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Dynamical behavior of a third-order rational fuzzy difference equation

Qianhong Zhang[1*], Jingzhong Liu[2] and Zhenguo Luo[3]

*Correspondence:
zqianhong68@163.com
[1]Key Laboratory of Economics System Simulation, School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, People's Republic of China
Full list of author information is available at the end of the article

Abstract

According to a generalization of division (g-division) of fuzzy numbers, this paper is concerned with the boundedness, persistence and global behavior of a positive fuzzy solution of the third-order rational fuzzy difference equation

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \quad n = 0, 1, \ldots,$$

where A and initial values x_0, x_{-1}, x_{-2} are positive fuzzy numbers. Moreover, some examples are given to demonstrate the effectiveness of the results obtained.

MSC: 39A10

Keywords: fuzzy difference equation; boundedness; persistence; global asymptotic behavior

1 Introduction

It is well known that difference equations appear naturally as discrete analogs and as numerical solutions of differential equations and delay differential equations having many applications in economics, biology, computer science, control engineering, *etc.* (see, for example, [1–11] and the references therein). Recently there has been a lot of work concerning the global asymptotic stability, the periodicity, and the boundedness of nonlinear difference equations. Moreover, similar results have been derived for systems of two nonlinear difference equations.

Papaschinopoulos and Schinas [12] investigated the global behavior for a system of the following two nonlinear difference equations:

$$x_{n+1} = A + \frac{y_n}{x_{n-p}}, \qquad y_{n+1} = A + \frac{x_n}{y_{n-q}}, \quad n = 0, 1, \ldots,$$

where A is a positive real number, p, q are positive integers, and $x_{-p}, \ldots, x_0, y_{-q}, \ldots, y_0$ are positive real numbers.

In 2005, Yang [13] studied the global behavior of the following system:

$$x_n = A + \frac{y_{n-1}}{x_{n-p}y_{n-q}}, \qquad y_n = A + \frac{x_{n-1}}{x_{n-r}y_{n-s}}, \quad n = 1, 2, \ldots,$$

where $p \geq 2$, $q \geq 2$, $r \geq 2$, $s \geq 2$, A is a positive constant, and initial values $x_{1-\max\{p,r\}}$, $x_{2-\max\{p,r\}}, \ldots, x_0$, $y_{1-\max\{q,s\}}, y_{2-\max\{q,s\}}, \ldots, y_0$ are positive real numbers.

In 2012, Zhang *et al.* [14] investigated the global behavior for a system of the following third-order nonlinear difference equations:

$$x_{n+1} = \frac{x_{n-2}}{B + y_{n-2}y_{n-1}y_n}, \qquad y_{n+1} = \frac{y_{n-2}}{A + x_{n-2}x_{n-1}x_n},$$

where $A, B \in (0, \infty)$, and initial values $x_{-i}, y_{-i} \in (0, \infty)$, $i = 0, 1, 2$.

Ibrahim and Zhang [15] studied dynamics of the third-order system of rational difference equations

$$x_{n+1} = \frac{\alpha_1 y_{n-2}}{\beta_1 + \gamma_1 x_n x_{n-1} x_{n-2}}, \qquad y_{n+1} = \frac{\alpha_2 x_{n-2}}{\beta_2 + \gamma_2 y_n y_{n-1} y_{n-2}},$$

$n = 0, 1, 2, \ldots$, where the parameters α_1, α_2, β_1, β_2, γ_1, γ_2 and initial conditions x_0, x_{-1}, x_{-2}, y_0, y_{-1}, y_{-2} are positive real numbers.

Although difference equations and a system of difference equations are sometimes very simple in their forms, they are extremely difficult to understand through the behavior of their solutions. On the other hand, these models inherently process uncertainty or vagueness. In order to consider these uncertain factors, fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in a mathematical model. Particularly, the use of fuzzy difference equations is a natural way to model the dynamical systems with embedded uncertainty.

Fuzzy difference equation is a difference equation where parameters and initial values are fuzzy numbers, and its solutions are sequences of fuzzy numbers. Due to the applicability of fuzzy difference equation for the analysis of phenomena where imprecision is inherent, this class of difference equations and its applications is a very important topic from theoretical point of view. Recently there has been an increasing interest in the study of fuzzy difference equations (see [16–26]). For example, fuzzy difference equations are suitable in finance problems. Chrysafis *et al.* [25] studied the fuzzy difference equation of finance. Their research is in finance which is about the alternative methodology to study the time value of money, the method of fuzzy difference equations. Studies have shown that the fuzzy difference equations have a potential to be applied in the theory of fuzzy time series, fuzzy differential equations and stochastic fuzzy differential equations. Readers can refer to [27–32].

Motivated by the discussions above, according to a generalization of division (g-division) of fuzzy numbers, we study the behavior of solutions of the following fuzzy difference equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \quad n = 0, 1, \ldots, \tag{1}$$

where A and initial conditions $x_{-i}, y_{-i} \in (0, \infty)$, $i = 0, 1, 2$, are positive fuzzy numbers.

The aim of this paper is to study the existence of positive solutions of (1). Furthermore, we give some conditions so that every positive solution of (1) is bounded and persistent. Finally, under some conditions we prove that (1) has a unique positive equilibrium x and every positive solution of (1) tends to x as $n \to \infty$. Our results extend the result of reference [20].

2 Preliminaries and definitions

For the convenience of the readers, we give the following definitions.

Definition 2.1 [20] $A : R \to [0,1]$ is said to be a fuzzy number if it satisfies conditions (i)-(iv) written below:

(i) A is normal, *i.e.*, there exists $x \in R$ such that $A(x) = 1$;

(ii) A is fuzzy convex, *i.e.*, for all $t \in [0,1]$ and $x_1, x_2 \in R$ such that

$$A\big(tx_1 + (1-t)x_2\big) \geq \min\big\{A(x_1), A(x_2)\big\};$$

(iii) A is upper semicontinuous;

(iv) the support of A, $\operatorname{supp} A = \overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = \overline{\{x : A(x) > 0\}}$ is compact.

For $\alpha \in (0,1]$, we define the α-cuts of fuzzy number A with $[A]_\alpha = \{x \in R : A(x) \geq \alpha\}$ and for $\alpha = 0$, the support of A is defined as $\operatorname{supp} A = [A]_0 = \overline{\{x \in R | A(x) > 0\}}$. It is clear that $[A]_\alpha$ is a closed interval. A fuzzy number is positive if $\operatorname{supp} A \subset (0,\infty)$.

It is obvious that if A is a positive real number, then A is a fuzzy number such that $[A]_\alpha = [A, A]$, $\alpha \in (0,1]$. Then we say that A is a trivial fuzzy number.

Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, $\alpha \in [0,1]$, and $k > 0$, we define addition and multiplication as follows:

$$[A + B]_\alpha = [A_{l,\alpha} + B_{l,\alpha}, A_{r,\alpha} + B_{r,\alpha}], \tag{2}$$

$$[kA]_\alpha = [kA_{l,\alpha}, kA_{r,\alpha}]. \tag{3}$$

The collection of all fuzzy numbers with addition and multiplication as defined by Eqs. (2) and (3) is denoted by E^1.

Definition 2.2 [20] The distance between two arbitrary fuzzy numbers A and B is defined as follows:

$$D(A,B) = \sup_{\alpha\in[0,1]} \max\big\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\big\}. \tag{4}$$

It is clear that (E^1, D) is a complete metric space.

Definition 2.3 [33] Let $A, B \in E^1$ have α-cuts $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, with $0 \notin [B]_\alpha$, $\forall\alpha \in [0,1]$. The g-division $\div_{\mathrm{g}}$ is the operation that calculates the fuzzy number $C = A \div_{\mathrm{g}} B$ having level cuts $[C]_\alpha = [C_{l,\alpha}, C_{r,\alpha}]$ (here $[A]_\alpha^{-1} = [1/A_{r,\alpha}, 1/A_{l,\alpha}]$) defined by

$$[C]_\alpha = [A]_\alpha \div_{\mathrm{g}} [B]_\alpha \quad \Longleftrightarrow \quad \begin{cases} \text{(i)} & [A]_\alpha = [B]_\alpha [C]_\alpha \\ \text{or} & \\ \text{(ii)} & [B]_\alpha = [A]_\alpha [C]_\alpha^{-1} \end{cases} \tag{5}$$

provided that C is a proper fuzzy number ($C_{l,\alpha}$ is nondecreasing, $C_{r,\alpha}$ is nondecreasing, $C_{l,1} \leq C_{r,1}$).

Remark 2.1 According to [33], in this paper the fuzzy number is positive, if $A \div_{\mathrm{g}} B = C \in E^1$ exists, it is easy to see that the following two cases are possible.

Case (i). If $A_{l,\alpha}B_{r,\alpha} \leq A_{r,\alpha}B_{l,\alpha}$, $\forall\alpha \in [0,1]$, then $C_{l,\alpha} = \frac{A_{l,\alpha}}{B_{l,\alpha}}$, $C_{r,\alpha} = \frac{A_{r,\alpha}}{Br,\alpha}$.

Case (ii). If $A_{l,\alpha}B_{r,\alpha} \geq A_{r,\alpha}B_{l,\alpha}$, $\forall\alpha \in [0,1]$, then $C_{l,\alpha} = \frac{A_{r,\alpha}}{B_{r,\alpha}}$, $C_{r,\alpha} = \frac{A_{l,\alpha}}{Bl,\alpha}$.

The fuzzy analog of the boundedness and persistence (see [18, 19]) is as follows: a sequence of positive fuzzy numbers (x_n) persists (resp. is bounded) if there exists a positive real number M (resp. N) such that

$$\operatorname{supp} x_n \subset [M,\infty) \qquad \bigl(\text{resp. } \operatorname{supp} x_n \subset (0,N]\bigr), \quad n = 1,2,\dots.$$

A sequence of positive fuzzy numbers (x_n) is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$\operatorname{supp} x_n \subset [M,N], \quad n = 1,2,\dots.$$

A sequence of positive fuzzy numbers (x_n), $n = 1,2,\dots$, is unbounded if the norm $\|x_n\|$, $n = 1,2,\dots$, is an unbounded sequence.

x_n is a positive solution of (1) if (x_n) is a sequence of positive fuzzy numbers which satisfies (1). A positive fuzzy number x is called a positive equilibrium of (1) if

$$x = A + \frac{x}{x^2}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x be a positive fuzzy number, $x_n \to x$ as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

3 Main results

3.1 Existence of solution of Eq. (1)

Firstly we study the existence of positive solutions of (1). We need the following lemma.

Lemma 3.1 [20] *Let $f : R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, A, B, C, D be fuzzy numbers. Then*

$$\bigl[f(A,B,C,D)\bigr]_\alpha = f\bigl([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha\bigr), \quad \alpha \in (0,1]. \tag{6}$$

Theorem 3.1 *Consider Eq.* (1) *where A is a positive fuzzy number. Then, for any positive fuzzy numbers x_{-2}, x_{-1}, x_0, there exists a unique positive solution x_n of* (1) *with initial conditions x_{-2}, x_{-1}, x_0.*

Proof The proof is similar to that of Proposition 2.1 in [19]. Suppose that there exists a sequence of fuzzy numbers (x_n) satisfying (1) with initial conditions x_{-2}, x_{-1}, x_0. Consider the α-cuts, $\alpha \in (0,1]$,

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad n = 0,1,2,\dots, \qquad [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]. \tag{7}$$

It follows from (1), (7) and Lemma 3.1 that

$$\begin{aligned}[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] &= \left[A + \frac{x_n}{x_{n-1}x_{n-2}}\right]_\alpha = [A]_\alpha + \frac{[x_n]_\alpha}{[x_{n-1}]_\alpha \times [x_{n-2}]_\alpha} \\ &= [A_{l,\alpha}, A_{r\alpha}] + \frac{[L_{n,\alpha}, R_{n,\alpha}]}{[L_{n-1,\alpha}L_{n-2,\alpha}, R_{n-1,\alpha}R_{n-2,\alpha}]}.\end{aligned}$$

Noting Remark 2.1, one of the following two cases holds.

Case (i)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, A_{r,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}\right]. \tag{8}$$

Case (ii)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}, A_{r,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}\right]. \tag{9}$$

If Case (i) holds true, it follows that for $n \in \{0,1,2,\dots\}$, $\alpha \in (0,1]$,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, \qquad R_{n+1,\alpha} = A_{r,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}. \tag{10}$$

Then it is obvious that for any initial condition $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2,-1,0$, $\alpha \in (0,1]$, there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$. Now we prove that $[L_{n,\alpha}, R_{n,\alpha}]$, $\alpha \in (0,1]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (10) with initial conditions $(L_{j,\alpha}, R_{j,\alpha})$, $j = -2,-1,0$, determines the solution x_n of (1) with initial conditions x_{-2}, x_{-1}, x_0 such that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0,1], n = 0,1,2,\dots. \tag{11}$$

From reference [18] and since x_j, $j = -2,-1,0$, are positive fuzzy numbers for any $\alpha_1, \alpha_2 \in (0,1]$, $\alpha_1 \le \alpha_2$, we have

$$0 < L_{j,\alpha_1} \le L_{j,\alpha_2} \le R_{j,\alpha_2} \le R_{j,\alpha_1}, \quad j = -2,-1,0. \tag{12}$$

We claim that

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}, \quad n = 0,1,2,\dots. \tag{13}$$

We prove it by induction. It is obvious from (12) that (13) holds true for $n = 0,1,2$. Suppose that (13) are true for $n \le k$, $k \in \{1,2,\dots\}$. Then, from (10), (12) and (13) for $n \le k$, it follows that

$$\begin{aligned} L_{k+1,\alpha_1} &= A_{l,\alpha_1} + \frac{L_{k,\alpha_1}}{L_{k-1,\alpha_1}L_{k-2,\alpha_1}} \le A_{l,\alpha} + \frac{L_{k,\alpha_2}}{L_{k-1,\alpha_2}L_{k-2,\alpha_2}} = L_{k+1,\alpha_2} \\ &= A_{l,\alpha} + \frac{L_{k,\alpha_2}}{L_{k-1,\alpha_2}L_{k-2,\alpha_2}} \le A_{r,\alpha} + \frac{R_{k,\alpha_2}}{R_{k-1,\alpha_2}R_{k-2,\alpha_2}} = R_{k+1,\alpha_2} \\ &= A_{r,\alpha} + \frac{R_{k,\alpha_2}}{R_{k-1,\alpha_2}R_{k-2,\alpha_2}} \le A_{r,\alpha} + \frac{R_{k,\alpha_1}}{R_{k-1,\alpha_1}R_{k-2,\alpha_1}} = R_{k+1,\alpha_1}. \end{aligned}$$

Therefore (13) are satisfied. Moreover, from (10) we have

$$L_{1,\alpha} = A_{l,\alpha} + \frac{L_{0,\alpha}}{L_{-2,\alpha}L_{-1,\alpha}}, \qquad R_{1,\alpha} = A_{r,\alpha} + \frac{R_{0,\alpha}}{R_{-2,\alpha}R_{-1,\alpha}}, \quad \alpha \in (0,1]. \tag{14}$$

Since x_j, $j = -2,-1,0$, are positive fuzzy numbers and A is a positive fuzzy number, then we have that $L_{0,\alpha}$, $R_{0,\alpha}$, $L_{-1,\alpha}$, $R_{-1,\alpha}$, $L_{-2,\alpha}$, $R_{-2,\alpha}$ are left continuous. So from (14) we have that $L_{1,\alpha}$, $R_{1,\alpha}$ are also left continuous. Inductively we can get that $L_{n,\alpha}$, $R_{n,\alpha}$, $n = 1, 2, \dots$, are left continuous.

Now we prove that the support of x_n, $\operatorname{supp} x_n = \overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, since x_j, $j = -2, -1, 0$, are positive fuzzy numbers and A is a positive fuzzy number, there exist constants $P > 0$, $Q > 0$, $M_j > 0$, $N_j > 0$, $j = -2, -1, 0$, such that for all $\alpha \in (0, 1]$,

$$[A_{l,\alpha}, A_{r,\alpha}] \subset [P, Q], \qquad [L_{j,\alpha}, R_{j,\alpha}] \subset [M_j, N_j], \quad j = -2, -1, 0. \tag{15}$$

Hence from (14) and (15) we can easily get

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[P + \frac{M_0}{M_{-1}M_{-2}}, Q + \frac{N_0}{N_{-1}N_{-2}}\right], \quad \alpha \in (0, 1], \tag{16}$$

from which it is obvious that

$$\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}] \subset \left[P + \frac{M_0}{M_{-1}M_{-2}}, Q + \frac{N_0}{N_{-1}N_{-2}}\right], \quad \alpha \in (0, 1]. \tag{17}$$

Therefore (17) implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. Deducing inductively we can easily obtain that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact, and

$$\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty), \quad n = 1, 2, \dots. \tag{18}$$

Therefore, (13), (18) and since $L_{n,\alpha}$, $R_{n,\alpha}$ are left continuous, we have that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers x_n such that (11) holds.

We prove now that x_n is the solution of (1) with initial conditions x_{-1}, x_0. Since for all $\alpha \in (0, 1]$,

$$\begin{aligned}
[x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\
&= \left[A_{l,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, A_{r,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}\right] \\
&= \left[A + \frac{x_n}{x_{n-1}x_{n-2}}\right]_\alpha,
\end{aligned}$$

we have that x_n is the solution of (1) with initial conditions x_{-2}, x_{-1}, x_0.

Suppose that there exists another solution $\overline{x}_n$ of (1) with initial conditions x_{-2}, x_{-1}, x_0. Then from arguing as above we can easily prove that

$$[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1], n = 0, 1, 2, \dots. \tag{19}$$

Then from (11) and (19) we have $[x_n]_\alpha = [\overline{x}_n]_\alpha$, $\alpha \in (0, 1]$, $n = 0, 1, 2, \dots$, from which it follows that $x_n = \overline{x}_n$, $n = 0, 1, \dots$.

If Case (ii) holds, the proof is similar to that of Case (i). Thus the proof of Theorem 3.1 is completed. □

3.2 Dynamics of Eq. (1)

To study the dynamical behavior of positive solutions of (1), according to Definition 2.3, we consider the following two cases.

Case (i)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, A_{r,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}\right].$$

Case (ii)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}, A_{r,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}\right].$$

Firstly, if Case (i) holds true, we give the following lemma.

Lemma 3.2 *Consider the system of difference equations*

$$y_{n+1} = p + \frac{y_n}{y_{n-1}y_{n-2}}, \qquad z_{n+1} = q + \frac{z_n}{z_{n-1}z_{n-2}}, \quad n = 0, 1, \ldots, \tag{20}$$

where $p, q \in (1, +\infty)$, $y_{-2}, y_{-1}, y_0, z_{-2}, z_{-1}, z_0 \in (0, +\infty)$. *Then, for* $n \geq 4$,

$$p \leq y_n \leq \frac{p^3}{p^2-1} + y_3, \qquad q \leq z_n \leq \frac{q^3}{q^2-1} + z_3. \tag{21}$$

Proof From (20) it is clear that $y_n > p$, $z_n > q$ for $n \geq 1$. In view of (20), we obtain for $n \geq 4$ that

$$y_n = p + \frac{y_{n-1}}{y_{n-2}y_{n-3}} \leq p + \frac{1}{p^2}y_{n-1}, \qquad z_n = q + \frac{z_{n-1}}{z_{n-2}z_{n-3}} \leq q + \frac{1}{q^2}z_{n-1}. \tag{22}$$

Working inductively, we conclude for $n - k \geq 3$ that

$$\begin{aligned} y_n &\leq p + \frac{1}{p} + \frac{1}{p^4}y_{n-2} \leq p + \frac{1}{p} + \frac{1}{p^3} + \frac{1}{p^6}y_{n-3} \leq p + \frac{1}{p} + \frac{1}{p^3} + \frac{1}{p^5} + \frac{1}{p^8}y_{n-4} \\ &\leq \cdots \leq \sum_{i=1}^{k} \frac{1}{p^{2i-3}} + \frac{y_{n-k}}{p^{2k}} = \frac{p}{1-1/p^2}\left[1 - \left(\frac{1}{p^2}\right)^k\right] + \frac{y_{n-k}}{p^{2k}} \\ &\leq \frac{p^3}{p^2-1} + y_{n-k}, \end{aligned} \tag{23}$$

$$\begin{aligned} z_n &\leq q + \frac{1}{q} + \frac{1}{q^4}z_{n-2} \leq q + \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^6}z_{n-3} \leq q + \frac{1}{q} + \frac{1}{q^3} + \frac{1}{q^5} + \frac{1}{q^8}z_{n-4} \\ &\leq \cdots \leq \sum_{i=1}^{k} \frac{1}{q^{2i-3}} + \frac{z_{n-k}}{q^{2k}} = \frac{q}{1-1/q^2}\left[1 - \left(\frac{1}{q^2}\right)^k\right] + \frac{z_{n-k}}{q^{2k}} \\ &\leq \frac{q^3}{q^2-1} + z_{n-k}. \end{aligned} \tag{24}$$

Notice that $n - k \geq 3$ is equivalent to $k \leq n - 3$. The assertion is true. □

Theorem 3.2 *Consider fuzzy difference equation* (1), *where A is a positive fuzzy number and the initial values x_{-1}, x_0 are positive fuzzy numbers. Suppose that there exist positive numbers P, Q for all $\alpha \in (0,1]$ such that $1 < P \leq A_{l,\alpha} \leq A_{r,\alpha} \leq Q$, then every positive solution x_n of* (1) *is bounded and persists.*

Proof (i) Let x_n be a positive solution of (1) such that (9) holds. From (8) it is obvious that

$$P \leq L_{n,\alpha}, \qquad P \leq R_{n,\alpha}, \quad n = 1, 2, \ldots, \alpha \in (0,1]. \tag{25}$$

Then from $A_{l,\alpha} \geq P > 1$, (25) and Lemma 3.2 we get

$$[L_{n,\alpha}, R_{n,\alpha}] \subset [P, T_\alpha], \quad n \geq 5, \tag{26}$$

where

$$T_\alpha = \max\left\{\frac{P^3}{P^2 - 1} + L_{3,\alpha}, \frac{Q^3}{Q^2 - 1} + R_{3,\alpha}\right\}.$$

Then since x_n is a positive fuzzy number, there exists a constant $T > 0$ such that for all $\alpha \in (0,1]$,

$$T_\alpha \leq T. \tag{27}$$

Therefore (25) and (26) imply that $[L_{n,\alpha}, R_{n,\alpha}] \subset [P, T]$, $n \geq 4$, from which we get for $n \geq 4$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [P, T]$, and so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [P, T]$. Thus the positive solution is bounded and persists. □

To show that every positive solution x_n of system (1) tends to the positive equilibrium x as $n \to \infty$, we need the following lemmas.

Lemma 3.3 *Consider the difference equation*

$$y_{n+1} = p + \frac{y_n}{y_{n-1}y_{n-2}}, \quad n = 0, 1, 2, \ldots. \tag{28}$$

Assume $p > \frac{2}{\sqrt{3}}$. Then the equilibrium point of (28) *is asymptotically stable.*

Proof Let $\overline{y}$ be an equilibrium point of (28), it is easy to get $\overline{y} = \frac{p+\sqrt{p^2+4}}{2}$. The linearized equation associated with (28) about equilibrium point $\overline{y}$ is

$$\begin{aligned} &y_{n+1} + \frac{2}{p + 2 + p\sqrt{p^2 + 4}}y_n - \frac{2}{p + 2 + p\sqrt{p^2 + 4}}y_{n-1} - \frac{2}{p + 2 + p\sqrt{p^2 + 4}}y_{n-2} \\ &\quad = 0, \quad n = 0, 1, 2, \ldots. \end{aligned} \tag{29}$$

Since $p > \frac{2}{\sqrt{3}}$, we can get

$$\frac{6}{p + 2 + p\sqrt{p^2 + 4}} < 1.$$

By virtue of Theorem 1.3.7 in [7], the equilibrium point of (28) is asymptotically stable. □

Lemma 3.4 *Consider the system of difference equations* (20), *and assume that* $q > p > \frac{2}{\sqrt{3}}$. *Then every positive solution of* (20) *converges to equilibrium* $(\overline{y}, \overline{z}) = (\frac{p+\sqrt{p^2+4}}{2}, \frac{q+\sqrt{q^2+4}}{2})$.

Proof It is clear that system (20) has a unique equilibrium $(\overline{y}, \overline{z}) = (\frac{p+\sqrt{p^2+4}}{2}, \frac{q+\sqrt{q^2+4}}{2})$. Let $\{y_n, z_n\}$ be an arbitrary positive solution of (20). Let

$$\Lambda_1 = \lim_{n\to\infty} \sup y_n, \qquad \lambda_1 = \lim_{n\to\infty} \inf y_n, \qquad \Lambda_2 = \lim_{n\to\infty} \sup z_n, \qquad \lambda_2 = \lim_{n\to\infty} \inf z_n.$$

From Lemma 3.2, we have $0 < p < \lambda_1 \leq \Lambda_1 < \infty$, $0 < q < \lambda_2 \leq \Lambda_2 < \infty$. This and (20) imply that

$$\Lambda_1 \leq p + \frac{\Lambda_1}{\lambda_1^2}, \qquad \Lambda_2 \leq q + \frac{\Lambda_2}{\lambda_2^2}, \qquad \lambda_1 \geq p + \frac{\lambda_1}{\Lambda_1^2}, \qquad \lambda_2 \geq q + \frac{\lambda_2}{\Lambda_2^2},$$

which can lead to

$$\Lambda_1 \leq \lambda_1, \qquad \Lambda_2 \leq \lambda_2.$$

Thus we have $\Lambda_1 = \lambda_1$ and $\Lambda_2 = \lambda_2$. Then $\lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} z_n$ exist. From the uniqueness of the positive equilibrium $(\overline{y}, \overline{z})$ of (20), we conclude that $\lim_{n\to\infty} y_n = \overline{y}$, $\lim_{n\to\infty} z_n = \overline{z}$. □

Theorem 3.3 *Suppose that for all* $\alpha \in (0,1]$, $A_{l,\alpha} > 2/\sqrt{3}$. *Then every positive solution* x_n *of* (1) *tends to the positive equilibrium* x *as* $n \to \infty$.

Proof Suppose that there exists a fuzzy number x such that

$$x = A + \frac{x}{x^2}, \qquad [x]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0,1], \tag{30}$$

where $L_\alpha, R_\alpha \geq 0$. Then from (30) we can prove that

$$L_\alpha = A_{l,\alpha} + \frac{L_\alpha}{L_\alpha^2}, \qquad R_\alpha = A_{r,\alpha} + \frac{R_\alpha}{R_\alpha^2}. \tag{31}$$

Hence from (31) we can have that

$$L_\alpha = \frac{A_{l,\alpha} + \sqrt{A_{l,\alpha}^2 + 4}}{2}, \qquad R_\alpha = \frac{A_{r,\alpha} + \sqrt{A_{r,\alpha}^2 + 4}}{2}.$$

Let x_n be a positive solution of (1) such that (7) holds. Since $A_{l,\alpha} > 2/\sqrt{3}$, $\alpha \in (0,1]$, we can apply Lemma 3.4 to system (10), and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \qquad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{32}$$

Therefore from (31) we have

$$\lim_{n\to\infty} D(x_n,x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \bigl\{\max\bigl\{|L_{n,\alpha}-L_\alpha|,|R_{n,\alpha}-R_\alpha|\bigr\}\bigr\} = 0.$$

This completes the proof of the theorem. □

Secondly, if Case (ii) holds true, it follows that for $n \in \{0,1,2,\ldots\}$, $\alpha \in (0,1]$,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}, \qquad R_{n+1,\alpha} = A_{r,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}. \tag{33}$$

We need the following lemmas.

Lemma 3.5 *Consider the system of difference equations*

$$y_{n+1} = p + \frac{z_n}{z_{n-1}z_{n-2}}, \qquad z_{n+1} = q + \frac{y_n}{y_{n-1}y_{n-2}}, \quad n = 0,1,\ldots, \tag{34}$$

where $p,q \in (1,+\infty)$, $y_{-2},y_{-1},y_0,z_{-2},z_{-1},z_0 \in (0,+\infty)$. *Then, for* $n \geq 4$,

$$p \leq y_n \leq \frac{p^2q}{pq-1} + y_2, \qquad q \leq z_n \leq \frac{q^2p}{pq-1} + z_2. \tag{35}$$

Proof From (34) it is clear that $y_n \geq p$, $z_n \geq q$ for $n \geq 1$. And for $n \geq 4$ we obtain that

$$y_n \leq p + \frac{1}{q^2}z_{n-1} \leq p + \frac{1}{q} + \frac{1}{p^2q^2}y_{n-2}, \qquad z_n \leq q + \frac{1}{p^2}y_{n-1} \leq q + \frac{1}{p} + \frac{1}{q^2p^2}z_{n-2}. \tag{36}$$

Working inductively, for $n - 2k \geq 2$, it can concluded that

$$\begin{aligned} y_n &\leq p + \frac{1}{q} + \frac{1}{p^2q^2}y_{n-2} \leq p + \frac{1}{q} + \frac{1}{p^2q^2}\left(p + \frac{1}{q^2}z_{n-3}\right) \\ &\leq p + \frac{1}{q} + \frac{1}{pq^2} + \frac{1}{p^2q^3} + \frac{1}{p^4q^4}y_{n-4} \\ &\leq \cdots \leq \sum_{i=1}^{2k} \frac{1}{p^{i-2}q^{i-1}} + \frac{1}{p^{2k}q^{2k}}y_{n-2k} = \frac{p}{1-1/(pq)}\left[1 - \left(\frac{1}{pq}\right)^{2k}\right] + \frac{1}{p^{2k}q^{2k}}y_{n-2k} \\ &\leq \frac{p^2q}{pq-1} + y_{n-2k}, \end{aligned} \tag{37}$$

$$\begin{aligned} z_n &\leq q + \frac{1}{p} + \frac{1}{q^2p^2}z_{n-2} \leq q + \frac{1}{p} + \frac{1}{q^2p^2}\left(q + \frac{1}{p^2}y_{n-3}\right) \\ &\leq q + \frac{1}{p} + \frac{1}{qp^2} + \frac{1}{q^2p^3} + \frac{1}{q^4p^4}z_{n-4} \\ &\leq \cdots \leq \sum_{i=1}^{2k} \frac{1}{q^{i-2}p^{i-1}} + \frac{1}{q^{2k}p^{2k}}z_{n-2k} = \frac{q}{1-1/(pq)}\left[1 - \left(\frac{1}{pq}\right)^{2k}\right] + \frac{1}{p^{2k}q^{2k}}z_{n-2k} \\ &\leq \frac{pq^2}{pq-1} + z_{n-2k}. \end{aligned} \tag{38}$$

Notice that $n - 2k \geq 2$ is equivalent to $k \leq (n-2)/2$. The assertion is true. □

Lemma 3.6 *Consider the system of difference equations* (34), *if*

$$p > 1, \qquad q > 1, \qquad \sqrt{3}pq > \max\{3p - q, 3q - p\} \tag{39}$$

are satisfied, then the unique positive equilibrium point $(\overline{y}, \overline{z})$ *is locally asymptotically stable.*

Proof From (34) the system of difference equations has a unique positive equilibrium point $(\overline{y}, \overline{z}) = (\frac{pq+\sqrt{p^2q^2+4pq}}{2p}, \frac{pq+\sqrt{p^2q^2+4pq}}{2q})$. The linearized equation of system (33) about the equilibrium point $(\overline{y}, \overline{z})$ is

$$\Psi_{n+1} = B\Psi_n, \tag{40}$$

where $\Psi_n = (y_n, y_{n-1}, y_{n-2}, z_n, z_{n-1}, z_{n-2})^T$, and

$$B = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\overline{z}^2} & -\frac{1}{\overline{z}^2} & -\frac{1}{\overline{z}^2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\overline{y}^2} & -\frac{1}{\overline{y}^2} & -\frac{1}{\overline{y}^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\lambda_1, \lambda_2, \dots, \lambda_6$ denote the eigenvalues of matrix B, let $D = \operatorname{diag}(d_1, d_2, \dots, d_6)$ be a diagonal matrix, where $d_1 = d_4 = 1$, $d_i = d_{3+i} = 1 - i\varepsilon$ $(i = 2, 3)$, and

$$0 < \varepsilon < \min\left\{\frac{1}{3} - \frac{1}{\overline{z}^2}, \frac{1}{3} - \frac{1}{\overline{y}^2}\right\}. \tag{41}$$

Clearly, D is invertible. Computing matrix DBD^{-1}, we obtain that

$$DBD^{-1} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\overline{z}^2}d_1d_4^{-1} & -\frac{1}{\overline{z}^2}d_1d_5^{-1} & -\frac{1}{\overline{z}^2}d_1d_6^{-1} \\ d_2d_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & d_3d_2^{-1} & 0 & 0 & 0 & 0 \\ \frac{1}{\overline{y}^2}d_4d_1^{-1} & -\frac{1}{\overline{y}^2}d_4d_2^{-1} & -\frac{1}{\overline{y}^2}d_4d_3^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_5d_4^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_6d_5^{-1} & 0 \end{pmatrix}.$$

From $d_1 > d_2 > d_3 > 0$, $d_4 > d_5 > d_6 > 0$ it follows that

$$d_2d_1^{-1} < 1, \qquad d_3d_2^{-1} < 1, \qquad d_5d_4^{-1} < 1, \qquad d_6d_5^{-1} < 1.$$

Furthermore, noting (41) we have

$$\begin{aligned} &\frac{1}{\overline{z}^2}d_1d_4^{-1} + \frac{1}{\overline{z}^2}d_1d_5^{-1} + \frac{1}{\overline{z}^2}d_1d_6^{-1} = \frac{1}{\overline{z}^2}\left(1 + \frac{1}{1-2\varepsilon} + \frac{1}{1-3\varepsilon}\right) < \frac{3}{\overline{z}^2(1-3\varepsilon)} < 1, \\ &\frac{1}{\overline{y}^2}d_4d_1^{-1} + \frac{1}{\overline{z}^2}d_4d_2^{-1} + \frac{1}{\overline{z}^2}d_4d_3^{-1} = \frac{1}{\overline{y}^2}\left(1 + \frac{1}{1-2\varepsilon} + \frac{1}{1-3\varepsilon}\right) < \frac{3}{\overline{y}^2(1-3\varepsilon)} < 1. \end{aligned}$$

It is well known that B has the same eigenvalues as DBD^{-1}, we have that

$$\begin{aligned}\max_{1\le i\le 6}|\lambda_i| &\le \|DBD^{-1}\|_\infty \\ &= \max\biggl\{d_2d_1^{-1}, d_3d_2^{-1}, d_5d_4^{-1}, d_6d_5^{-1}, \frac{1}{\overline{z}^2}d_1d_4^{-1}+\frac{1}{\overline{z}^2}d_1d_5^{-1}+\frac{1}{\overline{z}^2}d_1d_6^{-1}, \\ &\quad \frac{1}{\overline{y}^2}d_4d_1^{-1}+\frac{1}{\overline{z}^2}d_4d_2^{-1}+\frac{1}{\overline{z}^2}d_4d_3^{-1}\biggr\} \\ &< 1.\end{aligned}$$

This implies that the equilibrium $(\overline{y},\overline{z})$ of (34) is locally asymptotically stable. □

Lemma 3.7 *Consider the system of difference equations* (34) *if* $p,q\in(1,+\infty)$ *and* $\sqrt{3}pq>\max\{3p-q,3q-p\}$. *Then every positive solution of* (34) *converges to the equilibrium point* $(\overline{y},\overline{z})$.

Proof It is clear that system (34) has a unique positive equilibrium point

$$(\overline{y},\overline{z})=\left(\frac{pq+\sqrt{p^2q^2+4pq}}{2q},\frac{pq+\sqrt{p^2q^2+4pq}}{2p}\right).$$

Let $\{y_n,z_n\}$ be an arbitrary positive solution of (33). From (33)-(35) we have

$$\lim_{n\to\infty}\sup y_n=L_1,\qquad \lim_{n\to\infty}\inf y_n=l_1,\qquad \lim_{n\to\infty}\sup z_n=L_2,\qquad \lim_{n\to\infty}\inf z_n=l_2, \tag{42}$$

where $l_i,L_i\in(0,+\infty)$, $i=1,2$. Then from (34) and (42) we get

$$L_1\le p+\frac{L_2}{l_2^2},\qquad l_1\ge p+\frac{l_2}{L_2^2},\qquad L_2\le q+\frac{L_1}{l_1^2},\qquad l_2\ge q+\frac{l_1}{L_1^2},$$

from which we have

$$\bigl(L_1l_2^2-L_2\bigr)L_2^2\le\bigl(l_1L_2^2-l_2\bigr)l_2^2,\qquad \bigl(L_2l_1^2-L_1\bigr)L_1^2\le\bigl(l_2L_1^2-l_1\bigr)l_1^2. \tag{43}$$

We claim that

$$L_1=l_1,\qquad L_2=l_2. \tag{44}$$

Suppose on the contrary that $L_2>l_2$, then from the first inequality of (43) we have $L_1l_2<l_1L_2$, and so $L_1<l_1$, which is a contradiction. So $L_2=l_2$. Similarly we can prove that $L_1=l_1$. Hence from (34) and (43) there exist $\lim_{n\to\infty}y_n=\overline{y}$, $\lim_{n\to\infty}z_n=\overline{z}$. This completes the proof of Lemma 3.7. □

Combining Lemma 3.6 with Lemma 3.7, we obtain the following theorem.

Theorem 3.4 *Consider the system of difference equations* (34). *If relations* (39) *are satisfied, then the unique positive equilibrium* $(\overline{y},\overline{z})$ *is globally asymptotically stable.*

Theorem 3.5 *Suppose that*

$$A_{l,\alpha} > 1 \quad \text{and} \quad \sqrt{3}A_{l,\alpha}A_{r,\alpha} > 3A_{r,\alpha} - A_{l,\alpha}, \quad \forall\alpha \in (0,1]. \tag{45}$$

Then every positive solution of (1) *tends to the positive equilibrium x as $n \to +\infty$.*

Proof The proof is similar to that of Theorem 3.3. Suppose that there exists a fuzzy number x satisfying (30). Then from (30) we can get

$$L_\alpha = A_{l,\alpha} + \frac{R_\alpha}{R_\alpha^2}, \qquad R_\alpha = A_{r,\alpha} + \frac{L_\alpha}{L_\alpha^2}. \tag{46}$$

Hence we have from (46) that

$$L_\alpha = \frac{A_{l,\alpha}A_{r,\alpha} + \sqrt{A_{l,\alpha}^2 A_{r,\alpha}^2 + 4A_{l,\alpha}A_{r,\alpha}}}{2A_{r,\alpha}}, \qquad R_\alpha = \frac{A_{l,\alpha}A_{r,\alpha} + \sqrt{A_{l,\alpha}^2 A_{r,\alpha}^2 + 4A_{l,\alpha}A_{r,\alpha}}}{2A_{l,\alpha}}.$$

Let x_n be a positive solution of (1) such that (7) holds. Since (45) is satisfied, we can apply Lemma 3.6 and Lemma 3.7 to system (33), and so we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \qquad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{47}$$

Therefore from (46) we have

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \bigl\{\max\bigl\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\bigr\}\bigr\} = 0.$$

This completes the proof of the theorem. □

4 Numerical example

Example 4.1 Consider the following fuzzy difference equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \quad n = 0, 1, \ldots, \tag{48}$$

we take A and the initial values x_{-2}, x_{-1}, x_0 such that

$$x_{-2}(x) = \begin{cases} x - 5, & 5 \le x \le 6, \\ -\frac{1}{2}x + 4, & 6 \le x \le 8, \end{cases} \qquad x_{-1}(x) = \begin{cases} \frac{1}{2}x - 2, & 4 \le x \le 6, \\ -\frac{1}{2}x + 4, & 6 \le x \le 8, \end{cases} \tag{49}$$

$$A(x) = \begin{cases} 2x - 4, & 2 \le x \le 2.5, \\ -2x + 6, & 2.5 \le x \le 3, \end{cases} \qquad x_0(x) = \begin{cases} \frac{1}{2}x - 0.5, & 1 \le x \le 3, \\ -\frac{1}{2}x + 2.5, & 3 \le x \le 5. \end{cases} \tag{50}$$

From (49) we get

$$[x_{-2}]_\alpha = [5 + \alpha, 8 - 2\alpha], \qquad [x_{-1}]_\alpha = [4 + 2\alpha, 8 - 2\alpha], \quad \alpha \in (0,1]. \tag{51}$$

From (50) we get

$$[A]_\alpha = \left[2 + \frac{1}{2}\alpha, 3 - \frac{1}{2}\alpha\right], \qquad [x_0]_\alpha = [1 + 2\alpha, 5 - 2\alpha], \quad \alpha \in (0,1]. \tag{52}$$

Figure 1 Dynamics of system (54).

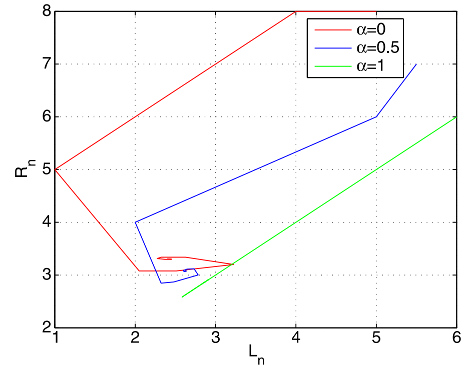
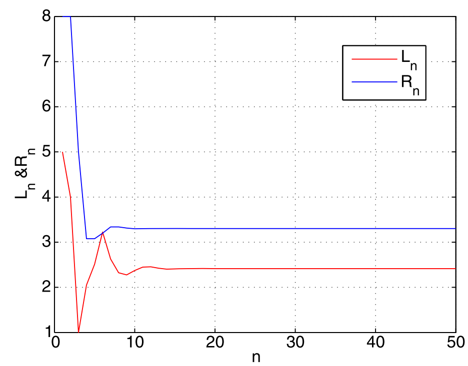


Figure 2 The solution of system (54) in $\alpha = 0$.



Therefore, it follows that

$$\begin{aligned} &\overline{\bigcup_{\alpha\in(0,1]} [x_{-2}]_\alpha} = [5,8], \qquad \overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]_\alpha} = [4,8], \\ &\overline{\bigcup_{\alpha\in(0,1]} [x_0]_\alpha} = [1,5], \qquad \overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = [2,3]. \end{aligned} \tag{53}$$

From (48), it results in a coupled system of difference equations with parameter α,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, \qquad R_{n+1,\alpha} = A_{r,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}, \quad \alpha \in (0,1]. \tag{54}$$

Therefore, $A_{l,\alpha} > 2/\sqrt{3}$, $\forall\alpha \in (0,1]$, and the initial values x_{-i} ($i = 0,1,2$) are positive fuzzy numbers. So from Theorem 3.2 we have that every positive solution x_n of Eq. (48) is bounded and persists. In addition, from Theorem 3.3, Eq. (48) has a unique positive equilibrium $\overline{x} = (2.4142, 2.8508, 3.3028)$. Moreover, every positive solution x_n of Eq. (48) converges to the unique equilibrium $\overline{x}$ with respect to D as $n \to \infty$ (see Figures 1-4).

Example 4.2 Consider the following fuzzy difference equation:

$$x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}, \quad n = 0,1,\ldots, \tag{55}$$

Figure 3 The solution of system (54) in $\alpha = 0.5$.

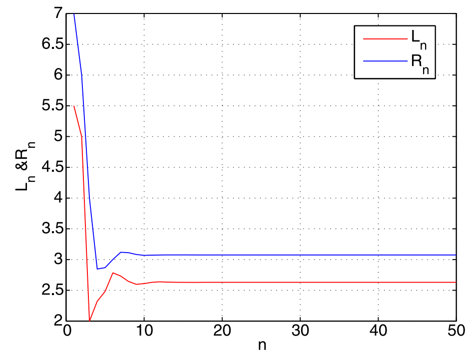
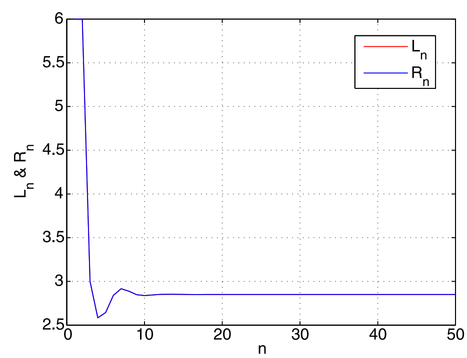


Figure 4 The solution of system (54) in $\alpha = 1$.



where A and the initial values x_{-2}, x_{-1}, x_0 are satisfied

$$A(x) = \begin{cases} 2x - 3, & 1.5 \le x \le 2, \\ -x + 3, & 2 \le x \le 3, \end{cases} \qquad x_{-2}(x) = \begin{cases} \frac{1}{4}x - 0.5, & 2 \le x \le 6, \\ -\frac{1}{2}x + 4, & 6 \le x \le 8, \end{cases} \tag{56}$$

$$x_{-1}(x) = \begin{cases} 2x - 4, & 2 \le x \le 2.5, \\ -2x + 6, & 2.5 \le x \le 3, \end{cases} \qquad x_0(x) = \begin{cases} \frac{1}{2}x - 0.5, & 1 \le x \le 3, \\ -\frac{1}{2}x + 2.5, & 3 \le x \le 5. \end{cases} \tag{57}$$

From (56) we get

$$[A]_\alpha = \left[1.5 + \frac{1}{2}\alpha, 3 - \alpha\right], \qquad [x_{-2}]_\alpha = [2 + 4\alpha, 8 - 2\alpha], \quad \alpha \in (0,1]. \tag{58}$$

From (57) we get

$$[x_{-1}]_\alpha = \left[2 + \frac{1}{2}\alpha, 3 - \frac{1}{2}\alpha\right], \qquad [x_0]_\alpha = [1 + 2\alpha, 5 - 2\alpha], \quad \alpha \in (0,1]. \tag{59}$$

Therefore, it follows that

$$\begin{aligned} &\overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = [1.5, 3], \qquad \overline{\bigcup_{\alpha\in(0,1]} [x_{-2}]_\alpha} = [2, 8], \\ &\overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]_\alpha} = [2, 3], \qquad \overline{\bigcup_{\alpha\in(0,1]} [x_0]_\alpha} = [1, 5]. \end{aligned} \tag{60}$$

From (55) it results in a coupled system of difference equations with parameter α,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{R_{n,\alpha}}{R_{n-1,\alpha}R_{n-2,\alpha}}, \qquad R_{n+1,\alpha} = A_{r,\alpha} + \frac{L_{n,\alpha}}{L_{n-1,\alpha}L_{n-2,\alpha}}, \quad \alpha \in (0,1]. \tag{61}$$

It is clear that (45) is satisfied and the initial values x_{-i} ($i = 0,1,2$) are positive fuzzy numbers, so from Theorem 3.5, Eq. (55) has a unique positive equilibrium $\overline{x} = (1.7808, 2.4142, 3.5616)$. Moreover, every positive solution x_n of Eq. (55) converges to the unique equilibrium $\overline{x}$ with respect to D as $n \to \infty$ (see Figures 5-8).

Figure 5 Dynamics of system (61).

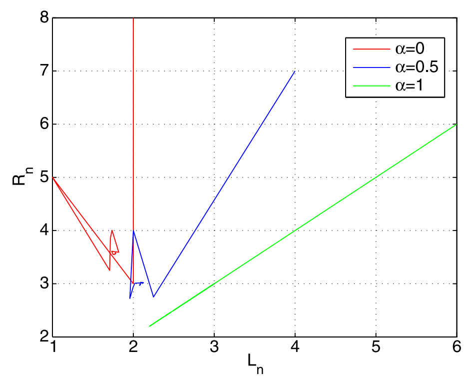


Figure 6 The solution of system (61) in $\alpha = 0$.

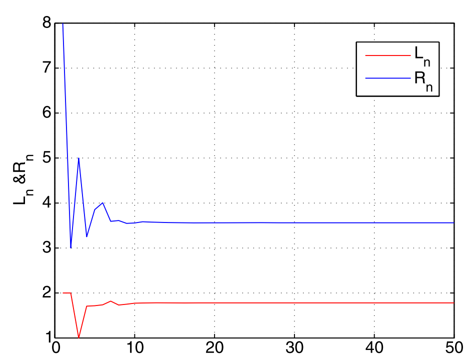


Figure 7 The solution of system (61) in $\alpha = 0.5$.

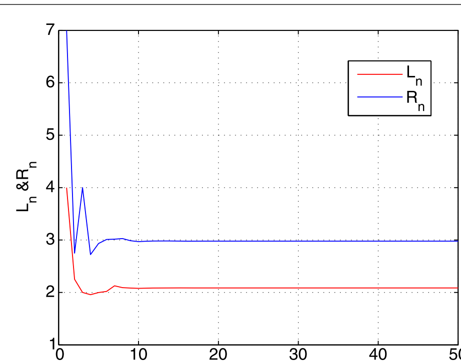
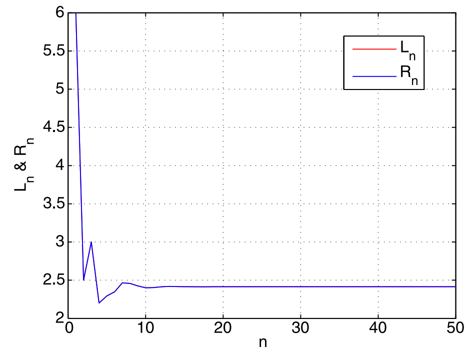


Figure 8 The solution of system (61) in $\alpha = 1$.



5 Conclusion

In this work, according to a generalization of division (g-division) of fuzzy numbers, we study the fuzzy difference equation $x_{n+1} = A + \frac{x_n}{x_{n-1}x_{n-2}}$. The existence of positive solution to (1) is investigated. Furthermore, we obtain the following results:

(i) The positive solution is bounded and persists if $A_{l,\alpha} > 1$, $\alpha \in (0,1]$, every solution x_n tends to the unique equilibrium x under condition $A_{l,\alpha} > 2/\sqrt{3}$, $\alpha \in (0,1]$ as $n \to \infty$.

(ii) If $A_{l,\alpha} > 1$ and $\sqrt{3}A_{l,\alpha}A_{r,\alpha} > 3A_{r,\alpha} - A_{l,\alpha}$, $\alpha \in (0,1]$, every solution x_n of (1) converges to the unique equilibrium x as $n \to \infty$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors indicated in parentheses made substantial contributions to the following tasks of research: drafting the manuscript (QHZ, JZL); participating in the design of the study (ZGL); writing and revision of paper (QHZ). All authors read and approved the final manuscript.

Author details

[1]Key Laboratory of Economics System Simulation, School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, Guizhou 550025, People's Republic of China. [2]Department of Mathematics and Physics, Hunan Institute of Technology, Hengyang, Hunan 421002, People's Republic of China. [3]Department of Mathematics, Hengyang Normal University, Hengyang, Hunan 421002, People's Republic of China.

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