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On (h,q)-Daehee numbers and polynomials

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Abstract

The p-adic q-integral (sometimes called q-Volkenborn integration) was defined by Kim. From p-adic q-integral equations, we can derive various q-extensions of Bernoulli polynomials and numbers. DS Kim and T Kim studied Daehee polynomials and numbers and their applications. Kim *et al.* introduced the q-analogue of Daehee numbers and polynomials which are called q-Daehee numbers and polynomials. Lim considered the modified q-Daehee numbers and polynomials which are different from the q-Daehee numbers and polynomials of Kim *et al.* In this paper, we consider (h,q)-Daehee numbers and polynomials and give some interesting identities. In case $h=0$, we cover the q-analogue of Daehee numbers and polynomials of Kim *et al.* In case $h=1$, we modify q-Daehee numbers and polynomials. We can find out various (h,q)-related numbers and polynomials which are studied by many authors.

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1 Introduction

Let p be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The p-adic norm is defined $|p|_p = \frac{1}{p}$.

When one talks of q-extension, q is variously considered as an indeterminate, complex $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for each $x \in \mathbb{Z}_p$. Throughout this paper, we use the notation

$$[x]_q = \frac{1-q^x}{1-q}.$$

Note that $\lim_{q\to 1}[x]_q = x$ for each $x \in \mathbb{Z}_p$.

Let $UD(\mathbb{Z}_p)$ be the space of a uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)q^x \quad \text{(see [1, 2])}. \tag{1}$$

Using this integration, the q-Daehee polynomials $D_{n,q}(x)$ are defined and studied by Kim *et al.* (see [3]), their generating function is as follows:

$$\frac{1-q+\frac{1-q}{\log q}\log(1+t)}{1-q-qt}(1+t)^x = \sum_{n=0}^{\infty} D_{n,q}(x)\frac{t^n}{n!}. \tag{2}$$

The generating function of the modified q-Daehee polynomials are defined and studied by Lim (see [4]).

$$F_q(x,t) = \frac{q-1}{\log q}\frac{\log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x|q)\frac{t^n}{n!} \quad \text{(see [1–16])}. \tag{3}$$

From (1), we have the following integral identity:

$$qI_q(f_1) - I_q(f) = \frac{q-1}{\log q}f'(0) + (q-1)f(0), \tag{4}$$

where $f_1(x) = f(x+1)$ and $\frac{d}{dx}f(x) = f'(x)$.

In a special case, for $h \in \mathbb{Z}_+$ $(= \mathbb{N} \cup \{0\})$, we apply $f(x) = q^{-hx}e^{tx}$ on (4), we have

$$\int_{\mathbb{Z}_p} q^{-hx}e^{xt}\,d\mu_q(x) = \frac{q^{h-1}(q-1)}{\log q}\frac{t-(h-1)\log q}{e^t - q^{h-1}}. \tag{5}$$

For $h \in \mathbb{Z}_+$, we define the (h,q)-Bernoulli number $B_n^{(h)}(q)$ as follows:

$$\sum_{n=0}^{\infty} B_n^{(h)}(q)\frac{t^n}{n!} = \frac{q^{h-1}(q-1)}{\log q}\frac{t-(h-1)\log q}{e^t - q^{h-1}}. \tag{6}$$

Indeed if $q \to 1$, we have $\lim_{q\to 1} B_n^{(h)}(q) = B_n$. So we call this $B_n^{(h)}(q)$ the nth (h,q)-Bernoulli number. And we define (h,q)-Bernoulli polynomials and the generating function to be

$$\frac{q^{h-1}(q-1)}{\log q}\frac{t-(h-1)\log q}{e^t - q^{h-1}}e^{xt} = \sum_{n=0}^{\infty} B_n^{(h)}(x|q)\frac{t^n}{n!}. \tag{7}$$

When $x = 0$, $B_n^{(h)}(0|q) = B_n^{(h)}(q)$ are the nth (h,q)-Bernoulli numbers.

From (4) and (7), we have

$$B_n^{(h)}(x|q) = \int_{\mathbb{Z}_p} q^{-hy}(x+y)^n\,d\mu_q(y).$$

From (7) we note that

$$B_n^{(h)}(x|q) = \sum_{l=0}^{n}\binom{n}{l}B_l^{(h)}(q)x^{n-l}. \tag{8}$$

For the case $|t|_p \leq p^{-\frac{1}{p-1}}$, the Daehee polynomials are defined as follows (see [3]):

$$\sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!} = \frac{\log(1+t)}{t}(1+t)^x. \tag{9}$$

From (2) and (3), if $q \to 1$, we have

$$\lim_{q\to 1} D_{n,q}(x) = D_n(x)$$

and

$$\lim_{q\to 1} D_n(x|q) = D_n(x).$$

The p-adic q-integral (or q-Volkenborn integration) was defined by Kim (see [1, 2]). From p-adic q-integral equations, we can derive various q-extensions of Bernoulli polynomials and numbers (see [1–24]). In [20], DS Kim and T Kim studied Daehee polynomials and numbers and their applications. In [3], Kim *et al.* introduced the q-analogue of Daehee numbers and polynomials which are called q-Daehee numbers and polynomials. Lim considered in [4] the modified q-Daehee numbers and polynomials which are different from the q-Daehee numbers and polynomials of Kim *et al.* In this paper, we consider (h,q)-Daehee numbers and polynomials and give some interesting identities. In case $h = 0$, we cover the q-analogue of Daehee numbers and polynomials of Kim *et al.* (see [3]). In case $h = 1$, we have modified q-Daehee numbers and polynomials in [4]. We can find out various (h,q)-related numbers and polynomials in [10, 13, 14].

2 (h,q)-Daehee numbers and polynomials

Let us now consider the p-adic q-integral representation as follows: for each $h \in \mathbb{Z}_+$,

$$\int_{\mathbb{Z}_p} q^{-hy}(x+y)_n \, d\mu_q(y) \quad \bigl(n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}\bigr), \tag{10}$$

where $(x)_n$ is known as the *Pochhammer symbol* (or *decreasing factorial*) defined by

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{11}$$

and here $S_1(n,k)$ is the Stirling number of the first kind (see [3, 20]).

From (10) we have

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-hy}(y)_n \, d\mu_q(y)\right)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{-hy}\left(\sum_{n=0}^{\infty}\binom{y}{n}t^n\right) d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-hy}(1+t)^y \, d\mu_q(y),\end{aligned} \tag{12}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

For $|t|_p < p^{-\frac{1}{p-1}}$, from (4) we have

$$\int_{\mathbb{Z}_p} q^{-hy}(1+t)^y\, d\mu_q(y) = \frac{q^{h-1}(q-1)}{\log q}\frac{\log\frac{1+t}{q^{h-1}}}{1+t-q^{h-1}}. \tag{13}$$

Let

$$F_q^{(h)}(t) = \frac{q^{h-1}(q-1)}{\log q}\frac{\log\frac{1+t}{q^{h-1}}}{1+t-q^{h-1}} = \sum_{n=0}^{\infty} D_n^{(h)}(q)\frac{t^n}{n!}. \tag{14}$$

Here, the numbers $D_n^{(h)}(q)$ are called the nth (h,q)-Daehee numbers of the first kind. Moreover, we have

$$D_n^{(h)}(q) = \int_{\mathbb{Z}_p} q^{-hy}(y)_n\, d\mu_q(y). \tag{15}$$

From (14) and (15), if $h=0$, $D_n^{(0)}(q)$ is just the q-Daehee numbers which are defined by Kim *et al.* in [3]. If $h=1$, $D_n^{(1)}(q)$ is just the modified q-Daehee numbers which are studied in [4].

On the other hand, we can derive (h,q)-Daehee polynomials

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-hy}(x+y)_n\, d\mu_q(y)\right)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p} q^{-hy}\left(\sum_{n=0}^{\infty}\binom{x+y}{n}t^n\right)d\mu_q(y)\\ &= \int_{\mathbb{Z}_p} q^{-hy}(1+t)^{x+y}\, d\mu_q(y)\\ &= \frac{q^{h-1}(q-1)}{\log q}\frac{\log(1+t)-(h-1)\log q}{1+t-q^{h-1}}(1+t)^x\\ &= \sum_{n=0}^{\infty} D_n^{(h)}(x|q)\frac{t^n}{n!},\end{aligned} \tag{16}$$

where $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$.

When $x=0$, $D_n^{(h)}(0|q) = D_n^{(h)}(q)$ is called the nth (h,q)-Daehee number.

Notice that $F_q^{(h)}(0,t)$ seems to be a new q-extension of the generating function for Daehee numbers of the first kind. Therefore, from (9) and the following fact, we get

$$\lim_{q\to 1} F_q^{(h)}(t) = \frac{\log(1+t)}{t}.$$

From (11) and (12), we have

$$D_n^{(h)}(x|q) = \int_{\mathbb{Z}_p} q^{-hy}(x+y)_n\, d\mu_q(y) = \sum_{k=0}^{n} S_1(n,k)B_k^{(h)}(x|q), \tag{17}$$

where $B_k^{(h)}(x|q)$ are the (h,q)-Bernoulli polynomials introduced in (7).

Thus we have the following theorem, which relates (h,q)-Bernoulli polynomials and (h,q)-Daehee polynomials.

Theorem 1 *For $n, m \in \mathbb{Z}_+$, we have the following equalities*:

$$D_n^{(h)}(x|q) = \sum_{k=0}^{n} S_1(n,k) B_k^{(h)}(x|q)$$

and

$$D_n^{(h)}(q) = \sum_{k=0}^{n} S_1(n,k) B_k^{(h)}(q).$$

From the generating function of the (h,q)-Daehee polynomials in $D_n^{(h)}(x|q)$ in (14), by replacing t to $e^t - 1$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} D_n^{(h)}(x|q) \frac{(e^t-1)^n}{n!} &= \frac{q^{h-1}(q-1)}{\log q} \frac{t-(h-1)\log q}{e^t - q^{h-1}} e^{xt} \\ &= \sum_{n=0}^{\infty} B_n^{(h)}(x|q) \frac{t^n}{n!}. \end{aligned} \tag{18}$$

On the other hand,

$$\sum_{n=0}^{\infty} D_n^{(h)}(x|q) \frac{(e^t-1)^n}{n!} = \sum_{m=0}^{\infty} D_m^{(h)}(x|q) \sum_{n=0}^{\infty} S_2(n,m) \frac{t^n}{n!}. \tag{19}$$

Here, $S_2(n,m)$ is the Stirling number of the second kind defined by the following generating series:

$$\sum_{n=m}^{\infty} S_2(n,m) \frac{t^n}{n!} = \frac{(e^t-1)^m}{m!} \quad \textit{cf.}\ [3, 20]. \tag{20}$$

Thus by comparing the coefficients of t^n, we have

$$B_n^{(h)}(x|q) = \sum_{m=0}^{n} D_m^{(h)}(x|q) S_2(n,m).$$

Therefore, we obtain the following theorem.

Theorem 2 *For $n, m \in \mathbb{Z}_+$, we have the following identity*:

$$B_n^{(h)}(x|q) = \sum_{m=0}^{n} D_m^{(h)}(x|q) S_2(n,m).$$

The increasing factorial sequence is known as

$$x^{(n)} = x(x+1)(x+2)\cdots(x+n-1) \quad (n \in \mathbb{Z}_+).$$

Let us define the (h,q)-Daehee numbers of the second kind as follows:

$$\widehat{D}_n^{(h)}(q) = \int_{\mathbb{Z}_p} q^{-hy} (-y)_n \, d\mu_q(y) \quad (n \in \mathbb{Z}_+). \tag{21}$$

It is easy to observe that

$$x^{(n)} = (-1)^n(-x)_n = \sum_{k=0}^{n} S_1(n,k)(-1)^{n-k}x^k. \tag{22}$$

From (21) and (22), we have

$$\begin{aligned}\widehat{D}_n^{(h)}(q) &= \int_{\mathbb{Z}_p} q^{-hy}(-y)_n \, d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-hy}y^{(n)}(-1)^n \, d\mu_q(y) \\ &= \sum_{k=0}^{n} S_1(n,k)(-1)^k B_k^{(h)}(q).\end{aligned} \tag{23}$$

Thus, we state the following theorem, which relates (h,q)-Daehee numbers and (h,q)-Bernoulli numbers.

Theorem 3 *The following holds true*:

$$\widehat{D}_n^{(h)}(q) = \sum_{k=0}^{n} S_1(n,k)(-1)^k B_k^{(h)}(q).$$

Let us now consider the generating function of (h,q)-Daehee numbers of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_n^{(h)}(q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p} q^{-hy}(-y)_n \, d\mu_q(y)\right)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} q^{-hy}\left(\sum_{n=0}^{\infty}\binom{-y}{n}t^n\right) d\mu_q(y) \\ &= \int_{\mathbb{Z}_p} q^{-hy}(1+t)^{-y} \, d\mu_q(y).\end{aligned} \tag{24}$$

From (4) and (24), we have the generating function for (h,q)-Daehee numbers of the second kind as follows:

$$\int_{\mathbb{Z}_p} q^{-hy}(1+t)^{-y} \, d\mu_q(y) = \frac{q^{h-1}(q-1)}{\log q}\frac{\log q - \log(1+t)}{1+t-q^{h-1}}. \tag{25}$$

Let us consider the (h,q)-Daehee polynomials of the second kind as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \widehat{D}_n^{(h)}(x|q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p} q^{-hy}(x-y)_n \, d\mu_q(y)\frac{t^n}{n!} \\ &= \int_{\mathbb{Z}_p} q^{-hy}(1+t)^{x-y} \, d\mu_q(y) \\ &= \frac{q^{h-1}(q-1)}{\log q}\frac{\log q - \log(1+t)}{1+t-q^{h}}(1+t)^x.\end{aligned} \tag{26}$$

From the (h,q)-Bernoulli polynomials in (7),

$$\begin{aligned} q^h \sum_{n=0}^{\infty} (-1)^n B_n^{(h)}\left(x|q^{-1}\right)\frac{t^n}{n!} &= q^h \frac{q^{1-h}(q^{-1}-1)}{\log q^{-1}} \frac{-t-\log q^{1-h}}{e^{-t}-q^{1-h}} e^{-xt} \\ &= \frac{q^{h-1}(q-1)}{\log q} \frac{t-\log q^{h-1}}{e^t - q^{h-1}} e^{(1-x)t} \\ &= \sum_{n=0}^{\infty} B_n^{(h)}(1-x|q)\frac{t^n}{n!}. \end{aligned} \tag{27}$$

Thus, we have

$$q^h(-1)^n B_n^{(h)}\left(x|q^{-1}\right) = B_n^{(h)}(1-x|q). \tag{28}$$

From (28), the value at $x = 1$, we have

$$q^h(-1)^n B_n^{(h)}\left(1|q^{-1}\right) = B_n^{(h)}(q).$$

On the other hand, we note that

$$(-x)_n = (-1)^n x^{(n)} = \sum_{l=0}^{n} S_1(n,l)(-x)^l = (-1)^n \sum_{l=0}^{n} \left|S_1(n,l)\right| x^l, \tag{29}$$

where $n \geq 0$ and $|S_1(n,k)|$ is the unsigned Stirling number of the first kind.
From (28) and (29),

$$\begin{aligned} \widehat{D}_n^{(h)}(x|q) &= \sum_{l=0}^{n} \left|S_1(n,l)\right|(-1)^l \int_{\mathbb{Z}_p} q^{-hy}(-x+y)^l \, d\mu_q(y) \\ &= \sum_{l=0}^{n} \left|S_1(n,l)\right|(-1)^l B_l^{(h)}(-x|q) \\ &= q^{-h} \sum_{l=0}^{n} \left|S_1(n,l)\right| B_l^{(h)}\left(x+1|q^{-1}\right). \end{aligned} \tag{30}$$

Thus, we have the following identity.

Theorem 4 *For $n \in \mathbb{Z}_+$, the following is true*:

$$\widehat{D}_n^{(h)}(x|q) = q^{-h} \sum_{l=0}^{n} \left|S_1(n,l)\right| B_l^{(h)}\left(x+1|q^{-1}\right).$$

On the other hand, we can check easily the following:

$$(x+y)_n = (-1)^n(-x-y+n-1)_n \tag{31}$$

and

$$\frac{(x+y)_n}{n!} = (-1)^n \binom{-x+y+n-1}{n}. \tag{32}$$

From (14), (26), (31) and (32), we have

$$
\begin{aligned}
(-1)^n \frac{D_n^{(h)}(x|q)}{n!} &= \int_{\mathbb{Z}_p} q^{-hy} \binom{-x-y+n-1}{n} \, d\mu_q(y) \\
&= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-hy} \binom{-x-y}{m} \, d\mu_q(y) \\
&= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h)}(-x|q)}{m!}
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
(-1)^n \frac{\widehat{D}_n^{(h)}(x|q)}{n!} &= (-1)^n \int_{\mathbb{Z}_p} q^{-hy} \binom{-x+y}{n} \, d\mu_q(y) \\
&= \int_{\mathbb{Z}_p} q^{-hy} \binom{-x+y+n-1}{n} \, d\mu_q(y) \\
&= \sum_{m=0}^{n} \binom{n-1}{n-m} \int_{\mathbb{Z}_p} q^{-hy} \binom{-x+y}{m} \, d\mu_q(y) \\
&= \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m^{(h)}(-x|q)}{m!}.
\end{aligned}
\tag{34}
$$

Therefore, we get the following theorem, which relates (h,q)-Daehee polynomials of the first and the second kind.

Theorem 5 *For $n \in \mathbb{N}$, the following equalities hold true*:

$$
(-1)^n \frac{D_n^{(h)}(x|q)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\widehat{D}_m^{(h)}(-x|q)}{m!}
$$

and

$$
(-1)^n \frac{\widehat{D}_n^{(h)}(x|q)}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m^{(h)}(-x|q)}{m!}.
$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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