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A Riccati-Bernoulli sub-ODE method for nonlinear partial differential equations and its application

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Abstract

The Riccati-Bernoulli sub-ODE method is firstly proposed to construct exact traveling wave solutions, solitary wave solutions, and peaked wave solutions for nonlinear partial differential equations. A Bäcklund transformation of the Riccati-Bernoulli equation is given. By using a traveling wave transformation and the Riccati-Bernoulli equation, nonlinear partial differential equations can be converted into a set of algebraic equations. Exact solutions of nonlinear partial differential equations can be obtained by solving a set of algebraic equations. By applying the Riccati-Bernoulli sub-ODE method to the Eckhaus equation, the nonlinear fractional Klein-Gordon equation, the generalized Ostrovsky equation, and the generalized Zakharov-Kuznetsov-Burgers equation, traveling solutions, solitary wave solutions, and peaked wave solutions are obtained directly. Applying a Bäcklund transformation of the Riccati-Bernoulli equation, an infinite sequence of solutions of the above equations is obtained. The proposed method provides a powerful and simple mathematical tool for solving some nonlinear partial differential equations in mathematical physics.

MSC: 35Q55; 35Q80; 35G25

Keywords: Riccati-Bernoulli sub-ODE method; Bäcklund transformation; traveling wave solution; solitary wave solution; peaked wave solution

1 Introduction

Nonlinear partial differential equations (NLPDEs) are known to describe a wide variety of phenomena not only in physics, but also in biology, chemistry, and several other fields. The investigation of traveling wave solutions for NLPDEs plays an important role in the study of nonlinear physical phenomena. In recent years, many powerful methods were used to construct traveling wave solutions of NLPDEs. For example, the inverse scattering method [1], the Bäcklund and Darboux transformation method [2], the homotopy perturbation method [3], the first integral method [4–6], the $(\frac{G'}{G})$-expansion method [7–9], the sub-equation method [10, 11], Hirota's method [12], the homogeneous balance method [13–15], the variational iteration method [16, 17], the tanh-sech method [18], the Jacobi elliptic function method [19], the modified simple equation method [20–23], the $\exp(-\Phi(\xi))$-expansion method [24], the alternative functional variable method [25], and so on.

Many well-known NLPDEs can be handled by those traditional methods. However, there is no unified method which can be used to deal with all types of NLPDEs. Moreover, we always encounter the fractional NLPDEs, the NLPDEs which have nonlinear terms of any order or peaked wave solutions. It is significant to construct traveling wave solutions of NLPDEs by a uniform method. Based on those problems, the Riccati-Bernoulli sub-ODE method is firstly presented.

In this paper, the Riccati-Bernoulli sub-ODE method is proposed to construct traveling wave solutions, solitary wave solutions, and peaked wave solutions of NLPDEs. By using a traveling wave transformation and the Riccati-Bernoulli equation, NLPDEs can be converted into a set of algebraic equations. Exact solutions of NLPDEs can be obtained by solving the set of algebraic equations. The Eckhaus equation, the nonlinear fractional Klein-Gordon equation, the generalized Ostrovsky equation, and the generalized Zakharov-Kuznetsov-Burgers (ZK-Burgers) equation are chosen to illustrate the validity of the Riccati-Bernoulli sub-ODE method. A Bäcklund transformation of the Riccati-Bernoulli equation is given. If we get a solution of NLPDEs, we can search for a new infinite sequence of solutions of the NLPDEs by using a Bäcklund transformation.

The remainder of this paper is organized as follows: the Riccati-Bernoulli sub-ODE method is described in Section 2. In Section 3, a Bäcklund transformation of the Riccati-Bernoulli equation is given. In Sections 4-7, we apply the Riccati-Bernoulli sub-ODE method to the Eckhaus equation, the nonlinear fractional Klein-Gordon equation, the generalized Ostrovsky equation, and the generalized ZK-Burgers equation, respectively. In Section 8, our results are compared with the first integral method, the $(\frac{G'}{G})$-expansion method, and physical explanations of the obtained solutions are discussed. In Section 9, some conclusions and directions for future work are given.

2 Description of the Riccati-Bernoulli sub-ODE method

Let there be given a NLPDE, say, in two variables,

$$P(u, u_t, u_x, u_{xx}, u_{xt}, \ldots) = 0, \tag{1}$$

where P is in general a polynomial function of its arguments, the subscripts denote the partial derivatives. The Riccati-Bernoulli sub-ODE method consists of three steps.

Step 1. Combining the independent variables x and t into one variable

$$\xi = k(x + Vt), \tag{2}$$

with

$$u(x, t) = u(\xi), \tag{3}$$

where the localized wave solution $u(\xi)$ travels with speed V, by using Eqs. (2) and (3), one can transform Eq. (1) to an ODE

$$P\big(u, u', u'', u''', \ldots\big) = 0, \tag{4}$$

where u' denotes $\frac{du}{d\xi}$.

Step 2. Suppose that the solution of Eq. (4) is the solution of the Riccati-Bernoulli equation

$$u' = au^{2-m} + bu + cu^m, \tag{5}$$

where a, b, c, and m are constants to be determined later.

From Eq. (5) and by directly calculating, we get

$$u'' = ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m + \left(2ac + b^2\right)u, \tag{6}$$

$$\begin{aligned} u''' = {} & \left(ab(3-m)(2-m)u^{1-m} + a^2(2-m)(3-2m)u^{2-2m}\right. \\ & \left. + m(2m-1)c^2u^{2m-2} + bcm(m+1)u^{m-1} + \left(2ac + b^2\right)\right)u', \end{aligned} \tag{7}$$

...

Remark When $ac \neq 0$ and $m = 0$, Eq. (5) is a Riccati equation. When $a \neq 0$, $c = 0$, and $m \neq 1$, Eq. (5) is a Bernoulli equation. Obviously, the Riccati equation and Bernoulli equation are special cases of Eq. (5). Because Eq. (5) is firstly proposed, we call Eq. (5) the Riccati-Bernoulli equation in order to avoid introducing new terminology.

Equation (5) has solutions as follows:

Case 1. When $m = 1$, the solution of Eq. (5) is

$$u(\xi) = Ce^{(a+b+c)\xi}. \tag{8}$$

Case 2. When $m \neq 1$, $b = 0$, and $c = 0$, the solution of Eq. (5) is

$$u(\xi) = \left(a(m-1)(\xi + C)\right)^{\frac{1}{m-1}}. \tag{9}$$

Case 3. When $m \neq 1$, $b \neq 0$, and $c = 0$, the solution of Eq. (5) is

$$u(\xi) = \left(-\frac{a}{b} + Ce^{b(m-1)\xi}\right)^{\frac{1}{m-1}}. \tag{10}$$

Case 4. When $m \neq 1$, $a \neq 0$, and $b^2 - 4ac < 0$, the solutions of Eq. (5) are

$$u(\xi) = \left(-\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a}\tan\left(\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}} \tag{11}$$

and

$$u(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a}\cot\left(\frac{(1-m)\sqrt{4ac - b^2}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}}. \tag{12}$$

Case 5. When $m \neq 1$, $a \neq 0$, and $b^2 - 4ac > 0$, the solutions of Eq. (5) are

$$u(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\coth\left(\frac{(1-m)\sqrt{b^2 - 4ac}}{2}(\xi + C)\right)\right)^{\frac{1}{1-m}} \tag{13}$$

and

$$u(\xi) = \left(-\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \tanh\left(\frac{(1-m)\sqrt{b^2 - 4ac}}{2}(\xi + C) \right) \right)^{\frac{1}{1-m}}. \tag{14}$$

Case 6. When $m \neq 1$, $a \neq 0$, and $b^2 - 4ac = 0$, the solution of Eq. (5) is

$$u(\xi) = \left(\frac{1}{a(m-1)(\xi + C)} - \frac{b}{2a} \right)^{\frac{1}{1-m}}, \tag{15}$$

where C is an arbitrary constant.

Step 3. Substituting the derivatives of u into Eq. (4) yields an algebraic equation of u. Noticing the symmetry of the right-hand item of Eq. (5) and setting the highest power exponents of u to equivalence in Eq. (4), m can be determined. Comparing the coefficients of u^i yields a set of algebraic equations for a, b, c, and V. Solving the set of algebraic equations and substituting m, a, b, c, V, and $\xi = k(x + Vt)$ into Eq. (8)-(15), we can get traveling wave solutions of Eq. (1).

In the subsequent section, we will give a Bäcklund transformation of the Riccati-Bernoulli equation and some applications to illustrate the validity of the Riccati-Bernoulli sub-ODE method.

3 Bäcklund transformation of the Riccati-Bernoulli equation

When $u_{n-1}(\xi)$ and $u_n(\xi)$ ($u_n(\xi) = u_n(u_{n-1}(\xi))$) are the solutions of Eq. (5), we get

$$\frac{du_n(\xi)}{d\xi} = \frac{du_n(\xi)}{du_{n-1}(\xi)} \frac{du_{n-1}(\xi)}{d\xi} = \frac{du_n(\xi)}{du_{n-1}(\xi)} \left(au_{n-1}^{2-m} + bu_{n-1} + cu_{n-1}^m \right),$$

namely

$$\frac{du_n(\xi)}{au_n^{2-m} + bu_n + cu_n^m} = \frac{du_{n-1}(\xi)}{au_{n-1}^{2-m} + bu_{n-1} + cu_{n-1}^m}.$$

Integrating above equation once with respect to ξ and simplifying it, we get

$$u_n(\xi) = \left(\frac{-cA_1 + aA_2(u_{n-1}(\xi))^{1-m}}{bA_1 + aA_2 + aA_1(u_{n-1}(\xi))^{1-m}} \right)^{\frac{1}{1-m}}, \tag{16}$$

where A_1 and A_2 are arbitrary constants.

Equation (16) is a Bäcklund transformation of Eq. (5). If we get a solution of Eq. (5), we can search for new infinite sequence of solutions of Eq. (5) by using Eq. (16). Then an infinite sequence of solutions of Eq. (1) is obtained.

4 Application to the Eckhaus equation

The Eckhaus equation reads

$$i\psi_t + \psi_{xx} + 2\left(|\psi|^2\right)_x \psi + |\psi|^4 \psi = 0, \tag{17}$$

where $\psi = \psi(x, t)$ is a complex-valued function of two real variables x, t.

The Eckhaus equation was found [26] as an asymptotic multiscale reduction of certain classes of nonlinear Schrödinger type equations. A lot of the properties of the Eckhaus equation were obtained [27]. The Eckhaus equation can be linearized by making some transformations of dependent variables [28]. An exact traveling wave solution of the Eckhaus equation was obtained by the $(\frac{G'}{G})$-expansion method [8] and the first integral method [5].

In this section, new type of exact traveling wave solutions of the Eckhaus equation are obtained by using the Riccati-Bernoulli sub-ODE method.

Using the traveling wave transformation

$$\psi(x,t) = u(\xi)e^{i(\alpha x+\beta t)}, \tag{18}$$

Eq. (17) is reduced to

$$k^2u'' - \left(\alpha^2+\beta\right)u + 4ku^2u' + u^5 = 0, \tag{19}$$

where

$$\xi = k(x-2\alpha t), \tag{20}$$

and k, α, β are real constants to be determined later.

Suppose that the solution of Eq. (19) is the solution of Eq. (5). Substituting Eqs. (5) and (6) into Eq. (19), we get

$$\begin{aligned} &k^2\left(ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m\right.\\ &\quad\left.+ \left(2ac+b^2\right)u\right) - \left(\alpha^2+\beta\right)u + 4ku^2\left(au^{2-m} + bu + cu^m\right) + u^5 = 0. \end{aligned} \tag{21}$$

Setting $m = -1$ and $c = 0$, Eq. (21) becomes

$$\left(k^2b^2 - \left(\alpha^2+\beta\right)\right)u + \left(4k^2ab + 4kb\right)u^3 + \left(3k^2a^2 + 4ka + 1\right)u^5 = 0. \tag{22}$$

Setting each coefficient of u^j $(j = 1,3,5)$ to zero, we get

$$k^2b^2 - \left(\alpha^2+\beta\right) = 0, \tag{23a}$$

$$4k^2ab + 4kb = 0, \tag{23b}$$

$$3k^2a^2 + 4ka + 1 = 0. \tag{23c}$$

Notice that $k \neq 0$, otherwise we can only get trivial solution.

Case A. If $b = 0$, from Eqs. (23a)-(23c) and (5), we get

$$u(x,t) = \pm\frac{1}{\sqrt{-2ka(x-2\alpha t)+C}}, \tag{24a}$$

$$\alpha^2 + \beta = 0, \tag{24b}$$

$$ka = -1\left(ka = -\frac{1}{3}\right), \tag{24c}$$

where C is an arbitrary real constant.

Case A-1. When $ka = -1$, we get exact traveling wave solutions of Eq. (17),

$$\psi_1(x,t) = \pm\frac{1}{\sqrt{2(x-2\alpha t)+C}}e^{i(\alpha x-\alpha^2 t)}, \tag{25}$$

where C and α are arbitrary real constants.

Case A-2. When $ka = -\frac{1}{3}$, we get exact traveling wave solutions of Eq. (17),

$$\psi_2(x,t) = \pm\frac{1}{\sqrt{\frac{2}{3}(x-2\alpha t)+C}}e^{i(\alpha x-\alpha^2 t)}, \tag{26}$$

where C and α are arbitrary real constants.

Case B. If $b \neq 0$, from Eqs. (23a)-(23c), we get

$$b = \pm\frac{\sqrt{\alpha^2+\beta}}{k}, \tag{27a}$$

$$a = -\frac{1}{k}. \tag{27b}$$

Case B-1. When $b = -\frac{\sqrt{\alpha^2+\beta}}{k}$ and $a = -\frac{1}{k}$, from Eqs. (10) and (18), we get an exact traveling wave solution of Eq. (17),

$$\psi_3(x,t) = \left(-\frac{1}{\sqrt{\alpha^2+\beta}} + Ce^{2\sqrt{\alpha^2+\beta}(x-2\alpha t)}\right)^{-\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{28}$$

where C, α, and β are arbitrary real constants.

Especially, if we choose $C = C_1 = \frac{1}{\sqrt{\alpha^2+\beta}}$, Eq. (28) becomes

$$\psi_4(x,t) = \left(\frac{\sqrt{\alpha^2+\beta}}{2}\left(-1+\coth\left(\sqrt{\alpha^2+\beta}(x-2\alpha t)\right)\right)\right)^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{29}$$

where α and β are arbitrary real constants.

If we choose $C = C_2 = -\frac{1}{\sqrt{\alpha^2+\beta}}$, Eq. (28) becomes

$$\psi_5(x,t) = \left(\frac{\sqrt{\alpha^2+\beta}}{2}\left(-1+\tanh\left(\sqrt{\alpha^2+\beta}(x-2\alpha t)\right)\right)\right)^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{30}$$

where α and β are arbitrary real constants.

Case B-2. When $b = \frac{\sqrt{\alpha^2+\beta}}{k}$ and $a = -\frac{1}{k}$, from Eqs. (10) and (18), we get an exact traveling wave solution of Eq. (17),

$$\psi_6(x,t) = \left(\frac{1}{\sqrt{\alpha^2+\beta}} + Ce^{-2\sqrt{\alpha^2+\beta}(x-2\alpha t)}\right)^{-\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{31}$$

where C, α, and β are arbitrary real constants.

Especially, if we choose $C = C_3 = \frac{1}{\sqrt{\alpha^2+\beta}}$, Eq. (31) becomes

$$\psi_7(x,t) = \left(\frac{\sqrt{\alpha^2+\beta}}{2}\left(1+\tanh\left(\sqrt{\alpha^2+\beta}(x-2\alpha t)\right)\right)\right)^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{32}$$

where α and β are arbitrary real constants.

If we choose $C = C_4 = -\frac{1}{\sqrt{\alpha^2+\beta}}$, Eq. (31) becomes

$$\psi_8(x,t) = \left(\frac{\sqrt{\alpha^2+\beta}}{2}\left(1+\coth\left(\sqrt{\alpha^2+\beta}(x-2\alpha t)\right)\right)\right)^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{33}$$

where α and β are arbitrary real constants.

Applying Eq. (16) to $\psi_j(x,t)$ $(j = 1,2,\ldots,8)$, we can get an infinite sequence of solutions of Eq. (17). For example, by applying Eq. (16) to Eq. (32), we get a new solution of Eq. (17),

$$\psi_7^*(x,t) = \left(\frac{A_2\sqrt{\alpha^2+\beta}(1+\tanh(\sqrt{\alpha^2+\beta}(x-2\alpha t)))}{2A_2+A_1\sqrt{\alpha^2+\beta}(-1+\tanh(\sqrt{\alpha^2+\beta}(x-2\alpha t)))}\right)^{\frac{1}{2}} e^{i(\alpha x+\beta t)}, \tag{34}$$

where A_1, A_2, α, and β are arbitrary real constants.

5 Application to the nonlinear fractional Klein-Gordon equation

The nonlinear fractional Klein-Gordon equation [23] reads

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2} + \beta u(x,t) + \gamma u^3(x,t), \quad t > 0, 0 < \alpha \le 1, \tag{35}$$

where β and γ are known constants.

As is well known, linear and nonlinear Klein-Gordon equations model many problems in classical and quantum mechanics, solitons and condensed matter physics. For example, the nonlinear sine Klein-Gordon equation models a Josephson junction, the motion of rigid pendula attached to a stretched wire, and dislocations in crystals [17, 29–31]. A nonlocal version of these equations are properly described by the fractional version of them. Exact traveling wave solutions of the nonlinear fractional Klein-Gordon equation were obtained by the homotopy perturbation method [29] and the first integral method [6].

In this section, exact traveling wave solutions of the nonlinear fractional Klein-Gordon equation are obtained by using the Riccati-Bernoulli sub-ODE method.

Using the transformation

$$u(x,t) = u(\xi), \tag{36}$$

with

$$\xi = lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}, \tag{37}$$

where l and λ are constants to be determined later, Eq. (35) becomes

$$u'' - \frac{\beta}{\lambda^2 - l^2}u - \frac{\gamma}{\lambda^2 - l^2}u^3 = 0. \tag{38}$$

Suppose that the solution of Eq. (38) is the solution of Eq. (5). Substituting Eq. (6) into Eq. (38), we get

$$ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m + \left(2ac+b^2\right)u - \frac{\beta}{\lambda^2-l^2}u - \frac{\gamma}{\lambda^2-l^2}u^3 = 0. \tag{39}$$

Setting $m = 0$, Eq. (39) is reduced to

$$3abu^2 + 2a^2u^3 + bc + \left(2ac+b^2\right)u - \frac{\beta}{\lambda^2-l^2}u - \frac{\gamma}{\lambda^2-l^2}u^3 = 0. \tag{40}$$

Setting each coefficient of u^i $(i = 0, 1, 2, 3)$ to zero, we get

$$bc = 0, \tag{41a}$$

$$2ac + b^2 - \frac{\beta}{\lambda^2-l^2} = 0, \tag{41b}$$

$$3ab = 0, \tag{41c}$$

$$2a^2 - \frac{\gamma}{\lambda^2-l^2} = 0. \tag{41d}$$

Solving Eqs. (41a)-(41d), we get

$$b = 0, \tag{42a}$$

$$ac = \frac{\beta}{2(\lambda^2-l^2)}, \tag{42b}$$

$$a = \pm\sqrt{\frac{\gamma}{2(\lambda^2-l^2)}}. \tag{42c}$$

Case A. When $\frac{\beta}{\lambda^2-l^2} > 0$, substituting Eqs. (42a)-(42c) and (37) into Eqs. (11) and (12), we get exact traveling wave solutions of Eq. (35),

$$u_{1,2}(x,t) = \pm\sqrt{\frac{\beta}{\gamma}}\tan\left(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right) + C\right), \tag{43a}$$

and

$$u_{3,4}(x,t) = \pm\sqrt{\frac{\beta}{\gamma}}\cot\left(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right) + C\right), \tag{43b}$$

where C, l, and λ are arbitrary constants.

Case B. When $\frac{\beta}{\lambda^2-l^2} < 0$, substituting Eqs. (42a)-(42c) and (37) into Eqs. (13) and (14), we get exact traveling wave solutions of Eq. (35),

$$u_{5,6}(x,t) = \pm\sqrt{-\frac{\beta}{\gamma}}\tanh\left(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}\left(lx - \frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right) + C\right), \tag{44a}$$

and

$$u_{7,8}(x,t) = \pm\sqrt{-\frac{\beta}{\gamma}}\coth\left(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}\left(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)}\right)+C\right), \tag{44b}$$

where C, l, and λ are arbitrary constants.

Applying Eq. (16) to $u_j(x,t)$ ($j = 1,2,\ldots,8$), we can get an infinite sequence of solutions of Eq. (35). For example, by applying Eq. (16) to $u_j(x,t)$ ($j = 1,2,\ldots,8$) once, we get new solutions of Eq. (35),

$$u^*_{1,2}(x,t) = \frac{-\frac{\beta}{\gamma} \pm A_3\sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)}{A_3 \pm \sqrt{\frac{\beta}{\gamma}}\tan(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)},$$

$$u^*_{3,4}(x,t) = \frac{-\frac{\beta}{\gamma} \pm A_3\sqrt{\frac{\beta}{\gamma}}\cot(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)}{A_3 \pm \sqrt{\frac{\beta}{\gamma}}\cot(\sqrt{\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)},$$

$$u^*_{5,6}(x,t) = \frac{-\frac{\beta}{\gamma} \pm A_3\sqrt{-\frac{\beta}{\gamma}}\tanh(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)}{A_3 \pm \sqrt{-\frac{\beta}{\gamma}}\tanh(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)},$$

$$u^*_{7,8}(x,t) = \frac{-\frac{\beta}{\gamma} \pm A_3\sqrt{-\frac{\beta}{\gamma}}\coth(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)}{A_3 \pm \sqrt{-\frac{\beta}{\gamma}}\coth(\sqrt{-\frac{\beta}{2(\lambda^2-l^2)}}(lx-\frac{\lambda t^{\alpha}}{\Gamma(1+\alpha)})+C)},$$

where A_1, A_2, C, l, and λ are arbitrary real constants.

6 Application to the generalized Ostrovsky equation

The generalized Ostrovsky equation reads

$$\left(u_t + 3uu_x - \frac{\beta}{4}u_{xxx}\right)_x - \frac{\varepsilon^2}{2}\left(u+\delta u^2\right) = 0, \tag{45}$$

where β, ε, and δ are known constants.

The generalized Ostrovsky equation is a model for the weakly nonlinear surface and internal waves in a rotating ocean. Exact peaked wave solutions were obtained by the undetermined coefficient method [32].

In this section, exact peaked wave solutions of the generalized Ostrovsky equation are obtained by using the Riccati-Bernoulli sub-ODE method.

Using the transformation

$$u(x,t) = u(\xi), \tag{46}$$

with

$$\xi = k(x+Vt), \tag{47}$$

where k and V are the wave number and wave speed, respectively, Eq. (45) becomes

$$k^2Vu'' + 3k^2\left(u'\right)^2 + 3k^2uu'' - \frac{\beta}{4}k^4u'''' - \frac{\varepsilon^2}{2}\left(u+\delta u^2\right) = 0. \tag{48}$$

Suppose that the solution of Eq. (48) is the solution of Eq. (5). From Eqs. (5) and (6), we get

$$\begin{aligned} u'''' &= \big(ab(3-m)(2-m)(1-m)u^{-m} + a^2(2-m)(3-2m)(2-2m) \\ &\quad \times u^{1-2m} + m(2m-1)(2m-2)c^2u^{2m-3} + bcm(m+1)(m-1)u^{m-2}\big) \\ &\quad \times \big(u'\big)^2 + \big(ab(3-m)(2-m)u^{1-m} + a^2(2-m)(3-2m)u^{2-2m} \\ &\quad + m(2m-1)c^2u^{2m-2} + bcm(m+1)u^{m-1} + \big(2ac+b^2\big)\big)u''. \end{aligned} \tag{49}$$

Substituting Eqs. (5), (6), and (49) into Eq. (48), we get

$$\begin{aligned} &\big(k^2V + 3k^2u\big)\big(ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} \\ &\quad + mc^2u^{2m-1} + bc(m+1)u^m + \big(2ac+b^2\big)u\big) + \frac{\beta}{4}k^4\Sigma = 0, \end{aligned} \tag{50}$$

where

$$\begin{aligned} \Sigma &= \big(au^{2-m} + bu + cu^m\big)^2\big(ab(3-m)(2-m)(1-m)u^{-m} + a^2(2-m)(3-2m) \\ &\quad \times (2-2m)u^{1-2m} + m(2m-1)(2m-2)c^2u^{2m-3} + bcm(m+1)(m-1)u^{m-2}\big) \\ &\quad + \big(ab(3-m)u^{2-m} + a^2(2-m)u^{3-2m} + mc^2u^{2m-1} + bc(m+1)u^m + \big(2ac+b^2\big)u\big) \\ &\quad \times \big(ab(3-m)(2-m)u^{1-m} + a^2(2-m)(3-2m)u^{2-2m} + m(2m-1)c^2u^{2m-2} \\ &\quad + bcm(m+1)u^{m-1} + \big(2ac+b^2\big)\big). \end{aligned}$$

Setting $m = 2$ and $c = 0$, Eq. (50) is reduced to

$$\begin{aligned} &\left(k^2Vab + 3k^2a^2 - \frac{\beta}{4}k^4ab^3\right) + \left(k^2Vb^2 - \frac{\beta}{4}k^4b^4 + 9k^2ab - \frac{1}{2}\varepsilon^2\right)u \\ &\quad + \left(6k^2b^2 - \frac{1}{2}\varepsilon^2\delta\right)u^2 = 0. \end{aligned} \tag{51}$$

Setting each coefficient of u^j $(j = 0,1,2)$ to zero, we get

$$6k^2b^2 - \frac{1}{2}\varepsilon^2\delta = 0, \tag{52a}$$

$$k^2Vb^2 - \frac{\beta}{4}k^4b^4 + 9k^2ab - \frac{1}{2}\varepsilon^2 = 0, \tag{52b}$$

$$k^2Vab + 3k^2a^2 - \frac{\beta}{4}k^4ab^3 = 0. \tag{52c}$$

Solving Eqs. (52a)-(52c), we get

$$-\frac{a}{b} = -\frac{1}{\delta}, \tag{53a}$$

$$kb = \pm\sqrt{\frac{\delta\varepsilon^2}{12}}, \tag{53b}$$

$$V = \frac{\beta\delta^2\varepsilon^2 - 144}{48\delta}. \tag{53c}$$

Substituting Eqs. (53a)-(53c) and (47) into Eq. (10), we get exact peaked wave solutions of Eq. (45),

$$u_{1,2}(x,t) = -\frac{1}{\delta} + Ce^{\pm\sqrt{\frac{\delta\varepsilon^2}{12}}|x+(\frac{\beta\delta^2\varepsilon^2-144}{48\delta})t|}, \tag{54}$$

where C is an arbitrary constant.

Similar to Sections 4 and 5, by using a Bäcklund transformation, we can get an infinite sequence of solutions of the generalized Ostrovsky equation. It being a similar process, we omit it.

7 Application to the generalized ZK-Burgers equation

The generalized ZK-Burgers equation [33] reads

$$u_t + \alpha u^{\lambda}u_x + \beta u_{xxx} + \gamma(u_{yy} + u_{zz}) + \sigma u_{xx} = 0, \tag{55}$$

where α, β, γ, σ, and λ are known constants.

The generalized ZK-Burgers equation retains the strong nonlinear aspects of the governing equation in many practical transport problems such as nonlinear waves in a medium with low-frequency pumping or absorption, transport and dispersion of pollutants in rivers, and sediment transport. Wang *et al.* obtained a solitary wave of the generalized ZK-Burgers equation with a positive fractional power term by using the HB method and with the aid of sub-ODEs [33].

In this section, exact traveling wave solutions of the generalized ZK-Burgers equation are obtained by using the Riccati-Bernoulli sub-ODE method.

Using the transformation

$$u(x,y,z,t) = u(\xi), \tag{56}$$

with

$$\xi = k(x + ly + nz + Vt), \tag{57}$$

where k, l, n, and V are constants to be determined later, Eq. (55) becomes

$$kVu' + k\alpha u^{\lambda}u' + k^3\beta u''' + k^3\gamma\left(l^2 + n^2\right)u''' + k^2\sigma u'' = 0. \tag{58}$$

Suppose that the solution of Eq. (55) is the solution of Eq. (5). Noticing $u' \neq 0$ and $k \neq 0$, otherwise we can only get trivial solution. Substituting Eqs. (5), (6), and (7) into Eq. (58), we get

$$\begin{aligned}&\rho k^2\big(ab(3-m)(2-m)u^{1-m} + a^2(2-m)(3-2m)u^{2-2m}\\&\quad + m(2m-1)c^2u^{2m-2} + bcm(m+1)u^{m-1} + \left(2ac + b^2\right)\big)\\&\quad + \sigma k\big(a(2-m)u^{1-m} + mcu^{m-1} + b\big) + V + \alpha u^{\lambda} = 0,\end{aligned} \tag{59}$$

where

$$\rho = \beta + \gamma\left(l^2 + n^2\right). \tag{60}$$

Setting $m = 1 - \frac{\lambda}{2}$ and $c = 0$, Eq. (59) is reduced to

$$\left(V + \rho k^2 b^2 + \sigma kb\right) + \left(\frac{(\lambda+2)(\lambda+4)\rho k^2 ab}{4} + \frac{(\lambda+2)\sigma ka}{2}\right)u^{\frac{\lambda}{2}} + \left(\alpha + \frac{(\lambda+2)(\lambda+1)\rho k^2 a^2}{2}\right)u^{\lambda} = 0. \tag{61}$$

Setting each coefficient of u^j $(j = 0, \frac{\lambda}{2}, \lambda)$ to zero, we get

$$V + \rho k^2 b^2 + \sigma kb = 0, \tag{62a}$$

$$\frac{(\lambda+2)ka}{2}\left(\frac{(\lambda+4)\rho kb}{2} + \sigma\right) = 0, \tag{62b}$$

$$\alpha + \frac{(\lambda+2)(\lambda+1)\rho k^2 a^2}{2} = 0. \tag{62c}$$

Solving Eqs. (62a)-(62c), we get

$$b = \frac{-2\sigma}{k\rho(\lambda+4)}, \tag{63a}$$

$$a = \pm\frac{1}{k}\sqrt{\frac{-2\alpha}{\rho(\lambda^2+3\lambda+2)}}, \tag{63b}$$

$$V = \frac{2\sigma^2(\lambda+2)}{\rho(\lambda+4)^2}. \tag{63c}$$

Substituting Eqs. (63a)-(63c) and (57) into Eq. (10), we get exact traveling wave solutions of Eq. (55),

$$u_{1,2}(x,t) = \left(\pm\frac{\rho(\lambda+4)}{\sigma}\sqrt{\frac{-\alpha}{2\rho(\lambda^2+3\lambda+2)}} + Ce^{\frac{\lambda\sigma}{\rho(\lambda+4)}(x+ly+nz+(\frac{2\sigma^2(\lambda+2)}{\rho(\lambda+4)^2})t)}\right)^{\frac{-2}{\lambda}}, \tag{64}$$

where C, l, and n are arbitrary constants.

Equation (64) is new type of traveling wave solution of the generalized ZK-Burgers equation. Especially, if we choose $C = C_1 = \frac{\rho(\lambda+4)}{\sigma}\sqrt{\frac{-\alpha}{2\rho(\lambda^2+3\lambda+2)}}$, we get the solitary wave solutions of Eq. (55),

$$u_3(x,t) = \left(\frac{1}{2C_1}\left(1 - \tanh\frac{\eta}{2}\right)\right)^{\frac{2}{\lambda}}, \tag{65}$$

$$u_4(x,t) = \left(\frac{1}{2C_1}\left(-1 + \coth\frac{\eta}{2}\right)\right)^{\frac{2}{\lambda}}, \tag{66}$$

where l, n are arbitrary constants and

$$\eta = \frac{\lambda\sigma}{\rho(\lambda+4)}\left(x + ly + nz + \left(\frac{2\sigma^2(\lambda+2)}{\rho(\lambda+4)^2}\right)t\right). \tag{67}$$

If we choose $C = C_2 = -\frac{\rho(\lambda+4)}{2\sigma}\sqrt{\frac{-2\alpha}{\rho(\lambda^2+3\lambda+2)}}$, we get the solitary wave solutions of Eq. (55),

$$u_5(x,t) = \left(\frac{1}{2C_2}\left(-1+\coth\frac{\eta}{2}\right)\right)^{\frac{2}{\lambda}}, \tag{68}$$

$$u_6(x,t) = \left(\frac{1}{2C_2}\left(1-\tanh\frac{\eta}{2}\right)\right)^{\frac{2}{\lambda}}, \tag{69}$$

where l and n are arbitrary constants.

Similar to Sections 4 and 5, by using a Bäcklund transformation, we can get an infinite sequence of solutions of the generalized ZK-Burgers equation. It being a similar process, we omit it.

8 Comparisons and explanations of the solutions

In this section, the physical interpretation of the results of Sections 4-7 are given, respectively. We will compare the Riccati-Bernoulli sub-ODE method with the $(\frac{G'}{G})$-expansion method, the first integral method, and so on. Some of our obtained exact solutions are in the figures represented with the aid of Maple software.

(1) The Eckhaus equation:

Applying the Riccati-Bernoulli sub-ODE method, Eqs. (25), (26), (28), (31), and (34) are new types of exact traveling wave solutions of the Eckhaus equation. Equations (29), (30), (32), and (33), which are expressed by the hyperbolic functions, are a kind of kink-type envelope solitary solutions. They could not be obtained by the method presented in Ref. [27]. Equation (26), which is expressed by the rational functions, could not be obtained by the $(\frac{G'}{G})$-expansion method [8] and the first integral method [5].

(2) The nonlinear fractional Klein-Gordon equation:

Applying the Riccati-Bernoulli sub-ODE method and comparing our results with Golmankhaneh's results [29], it is easy to find that $u_j(x,t)$ $(j = 1,\dots,8)$ are new and identical to results by the first integral method [6]. $u_j(x,t)$ $(j = 1,2,3,4)$, which are expressed by the trigonometric functions, are periodic wave solutions. $u_j(x,t)$ $(j = 5,6,7,8)$, which are expressed by the hyperbolic functions, are a kind of kink-type envelope solitary solutions. The shape of $u = u_1(x,t)$ is represented in Figure 1 with $\alpha = \frac{1}{90}$, $\beta = 1$, $\gamma = 1$, $\lambda = \frac{9}{5}$, $C = 0$ and $l = \frac{3}{2}$ within the interval $-5 \le x \le 5$ and $0 \le t \le \frac{1}{2}$. The shape of $u = u_5(x,t)$ is represented in Figure 2 with $\alpha = \frac{1}{5}$, $\beta = -1$, $\gamma = 1$, $\lambda = 2$, $C = 0$, and $l = 1$ within the interval $-6 \le x \le 6$ and $0 \le t \le 6$.

(3) The generalized Ostrovsky equation:

Applying the Riccati-Bernoulli sub-ODE method, it is easy to find that our results are identical to results presented in Ref. [32]. $u = u_{1,2}(x,t)$ are peaked wave solutions of the generalized Ostrovsky equation. The shape of $u = u_1(x,t)$ is represented in Figure 3 with $\delta = 6$, $\beta = 6$, $\varepsilon = 1$, $\lambda = 2$, and $C = \frac{1}{10}$ within the interval $-5 \le x,t \le 5$.

(4) The generalized ZK-Burgers equation:

By applying the Riccati-Bernoulli sub-ODE method to the generalized ZK-Burgers equation, we find that if λ is a positive fraction, our results degenerate to the results of Ref. [33]. Moreover, we enlarge the value range of parameters λ of the generalized ZK-Burgers equation so that the parameter λ can be an arbitrary constant $(\lambda \neq -1,-2,-4)$. $u_j(x,t)$ $(j = 1,\dots,6)$ are exact traveling wave solutions of the generalized ZK-Burgers equation.

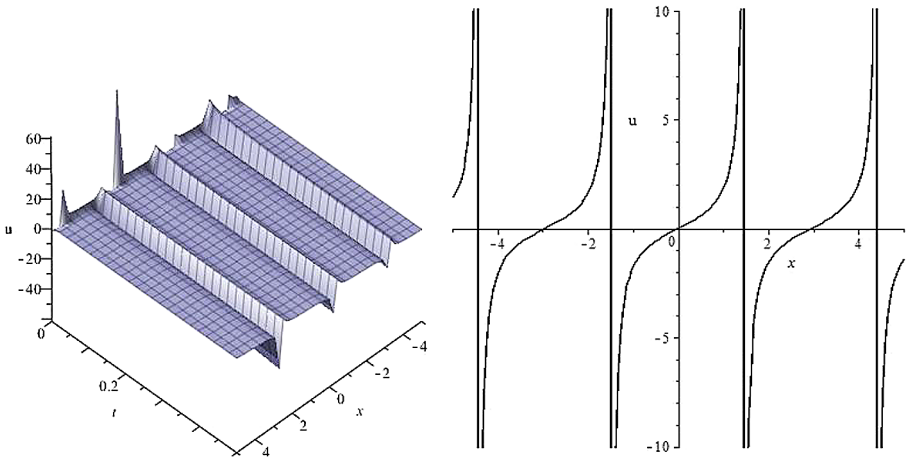

Figure 1 Graph of solution $u = u_1(x,t)$ of the nonlinear fractional Klein-Gordon equation for $\alpha = \frac{1}{90}$, $\beta = 1$, $\gamma = 1$, $\lambda = \frac{9}{5}$, $C = 0$, and $l = \frac{3}{2}$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$.

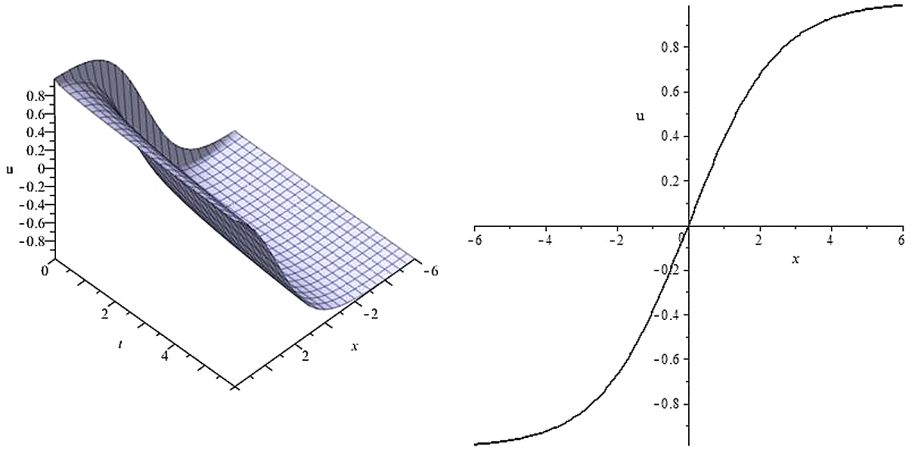

Figure 2 Graph of solution $u = u_5(x,t)$ of the nonlinear fractional Klein-Gordon equation for $\alpha = \frac{1}{5}$, $\beta = -1$, $\gamma = 1$, $\lambda = 2$, $C = 0$, and $l = 1$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$.

$u_j(x,t)$ $(j = 3,4,5,6)$, which are expressed by the hyperbolic functions, are a kind of kink-type envelope solitary solutions. The shape of $u = u_1(x,t)$ is represented in Figure 4 with $\alpha = \beta = \gamma = l = n = y = z = 1$, $\lambda = -\sqrt{2}$ and $\sigma = 2$ within the interval $-5 \le x,t \le 5$.

Moreover, by using a Bäcklund transformation, we can get an infinite sequence of solutions of these NLPDEs which cannot be obtained by the $(\frac{G'}{G})$-expansion method and the first integral method. The graphical demonstrations of some obtained solutions are shown in Figures 1-4.

9 Conclusions

The Riccati-Bernoulli sub-ODE method is successfully used to establish exact traveling wave solutions, solitary wave solutions and peaked wave solutions of NLPDEs. A Bäcklund transformation of the Riccati-Bernoulli equation is given. By applying a Bäcklund transfor-

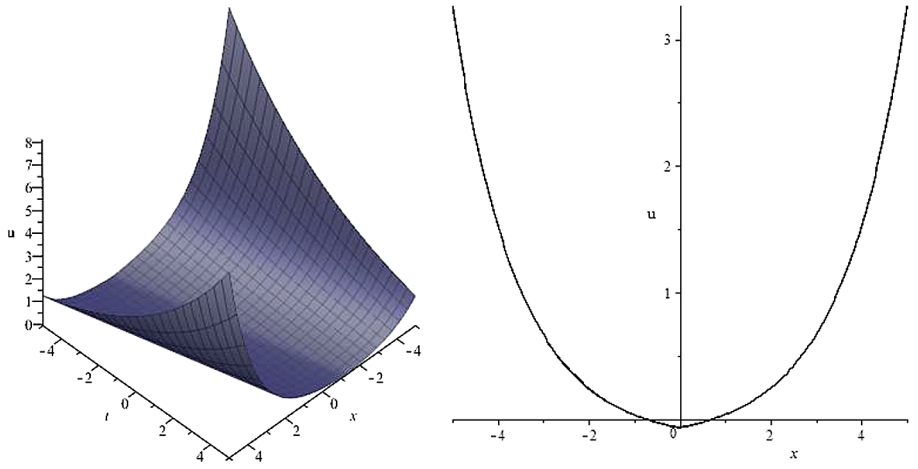

Figure 3 Graph of solution $u = u_5(x,t)$ of the generalized Ostrovsky equation for $\delta = 6$, $\beta = 6$, $\varepsilon = 1$, $\lambda = 2$, and $C = \frac{1}{10}$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 0$.

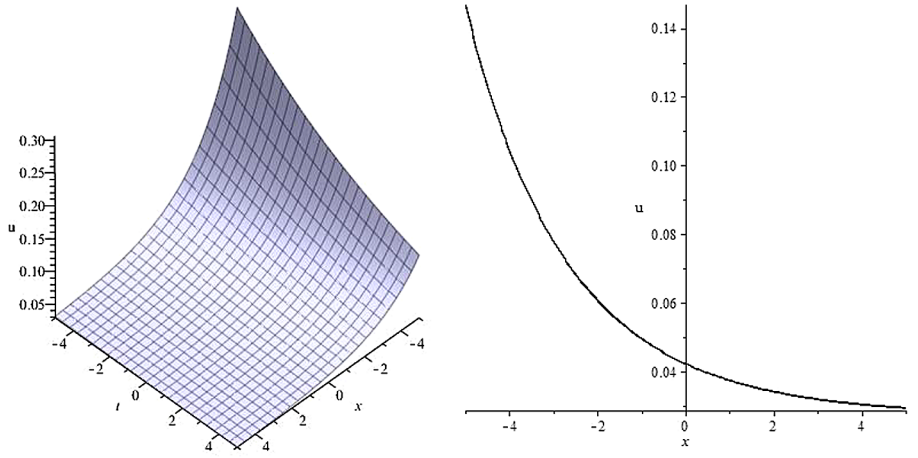

Figure 4 Graph of solution $u = u_5(x,t)$ of the generalized ZK-Burgers equation for $\alpha = \beta = \gamma = l = n = y = z = 1$, $\lambda = -\sqrt{2}$, and $\sigma = 2$. The left figure shows the 3-D plot and the right figure shows the 2-D plot for $t = 3$.

mation of the Riccati-Bernoulli equation to the NLPDEs, an infinite sequence of solutions of the NLPDEs is obtained. The Eckhaus equation, the nonlinear fractional Klein-Gordon equation, the generalized Ostrovsky equation, and the generalized ZK-Burgers equation are chosen to illustrate the validity of the Riccati-Bernoulli sub-ODE method. Many well-known NLPDEs can be handled by this method. The performance of this method is found to be simple and efficient. The availability of computer systems like Maple facilitates the tedious algebraic calculations. The Riccati-Bernoulli sub-ODE method is also a standard and computable method, which allows us to perform complicated and tedious algebraic calculations.

It is well known that it is difficult to propose an uniform analytical method for all types of the NLPDEs, and the Riccati-Bernoulli sub-ODE method is no exception. Similar to the first integral method, the $(\frac{G'}{G})$-expansion method and the homogeneous balance method,

the Riccati-Bernoulli sub-ODE method is used to obtain exact solutions of the form of Eq. (1). Constructing more powerful sub-ODE and Bäcklund transformations is future work and aims to search for exact solutions of NLPDEs

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors completed the paper together. All authors read and approved the final manuscript.

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