x_0^*),f^k(y_0^*)) > \frac{7}{8}\delta > \frac{1}{2}\delta$, which implies that $k \in N_f(U,\frac{\delta}{2})$. Furthermore, $N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta) \subset N_f(U,\frac{\delta}{2})$. This shows that $N_f(U,\frac{\delta}{2})$ is cofinite, *i.e.*, $f$ is cofinitely sensitive. □

**Theorem 3.4** *Let $(X,f)$ be a dynamical system and $f: X \to X$ be a surjective map. Then $f$ is ergodically sensitive if and only if so is $\sigma_f$.*

*Proof* Necessity. Suppose that $f$ is ergodically sensitive with sensitive constant $\delta > 0$. We shall prove that $N_{\sigma_f}(\widetilde{U},\delta)$ has positive upper density for any nonempty open subset $\widetilde{U}$ in $\lim_{\leftarrow}(X,f)$.

Let $\widetilde{U}$ be any nonempty open subset in $\lim_{\leftarrow}(X,f)$. Then $\pi_0(\widetilde{U})$ is a nonempty open subset in $X$ because $\pi_0$ is an open map. Since $f$ is ergodically sensitive with sensitive constant $\delta > 0$, $N_f(\pi_0(\widetilde{U}),\delta)$ has positive upper density, *i.e.*,

$$d^*\big(N_f\big(\pi_0(\widetilde{U}),\delta\big)\big) = \lim_{k\to\infty}\sup\frac{1}{k+1}\operatorname{card}\big\{0 \le j \le k : j \in N_f\big(\pi_0(\widetilde{U}),\delta\big)\big\} > 0.$$

For any $k \in N_f(\pi_0(\widetilde{U}),\delta)$, there exist $x_0, y_0 \in \pi_0(\widetilde{U})$ such that $d(f^k(x_0),f^k(y_0)) > \delta$. Let $\underline{x} = (x_0,x_1,\ldots) \in \pi_0^{-1}(x_0) \cap \widetilde{U}$ and $\underline{y} = (y_0,y_1,\ldots) \in \pi_0^{-1}(y_0) \cap \widetilde{U}$. Then

$$\underline{d}\big(\sigma_f^k(\underline{x}),\sigma_f^k(\underline{y})\big) = \sum_{i=0}^{\infty}\frac{d(f^k(x_i),f^k(y_i))}{2^i} \ge d\big(f^k(x_0),f^k(y_0)\big) > \delta.$$

Hence, $k \in N_{\sigma_f}(\widetilde{U},\delta)$ and $N_f(\pi_0(\widetilde{U}),\delta) \subset N_{\sigma_f}(\widetilde{U},\delta)$. Furthermore,

$$d^*\big(N_{\sigma_f}(\widetilde{U},\delta)\big) \ge d^*\big(N_f\big(\pi_0(\widetilde{U}),\delta\big)\big).$$

Moreover, $d^*(N_f(\pi_0(\widetilde{U}),\delta)) > 0$, so $d^*(N_{\sigma_f}(\widetilde{U},\delta)) > 0$. This shows $N_{\sigma_f}(\widetilde{U},\delta)$ positive upper density, *i.e.*, $\sigma_f$ is ergodically sensitive.

Sufficiency. Suppose that $\sigma_f$ is ergodically sensitive with sensitive constant $\delta > 0$. We shall prove that $N_f(U,\frac{\delta}{2})$ has positive upper density for any nonempty subset $U$ in $X$.

Let $U$ be a nonempty subset in $X$. Then $\pi_0^{-1}(U)$ is a nonempty subset in $\lim_{\leftarrow}(X,f)$ because $\pi_0$ is a continuous map. Take $\underline{x} \in \pi_0^{-1}(U)$, then there exists $m > 8$ such that $B(\underline{x},\frac{\delta}{m}) \subset \pi_0^{-1}(U)$. Since $\sigma_f$ is ergodically sensitive with sensitive constant $\delta > 0$,

$$d^*\left(N_{\sigma_f}\left(B\left(\underline{x},\frac{\delta}{m}\right),\delta\right)\right) = \lim_{k\to\infty}\sup\frac{1}{k+1}\operatorname{card}\left\{0 \le j \le k : j \in B\left(\underline{x},\frac{\delta}{m}\right)\right\} > 0.$$

For any $k \in N_{\sigma_f}(B(\underline{x},\frac{\delta}{m}),\delta)$, there exist $\underline{x}^*,\underline{y}^* \in B(\underline{x},\frac{\delta}{m})$ such that $\underline{d}(\sigma_f^k(\underline{x}^*),\sigma_f^k(\underline{y}^*)) > \delta$. Since $\sigma_f^{k-1}$ is continuous for $\underline{x}$, there exists $\frac{\delta}{m} < \delta' < \frac{\delta}{8}$, when $\underline{x}' \in B(\underline{x},\delta')$, we have $\underline{d}(\sigma_f^{k-1}(\underline{x}'),\sigma_f^{k-1}(\underline{x})) < \frac{\delta}{8}$. By the triangular inequality, $\underline{d}(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)) < \frac{\delta}{4}$.

Let $\underline{x}^* = (x_0^*,x_1^*,\ldots)$ and $\underline{y}^* = (y_0^*,y_1^*,\ldots)$. Then $x_0^* = \pi_0(\underline{x}^*) \in U$ and $y_0^* = \pi_0(\underline{y}^*) \in U$. Since

$$\begin{aligned}
\underline{d}\big(\sigma_f^k(\underline{x}^*),\sigma_f^k(\underline{y}^*)\big) &= \sum_{i=0}^{\infty}\frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \sum_{i=1}^{\infty}\frac{d(f^k(x_i^*),f^k(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{2}\sum_{i=0}^{\infty}\frac{d(f^{k-1}(x_i^*),f^{k-1}(y_i^*))}{2^i} \\
&= d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{2}\underline{d}\big(\sigma_f^{k-1}(\underline{x}^*),\sigma_f^{k-1}(\underline{y}^*)\big) \\
&\le d\big(f^k(x_0^*),f^k(y_0^*)\big) + \frac{1}{8}\delta,
\end{aligned}$$

we have $d(f^k(x_0^*), f^k(y_0^*)) > \frac{7}{8}\delta > \frac{1}{2}\delta$, which implies that $k \in N_f(U, \frac{\delta}{2})$. Furthermore, $N_{\sigma_f}(B(\underline{x}, \frac{\delta}{m}), \delta) \subset N_f(U, \frac{\delta}{2})$, which implies that $d^*(N_f(U, \frac{\delta}{2})) \geq d^*(N_{\sigma_f}(B(\underline{x}, \frac{\delta}{m}), \delta))$. This shows that $d^*(N_f(U, \frac{\delta}{2})) > 0$, *i.e.*, $f$ is ergodically sensitive. □

**Theorem 3.5** *Let $(X, f)$ be a dynamical system and $f: X \to X$ be a surjective map. Then $f$ is multi-sensitive if and only if so is $\sigma_f$.*

*Proof* Necessity. Suppose that $f$ is multi-sensitive with sensitive constant $\delta > 0$. We shall prove that $\bigcap_{i=1}^p N_{\sigma_f}(\widetilde{U}_i, \delta) \neq \emptyset$ for every $p \in N$ and any nonempty open subset $\widetilde{U}_i$ ($i = 1, 2, \ldots, p$) in $\lim_{\leftarrow}(X, f)$.

Let $\widetilde{U}_i$ ($i = 1, 2, \ldots, p$) be any nonempty open subset in $\lim_{\leftarrow}(X, f)$. Then $\pi_0(\widetilde{U}_i)$ ($i = 1, 2, \ldots, p$) is a nonempty open subset in $X$ because $\pi_0$ is an open map. Since $f$ is multi-sensitive with sensitive constant $\delta > 0$, then $\bigcap_{i=1}^p N_f(\pi_0(\widetilde{U}_i), \delta) \neq \emptyset$. For any $k \in \bigcap_{i=1}^p N_f(\pi_0(\widetilde{U}_i), \delta)$, there exist $x_{i0}, y_{i0} \in \pi_0(\widetilde{U}_i)$ such that $d(f^k(x_{i0}), f^k(y_{i0})) > \delta$ for $i = 1, 2, \ldots, p$. Let $\underline{x_i} = (x_{i0}, x_{i1}, \ldots) \in \pi_0^{-1}(x_{i0}) \cap \widetilde{U}$ and $\underline{y_i} = (y_{i0}, y_{i1}, \ldots) \in \pi_0^{-1}(y_{i0}) \cap \widetilde{U}$ for $i = 1, 2, \ldots, p$. Then

$$\underline{d}\big(\sigma_f^k(\underline{x_i}), \sigma_f^k(\underline{y_i})\big) = \sum_{j=0}^{\infty} \frac{d(f^k(x_{ij}), f^k(y_{ij}))}{2^j} \geq d\big(f^k(x_{i0}), f^k(y_{i0})\big) > \delta \quad \text{for } i = 1, 2, \ldots, p.$$

Hence, $k \in N_{\sigma_f}(\widetilde{U}_i, \delta)$ and $N_f(\pi_0(\widetilde{U}_i), \delta) \subset N_{\sigma_f}(\widetilde{U}_i, \delta)$ for $i = 1, 2, \ldots, p$. Furthermore, $k \in \bigcap_{i=1}^p (N_{\sigma_f}(\widetilde{U}_i, \delta))$. This shows $\bigcap_{i=1}^p (N_{\sigma_f}(\widetilde{U}_i, \delta)) \neq \emptyset$, *i.e.*, $\sigma_f$ is multi-sensitive.

Sufficiency. Suppose that $\sigma_f$ is multi-sensitive with sensitive constant $\delta > 0$. We shall prove that $\bigcap_{i=1}^p N_f(U_i, \frac{\delta}{2}) \neq \emptyset$ for any nonempty open subset $U_i$ ($i = 1, 2, \ldots, p$) in $X$.

Let $U_i$ ($i = 1, 2, \ldots, p$) be a nonempty open subset in $X$. Then $\pi_0^{-1}(U_i)$ is a nonempty open subset in $\lim_{\leftarrow}(X, f)$ because $\pi_0$ is a continuous map. Take $\underline{x_i} \in \pi_0^{-1}(U_i)$, then there exists $m > 8$ such that $B(\underline{x_i}, \frac{\delta}{m}) \subset \pi_0^{-1}(U_i)$ for $i = 1, 2, \ldots, p$. Since $\sigma_f$ is multi-sensitive with sensitive constant $\delta > 0$, we have $\bigcap_{i=1}^p N_{\sigma_f}(B(\underline{x_i}, \frac{\delta}{m}), \delta) \neq \emptyset$. For any $k \in N_{\sigma_f}(B(\underline{x_i}, \frac{\delta}{m}), \delta)$ ($i = 1, 2, \ldots, p$), there exist $\underline{x_i^*}, \underline{y_i^*} \in B(\underline{x}, \frac{\delta}{m})$ such that $\underline{d}(\sigma_f^k(\underline{x_i^*}), \sigma_f^k(\underline{y_i^*})) > \delta$. Since $\sigma_f^{k-1}$ is continuous for $\underline{x_i}$, there exists $\frac{\delta}{m} < \delta' < \frac{\delta}{8}$, when $\underline{x_i'} \in B(\underline{x_i}, \delta')$, we have $\underline{d}(\sigma_f^{k-1}(\underline{x_i'}), \sigma_f^{k-1}(\underline{x_i})) < \frac{\delta}{8}$ for $i = 1, 2, \ldots, p$. By the triangular inequality, $\underline{d}(\sigma_f^{k-1}(\underline{x_i^*}), \sigma_f^{k-1}(\underline{y_i^*})) < \frac{\delta}{4}$ for $i = 1, 2, \ldots, p$.

Let $\underline{x_i^*} = (x_{i0}^*, x_{i1}^*, \ldots)$ and $\underline{y_i^*} = (y_{i0}^*, y_{i1}^*, \ldots)$ for $i = 1, 2, \ldots, p$. Then $x_{i0}^* = \pi_0(\underline{x_i^*}) \in U_i$ and $y_{i0}^* = \pi_0(\underline{y_i^*}) \in U_i$ for $i = 1, 2, \ldots, p$. Since

$$\begin{aligned}
\underline{d}\big(\sigma_f^k(\underline{x_i^*}), \sigma_f^k(\underline{y_i^*})\big) &= \sum_{j=0}^{\infty} \frac{d(f^k(x_{ij}^*), f^k(y_{ij}^*))}{2^j} \\
&= d\big(f^k(x_{i0}^*), f^k(y_{i0}^*)\big) + \sum_{j=1}^{\infty} \frac{d(f^k(x_{ij}^*), f^k(y_{ij}^*))}{2^j} \\
&= d\big(f^k(x_{i0}^*), f^k(y_{i0}^*)\big) + \frac{1}{2}\sum_{j=0}^{\infty} \frac{d(f^{k-1}(x_{ij}^*), f^{k-1}(y_{ij}^*))}{2^j} \\
&= d\big(f^k(x_{i0}^*), f^k(y_{i0}^*)\big) + \frac{1}{2}\underline{d}\big(\sigma_f^{k-1}(\underline{x_i^*}), \sigma_f^{k-1}(\underline{y_i^*})\big) \\
&\leq d\big(f^k(x_{i0}^*), f^k(y_{i0}^*)\big) + \frac{1}{8}\delta,
\end{aligned}$$

we have $d(f^k(x_{i0}^*), f^k(y_{i0}^*)) > \frac{7}{8}\delta > \frac{1}{2}\delta$, which implies that $k \in N_f(U_i, \frac{\delta}{2})$ for $i = 1, 2, \ldots, p$. Furthermore, $k \in \bigcap_{i=1}^{p} N_f(U_i, \frac{\delta}{2})$, which implies that $\bigcap_{i=1}^{p} N_f(U_i, \frac{\delta}{2}) \neq \emptyset$. This shows that $f$ is multi-sensitive. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
HZ and LL (the first and second authors) carried out the study of stronger forms of sensitivity for inverse limit dynamical systems and drafted the manuscript. JW (the third author) helped to draft the manuscript. All authors read and approved the final manuscript.

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