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Periodic solutions for nonautonomous first order delay differential systems via Hamiltonian systems

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Abstract

The existence of the nontrivial periodic solutions for the nonautonomous first order delay differential equation $x'(t) = -[f(t, x(t - 1)) + f(t, x(t - 2)) + \cdots + f(t, x(t - (2N - 1)))]$ is investigated, where $f \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ is 2N-periodic in t and odd in x, N is a positive integer. We prove several new existence results by some recent critical point theorems.

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1 Introduction

It is well known that critical point theory is a powerful tool that deals with the multiplicity of periodic solutions to ordinary differential systems as well as partial differential equations (see [1–6]). In 1998, Li and He [7] firstly applied the critical point theory to study the multiplicity of periodic solutions to delay differential equations. Especially, in 2005, Guo and Yu [8] established a variational framework for autonomous systems. In the past several years, some results on the existence of periodic solutions for the functional differential equation were obtained by critical point theory (see [7–21]).

In 2006, Fei [11, 12] considered the following autonomous functional differential equation:

$$(*) \quad x'(t) = -\big[f\big(x(t-1)\big) + f\big(x(t-2)\big) + \cdots + f\big(x\big(t - (n-1)\big)\big)\big],$$

where f is odd and $n \geq 2$ is an integer. Using the Yorke-Kaplan technique, $(*)$ is changed into a Hamiltonian system. Periodic solutions of the Hamiltonian systems are still obtained as critical points of a function φ over a Hilbert space E. However, instead of finding critical points of φ over E directly, the author worked on a subspace of E which has a symmetric structure. When n is even, the function φ is invariant and φ is equivariant about a compact group. This allows one to find critical points of φ on a subspace of E which is invariant under the mentioned group action. Then one can apply the pseudo-index theory [2] to obtain periodic solutions in this subspace, which surely have the required symmetric structure and give solutions to $(*)$. When n is odd, the function φ is still invariant about a similar

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compact group. However, φ is not equivariant about this compact group action anymore. Therefore one cannot directly apply the same idea as in the case when n is even. In order to overcome the difficulty, one has to construct equivariant pseudo-gradient vector fields and prove a new deformation theorem. Then one can combine a Galerkin approximation with the S^1-index theory ([4], Chapter 6) to obtain critical points of φ with the required symmetric structure (for details see [12]).

But most of these functional differential equation are autonomous, and the results on the nonautonomous functional differential equation are relatively few (see [9, 15–19]).

Motivated by the work of [10–12, 15–18], we consider a class of nonautonomous first order delay differential equations,

$$x'(t) = -\big[f\big(t,x(t-1)\big) + f\big(t,x(t-2)\big) + \cdots + f\big(t,x\big(t-(2N-1)\big)\big)\big], \tag{1.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}, \mathbf{R})$ is $2N$-periodic in t and odd in $x, N \in \mathbf{Z}^+$ and $\mathbf{Z}^+$ is the set of all positive integers.

More precisely, if the solution $x(t)$ of (1.1) satisfies $x(t) = -x(t-2N)$, let

$$x_1(t) = x(t), \quad x_2(t) = x(t-1), \quad \ldots, \quad x_{2N}(t) = x\big(t-(2N-1)\big), \tag{1.2}$$

then $X(t) = (x_1(t), x_2(t), \ldots, x_{2N}(t))^T$ satisfies

$$\frac{d}{dt}X(t) = A_{2N}G(t,X), \quad \text{where } A_{2N} = \begin{pmatrix} 0 & -1 & \cdots & -1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}, \tag{1.3}$$

i.e., A_{2N} is a $2N \times 2N$ skew symmetric matrix, and

$$G(t,X) = \big(f(t,x_1), f(t,x_2), \ldots, f(t,x_{2N})\big)^T.$$

Also

$$X(t) = \big(x_1(t), x_2(t), \ldots, x_{2N}(t)\big)^T$$

satisfies the following symmetric structure:

$$x_1(t) = -x_{2N}(t-1), \quad x_2(t) = x_1(t-1), \quad x_3(t) = x_2(t-1), \quad \ldots,$$

$$x_{2N}(t) = x_{2N-1}(t-1). \tag{1.4}$$

In this paper, we have the following conditions on f.

(f1)   There exist two $2N$-periodic and continuous functions α, β such that

$$\alpha(t) = \lim_{x \to 0} \frac{f(t,x)}{x}, \qquad \beta(t) = \lim_{x \to \infty} \frac{f(t,x)}{x}$$

uniformly for $t \in [0, 2N]$.

(f2$^\pm$) There exist $d_1, d_2 > 0$ such that

$$\pm\big[2F(t,x) - xf(t,x)\big] \geq d_1|x| - d_2, \quad (t,x) \in [0, 2N] \times \mathbf{R},$$

where $F(t,x) = \int_0^x f(t,s)\,ds$.

Denote $m^-(t) = 1$ if all $t < 0$; $m^-(t) = 0$ otherwise.

Theorem 1.1 *Assume that f satisfies* (f1)-(f2$^-$) *with*

$$\beta(t) > \frac{\pi}{2N} \tan \frac{\pi}{4N}, \quad for\ t \in [0, 2N].$$

Then (1.1) *possesses at least 2m pairs 4N-periodic solutions with* $x(t) = -x(t - 2N)$, *where*

$$m = \sum_{j=1}^{\infty} m^- \left[\frac{\pi(2j-1)}{2N} \tan \frac{\pi(2j-1)}{4N} - \beta(t) \right],$$

for

$$\tan \frac{\pi(2j-1)}{4N} > 0, \quad j \in \mathbf{Z}^+ \ and\ for\ all\ t \in [0, 2N].$$

Theorem 1.2 *Assume that f satisfies* (f1)-(f2$^+$) *with*

$$\beta(t) < \frac{\pi(4N-1)}{2N} \tan \frac{\pi(4N-1)}{4N}, \quad for\ t \in [0, 2N].$$

Then (1.1) *possesses at least 2m pairs 4N-periodic solutions with* $x(t) = -x(t - 2N)$, *where*

$$m = \sum_{j=1}^{\infty} m^- \left[\beta(t) - \frac{\pi(2j-1)}{2N} \tan \frac{\pi(2j-1)}{4N} \right],$$

for

$$\tan \frac{\pi(2j-1)}{4N} < 0, \quad j \in \mathbf{Z}^+ \ and\ for\ all\ t \in [0, 2N].$$

In this paper, the main purpose is to study the multiplicity of periodic solutions for the systems (1.1) via some recent critical point theorems for strongly indefinite functionals. In order to achieve this, some preliminaries are necessary. Let X and Y be Banach spaces with X being separable and reflexive, and set $E = X \oplus Y$. Let $\mathcal{S} \subset X^*$ be a dense subset. For each $s \in \mathcal{S}$, there is a semi-norm on E defined by

$$p_s : E \to \mathbf{R}, \quad p_s(u) = \big|s(x)\big| + \|y\| \quad for\ u = x + y \in X \oplus Y.$$

We denote by $\mathcal{T}_\mathcal{S}$ the topology on E induced by a semi-norm family $\{p_s\}$, and let ω and ω^* denote the weak topology and weak* topology, respectively. Clearly, the $\mathcal{T}_\mathcal{S}$ topology contains the product topology on $E = X \oplus Y$ produced by the weak topology on X and the strong topology on Y.

For a functional $\Phi \in C^1(E, \mathbf{R})$ we write $\Phi_a = \{u \in E : \Phi(u) \geq a\}$. Recall that Φ' is weakly sequentially continuous if $u_k \rightharpoonup u$ in E, and one has $\lim_{k\to\infty} \Phi'(u_k)v \to \Phi'(u_k)v$ for each

$v \in E$, i.e. $\Phi' : (E, \omega) \to (E^*, \omega^*)$ is sequentially continuous. For $c \in \mathbf{R}$ we say that Φ satisfies the $(C)_c$ condition if any sequence $\{u_k\} \subset E$ such that $\Phi(u_k) \to c$ and $(1 + \|u_k\|)\Phi'(u_k) \to 0$ as $k \to \infty$ contains a convergent subsequence.

Suppose that

(Φ_0) $\Phi \in C^1(E, \mathbf{R})$, Φ_c is $\mathcal{T}_S$-closed for every $c \in \mathbf{R}$, and $\Phi' : (\Phi_c, \mathcal{T}_S) \to (E^*, \omega^*)$ is continuous.

(Φ_1) There exists a $\rho > 0$ such that $\kappa := \inf \Phi(B_\rho \cap Y) > 0 = \Phi(0)$, where $B_\rho = \{u \in E : \|u\| = \rho\}$.

(Φ_2) There exist a finite dimensional subspace $Y_0 \subset Y$ and $R > \rho$ such that $\bar{c} := \sup \Phi(E_0) < \infty$ and $\sup \Phi(E_0 \setminus S_0) < \inf \Phi(B_\rho \cap Y)$, where $E_0 := X \oplus Y_0$, and $S_0 = \{u \in E_0 : \|u\| \leq R\}$.

The following critical point theorem will be used later (see [1, 5, 15]).

Theorem A *Assume that Φ is even and (Φ_0)-(Φ_2) are satisfied. Then Φ has at least $m =$ dim Y_0 pairs of critical points with critical values less than or equal to $\bar{c}$ provided Φ satisfies the $(C)_c$ condition for all $c \in [\kappa, \bar{c}]$.*

2 Preliminaries

It is easy to see that A_{2N} is a nonsingular skew symmetric Hamiltonian matrix, and (1.3) becomes a classical Hamiltonian system,

$$z'(t) = A_{2N} \nabla H(t, z), \tag{2.1}$$

where $\nabla H(t, z)$ denotes the gradient of $H(t, z)$ with respect to the z variable, and

$$H(t, z) = F(t, z_1) + F(t, z_2) + \cdots + F(t, z_{2N}), \tag{2.2}$$

for any $z = (z_1, z_2, \ldots, z_{2N})^T \in \mathbf{R}^{2N}$.

For $S^1 = \mathbf{R}/(4N\mathbf{Z})$, let $H^{1/2} = W^{1/2,2}(S^1, \mathbf{R}^{2N})$. Then $H^{1/2}$ is a Hilbert space with norm $\| \cdot \|$ and inner product $\langle, \rangle$, and $H^{1/2}$ consists of those $z(t)$ in $L^2(S^1, \mathbf{R}^{2N})$ whose Fourier series,

$$z(t) = a_0 + \sum_{j=1}^{\infty} \left[a_j \cos\left(\frac{\pi}{2N} jt \right) + a_j \cos\left(\frac{\pi}{2N} jt \right) \right],$$

satisfies

$$\|z\| = 4N|a_0|^2 + 2N \sum_{j=1}^{\infty} j\left(|a_j|^2 + |b_j|^2 \right) < \infty,$$

where $a_j, b_j \in \mathbf{R}^{2N}$. By Proposition 6.6 [4], we know that $H^{1/2}$ is compactly embedded in $L^s(S^1, \mathbf{R}^{2N})$, where $s \in [1, \infty)$.

Since A_{2N} is a nonsingular skew, so is A_{2N}^{-1}. We define an operator L by extending the bilinear form

$$\langle Lz, y \rangle = \int_0^{4N} \left(-A_{2N}^{-1} \dot{z}(t), y(t) \right) dt. \tag{2.3}$$

By direct computation, L is a bounded self-adjoint linear operator on $H^{1/2}$.

By (f1), one can show that $H(t,z) \in C^1(\mathbf{R} \times \mathbf{R}^{2N}, \mathbf{R})$ and there exist two continuous functions $a(t) > 0$, $b(t) \geq 0$ such that

$$H(t,z) \leq a(t)|z|^2 + b(t), \quad \forall z \in \mathbf{R}^{2N}.$$

By using similar arguments as in [4], Chapter 6, let

$$\varphi(z) = \frac{1}{2}\langle Lz, z \rangle + \psi(z), \tag{2.4}$$

where

$$\psi(z) = \int_0^{4N} H\big(t, z(t)\big)\, dt. \tag{2.5}$$

Thus the critical points of $\varphi(z)$ in $H^{1/2}$ are classical solutions of (2.1).

Let T_{2N} be the $2N \times 2N$ matrix given

$$T_{2N} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \ddots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

For $z(t) \in H^{1/2}$, define by

$$\delta z(t) = T_{2N} z(t-1). \tag{2.6}$$

Then $\delta^{2N} z(t) = -z(t-2N)$, $\delta^{4N} z(t) = z(t)$, and $G = \{\delta, \delta^2, \ldots, \delta^{4N}\}$ is a compact group action over $H^{1/2}$. Moreover, if $\delta z(t) = z(t)$ holds, $z(t)$ has the symmetric structure (1.4).

Lemma 2.1 [11] *Denote $E = \{z(t) \in H^{1/2} : \delta z(t) = z(t)\}$. Then*

$$E = \left\{ z(t) \in H^{1/2}\big(S^1, \mathbf{R}^{2N}\big) : z(t) = \sum_{j=1}^{\infty} \left[a_j \cos\left(\frac{\pi}{2N}(2j-1)t\right) + b_j \sin\left(\frac{\pi}{2N}(2j-1)t\right) \right] : \right.$$

$$\left. \begin{pmatrix} a_j \\ b_j \end{pmatrix} \in \operatorname{span}\left\{ \begin{pmatrix} u_j \\ w_j \end{pmatrix}, \begin{pmatrix} -w_j \\ u_j \end{pmatrix} \right\} \right\},$$

where $\beta_j = \frac{\pi}{2N}(2j-1)$ and

$$u_j = \big(1, \cos\beta_j, \cos(2\beta_j), \ldots, \cos(2N-1)\beta_j\big)^T,$$

$$w_j = \big(0, \sin\beta_j, \sin(2\beta_j), \ldots, \sin(2N-1)\beta_j\big)^T.$$

Then we have the following lemma by using similar arguments [11].

Lemma 2.2 *If $z(t)$ is a critical point of φ in E, then $z(t)$ is a critical point of φ in $H^{1/2}$.*

Proof By (2.2) and direct computation, we have

$$A_{2N}T_{2N} = T_{2N}A_{2N}, \qquad H(t, T_{2N}z) = H(t, z), \qquad \nabla H(t, T_{2N}z) = T_{2N}\nabla H(t, z).$$

Combing these with (2.6) and the fact that any $z(t) \in E$ is $4N$-periodic, one can easily verify that

$$\varphi(\delta z) = \varphi(z), \qquad \varphi'(\delta z) = \delta\varphi'(z),$$

i.e., φ is G-invariant, and φ' is G-equivariant. The conclusion follows directly. □

Remark 2.1 By Lemma 2.2, the function φ is invariant and φ is equivariant about the compact group G. This allows to find critical points of φ on a subspace of E which is invariant under the group G. Then we can apply Theorem A to obtain periodic solutions in this subspace, which surely have the required symmetric structure and give solutions to (1.1).

Moreover, we also denote by $M^+(\cdot)$, $M^-(\cdot)$ and $M^0(\cdot)$ the positive definite, negative definite, and null subspaces of the self-adjoint linear operator defining it, respectively.

Then E has an orthogonal decomposition

$$E = M^+(L) \oplus M^-(L) \oplus M^0(L).$$

By Lemma 2.3 [11], for $z_j(t) = a_j \cos(\frac{\pi}{2N}(2j-1)t) + b_j \sin(\frac{\pi}{2N}(2j-1)t)$ $(j = 1, 2, \ldots)$, a_j and b_j are defined in Lemma 2.1, we have

$$\langle Lz_j, z_j \rangle = -\frac{\pi(2j-1)}{2N} \tan \frac{\pi(2j-1)}{4N} \big[(a_j, a_j) + (b_j, b_j)\big]. \tag{2.7}$$

Let

$$\sigma = \min\left\{ \left| \frac{\pi(2j-1)}{2N} \tan \frac{\pi(2j-1)}{4N} \right| : j \in Z^+ \right\} > 0.$$

So we have

$$\langle Lz_j, z_j \rangle \geq \sigma \|z_j\|^2, \quad \text{for } \gamma_j > 0,$$

$$\langle Lz_j, z_j \rangle \leq -\sigma \|z_j\|^2, \quad \text{for } \gamma_j < 0,$$

where $\gamma_j = -\tan \frac{\pi(2j-1)}{4N}$. Therefore, we get

$$\langle Lz, z \rangle \geq \sigma \|z\|^2, \quad \text{for } z \in M^+(L), \tag{2.8}$$

$$\langle Lz, z \rangle \leq -\sigma \|z\|^2, \quad \text{for } z \in M^-(L). \tag{2.9}$$

3 Proofs of theorems

In this section, c_i stand for different positive constants for $i \in Z^+$.

By direct computation, (f1) implies that $H(t,z)$ is even and satisfies

$$\nabla H(t,z) = \alpha(t)z + o(|z|) \quad \text{as } |z| \to 0, \tag{3.1}$$

$$\nabla H(t,z) = \beta(t)z + o(|z|) \quad \text{as } |z| \to \infty, \tag{3.2}$$

uniformly for $t \in [0, 2N]$.

(f2$^{\pm}$) implies that

$$2H(t,z) - \big(\nabla H(t,z), z\big) \geq d_1|z| - 2Nd_2, \quad \forall(t,z) \in [0, 2N] \times \mathbf{R}^{2N} \tag{3.3}$$

and

$$-\big[2H(t,z) - \big(\nabla H(t,z), z\big)\big] \geq d_1|z| - 2Nd_2, \quad \forall(t,z) \in [0, 2N] \times \mathbf{R}^{2N}. \tag{3.4}$$

Remark 3.1 Here we prove that (3.3) and (3.4) hold. By (2.2) and (f2$^{\pm}$), we have

$$\pm\big[2H(t,z) - \big(\nabla H(t,z), z\big)\big] = \pm \sum_{i=1}^{2N}\big[2F(t,z_i) - z_if(t,z_i)\big]$$

$$\geq d_1 \sum_{i=1}^{2N}|z_i| - 2Nd_2 \geq d_1|z| - 2Nd_2.$$

Lemma 3.1 *Suppose that f satisfies* (f1)-(f2$^{\pm}$)*. Then the function φ satisfies the* (C)$_c$ *condition for any $c \in \mathbf{R}$.*

Proof First, we need some notations. For any $z, y \in E$, define an operator B by extending the bilinear form

$$\langle Bz, y \rangle = \int_0^{4N} \big(\beta(t)z(t), y(t)\big)\,dt.$$

By direct computation, B is a bounded self-adjoint linear operator on E. Thus $L + B$ is also a self-adjoint linear operator on E.

Let

$$E = M^+(L + B) \oplus M^-(L + B) \oplus M^0(L + B).$$

Also there exists $\sigma_1 > 0$ such that

$$\big\langle (L + B)z, z \big\rangle \geq \sigma_1 \|z\|^2, \quad \text{for } z \in M^+(L + B), \tag{3.5}$$

$$\big\langle (L + B)z, z \big\rangle \leq -\sigma_1 \|z\|^2, \quad \text{for } z \in M^-(L + B). \tag{3.6}$$

So we have

$$\Phi(z) = -\varphi(z) = -\frac{1}{2}\big\langle (L + B)z, z \big\rangle - \int_0^{4N}\left[H(t,z) - \frac{1}{2}\big(\beta(t)z, z\big)\right]dt$$

and

$$\big\langle \Phi'(z), y \big\rangle = -\big\langle (L + B)z, y \big\rangle - \int_0^{4N}\big(\nabla H(t,z) - \beta(t)z, y\big)\,dt.$$

Let $\{z_k\} \subset E$ be any sequence such that

$$\Phi(z_k) \to c, \quad \left(1 + \|z_k\|\right)\Phi'(z_k) \to 0 \quad \text{as } k \to \infty. \tag{3.7}$$

We first prove that $\{z_k\}$ is bounded. Since $z_k \in E$, we have

$$z_k = z_k^+ + z_k^- + z^0 \in M^+(L + B) \oplus M^-(L + B) \oplus M^0(L + B).$$

Let

$$h(z) = \int_0^{4N} \tilde{H}(t, z)\, dt, \qquad \tilde{H}(t, z) = H(t, z) - \frac{1}{2}\left(\beta(t)z, z\right), \quad z \in E.$$

By (3.2), we have

$$\frac{|\nabla \tilde{H}(t, z)|}{|z|} \to 0 \quad \text{as } |z| \to \infty,$$

uniformly for $t \in [0, 2N]$. This means that, for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\left|\nabla \tilde{H}(t, z)\right| \le \varepsilon |z| + M, \quad \text{for all } (t, z) \in [0, 2N] \times \mathbf{R}^{2N}. \tag{3.8}$$

Therefore, for any $y \in E$,

$$
\begin{aligned}
\left|\left\langle h'(z_k), y\right\rangle\right| &= \left|\int_0^{4N} \left(\nabla \tilde{H}\left(t, z_k(t)\right), y(t)\right) dt\right| \\
&\le \int_0^{4N} \left|\nabla \tilde{H}\left(t, z_k(t)\right)\right|\left|y(t)\right| dt \\
&\le \left(\int_0^{4N} \left|\nabla \tilde{H}\left(t, z_k(t)\right)\right|^2 dt\right)^{1/2} \|y\|_{L^2} \\
&\le \left(2\varepsilon^2 \|z_k\|_{L^2}^2 + 8NM^2\right)^{1/2} \|y\|_{L^2} \\
&\le \left(2\varepsilon^2 \|z_k\|^2 + 8NM^2\right)^{1/2} \|y\|.
\end{aligned}
$$

This implies

$$\lim_{k \to \infty} \frac{\|h'(z_k)\|}{\|z_k\|} \le \sqrt{2}\varepsilon, \quad \text{for any } \varepsilon > 0,$$

i.e.

$$\frac{\|h'(z_k)\|}{\|z_k\|} \to 0 \quad \text{as } k \to \infty. \tag{3.9}$$

Then

$$
\begin{aligned}
\left\langle -\Phi'(z_k), z_k^+\right\rangle &= \left\langle (L + B)z_k^+, z_k^+\right\rangle + \left\langle h'(z_k), z_k^+\right\rangle \\
&\ge \sigma_1 \left\|z_k^+\right\|^2 - \left\|h'(z_k)\right\|\left\|z_k^+\right\|.
\end{aligned}
$$

By (3.7) and (3.9), we have

$$\frac{\|z_k^+\|}{\|z_k\|} \to 0 \quad \text{as } k \to \infty. \tag{3.10}$$

Similarly,

$$\frac{\|z_k^-\|}{\|z_k\|} \to 0 \quad \text{as } k \to \infty. \tag{3.11}$$

Case (i): (3.3) holds. We have

$$\langle \Phi'(z_k), z_k \rangle - 2\Phi(z_k)$$

$$= \int_0^{4N} \left[2H(t, z_k) - \langle \nabla H(t, z_k), z_k \rangle \right] dt$$

$$\geq d_1 \int_0^{4N} |z_k| \, dt - 8N^2 d_2$$

$$\geq d_1 \int_0^{4N} |z_k^0| \, dt - d_1 \int_0^{4N} \left(|z_k^+| + |z_k^-| \right) dt - 8N^2 d_2$$

$$\geq d_1 \|z_k^0\| - d_1 \left(\|z_k^+\| + \|z_k^-\| \right) - 8N^2 d_2.$$

Here we used the fact that $M^0(L + B)$ is finite dimensional. By (3.8), (3.10), and (3.11), we have

$$\frac{\|z_k^0\|}{\|z_k\|} \to 0 \quad \text{as } k \to \infty. \tag{3.12}$$

But this implies the following contradiction:

$$1 = \frac{\|z_k\|}{\|z_k\|} \leq \frac{\|z_k^0\| + \|z_k^-\| + \|z_k^+\|}{\|z_k\|} \to 0 \quad \text{as } k \to \infty. \tag{3.13}$$

Therefore $\{\|z_k\|\}$ must be bounded.

Case (ii): (3.4) holds. Similar to Case (i), we have

$$\langle 2\Phi(z_k) - \Phi'(z_k), z_k \rangle$$

$$= \int_0^{4N} \left[\langle \nabla H(t, z_k), z_k \rangle - 2H(t, z_k) \right] dt$$

$$\geq d_1 \int_0^{4N} |z_k| \, dt - 8N^2 d_2$$

$$\geq d_1 \int_0^{4N} |z_k^0| \, dt - d_1 \int_0^{4N} \left(|z_k^+| + |z_k^-| \right) dt - 8N^2 d_2$$

$$\geq d_1 \|z_k^0\| - d_1 \left(\|z_k^+\| + \|z_k^-\| \right) - 8N^2 d_2.$$

This implies (3.12) and (3.13). Thus $\{\|z_k\|\}$ must be bounded.

So we see that $\{z_k\}$ is bounded, and going if necessary to a subsequence, we can assume that $z_k \rightharpoonup z$ in E and $z_k \to z$ in $L(S^1, \mathbf{R}^{2N})$. Write $z_k = z_k^+ + z_k^- + z_k^-$ and $z = z^+ + z^- + z^0$, then $z_k^\pm \rightharpoonup z^\pm$ in E, $z_k^0 \rightharpoonup z^0$ in E, and $z_k^\pm \to z^\pm$ in $L(S^1, \mathbf{R}^{2N})$.

In view of (3.8) and $z_k^- \to z^-$ in $L(S^1, \mathbf{R}^{2N})$, it is easy to verify

$$\int_0^{4N} \left(\nabla H(t, z_k) - \beta(t) z_k, z_k^- - z^- \right) dt \to 0$$

and

$$\int_0^{4N} \left(\nabla H(t, z) - \beta(t) z, z_k^- - z^- \right) dt \to 0.$$

But then $\langle \Phi'(z_k) - \Phi'(z), z_k^- - z^- \rangle \to 0$ as $k \to \infty$, and

$$
\begin{aligned}
&\left\langle \Phi'(z_k) - \Phi'(z), z_k^- - z^- \right\rangle \\
&\quad = -\left\langle (L + B)\left(z_k^- - z^- \right), z_k^- - z^- \right\rangle \\
&\qquad - \int_0^{4N} \left(\nabla H(t, z_k) - \beta(t) z_k, z_k^- - z^- \right) dt + \int_0^{4N} \left(\nabla H(t, z) - \beta(t) z, z_k^- - z^- \right) dt \\
&\quad \geq \sigma_1 \left\| z_k^- - z^- \right\|^2 - \int_0^{4N} \left(\nabla H(t, z_k) - \beta(t) z_k, z_k^- - z^- \right) dt \\
&\qquad + \int_0^{4N} \left(\nabla H(t, z) - \beta(t) z, z_k^- - z^- \right) dt.
\end{aligned}
$$

This yields $z_k^- \to z^-$ in E. Similarly, $z_k^+ \to z^+$ in E and hence $z_k \to z$ in E, that is, Φ satisfies the $(C)_c$ condition. Thus φ satisfies the $(C)_c$ condition. The proof of Lemma 3.1 is complete. □

Proof of Theorem 1.1 Let $X = M^+(L) \oplus M^0(L)$, $Y = M^-(L)$, $E = X \oplus Y$, and

$$\Phi(z) = -\varphi(z), \qquad \Psi(z) = \psi(z),$$

where $\varphi(z)$ and $\psi(z)$ are defined in Section 2.

In order to obtain this theorem, we will apply Theorem A to the functional $\Phi(z)$. Let $\mathcal{S} = X^*$, $\mathcal{T}_{\mathcal{S}} = E = X \oplus Y$. The proof of this theorem is divided into the following three steps.

Step 1. Φ satisfies (Φ_0).

We first check that Φ_c is $\mathcal{T}_{\mathcal{S}}$-closed for any $c \in \mathbf{R}$. Let $\{z_k\}$ be any sequence $\mathcal{T}_{\mathcal{S}}$-converging to some $z \in E$. Write $z_k = z_k^+ + z_k^0 + z_k^-$ and $z = z^+ + z^0 + z^-$, then $z_k^- \to u^-$ in E and hence $\{z^-\}$ is bounded in the norm topology.

Note that $\beta(t) > \frac{\pi}{2N} \tan \frac{\pi}{4N}$ for $t \in [0, 2N]$, then for any $z \in E$ and small $\varepsilon > 0$, by (3.8), we have

$$
\begin{aligned}
\Psi(z) &= \int_0^{4N} H(t, z) \, dt = \int_0^{4N} \int_0^1 \left(\nabla H(t, sz), z \right) ds \, dt \\
&\geq \int_0^{4N} \int_0^1 \left(\beta(t) sz, z \right) ds \, dt
\end{aligned}
$$

$$- \int_0^{4N} \int_0^1 \left| \nabla H(t, sz) - \left(\beta(t)sz, z \right) \right| |z| \, ds \, dt$$

$$\geq \frac{1}{2} \int_0^{4N} \left(\beta(t)z, z \right) dt - \int_0^{4N} \int_0^1 \left(\varepsilon |sz| + M \right) |z| \, ds \, dt$$

$$\geq \frac{1}{2} \left(\frac{\pi}{2N} \tan \frac{\pi}{4N} - \varepsilon \right) \|z\|_{L^2}^2 - M \|z\|_{L^2} \geq c_1 \|z\|_{L^2}^2 - c_2 \|z\|_{L^2},$$

which implies that Ψ is bounded from below on E. Consequently, combining $z_k \in \Phi_c$ and $\frac{1}{2} \langle Lz_k^+, z_k^+ \rangle = -\frac{1}{2} \langle Lz_k^-, z_k^- \rangle - \Phi(z_k) - \Psi(z_k)$ shows that $\{z_k^+\}$ is bounded in E by (2.7) and hence

$$\Psi(z_k) = -\frac{1}{2} \langle Lz_k^+, z_k^+ \rangle - \frac{1}{2} \langle Lz_k^-, z_k^- \rangle - \Phi(z_k)$$

$$\leq -\frac{1}{2} \langle Lz_k^+, z_k^+ \rangle - \frac{1}{2} \langle Lz_k^-, u_k^- \rangle - c \leq c_3. \tag{3.14}$$

Moreover, since $\|z_k\|_{L^2}^2 = \|z_k^+\|_{L^2}^2 + \|z_k^0\|_{L^2}^2 + \|z_k^-\|_{L^2}^2$, we have

$$\Psi(z_k) \geq c_1 \|z_k\|_{L^2}^2 - c_2 \|z_k\|_{L^2} \geq \|z_k\|_{L^2} - c_4 \geq \left\| z_k^0 \right\|_{L^2} - c_4. \tag{3.15}$$

It follows from (3.14) and (3.15) that $\{z_k^0\}$ is also bounded in E since all norms are equivalent in a finite dimensional space. Then $\|z_k\|^2$ is bounded, and hence we can assume that $\{z_k\}$ converges weakly to $z = z^+ + z^0 + z^-$ in E. Thus we have $\Psi(z_k) \to \Psi(z)$. Note that $\langle Ly, y \rangle^{1/2}$ is an equivalent norm on H^+. By the lower semi-continuity of the norm, we get

$$c \leq \limsup_{k \to \infty} \Phi(z_k) = \limsup_{k \to \infty} \left(-\frac{1}{2} \langle Lz_k^+, z_k^+ \rangle - \frac{1}{2} \langle Lz_k^-, z_k^- \rangle - \Psi(z_k) \right)$$

$$= -\liminf_{k \to \infty} \frac{1}{2} \langle Lz_k^+, z_k^+ \rangle - \frac{1}{2} \langle Lz^-, z^- \rangle - \Psi(z) \leq \Phi(z),$$

that is, $z \in \Phi_c$ and hence Φ_c is $\mathcal{T}_S$-closed.

Next, we prove that $\Phi' : (\Phi_c, \mathcal{T}_S) \to (E^*, \omega^*)$ is continuous. To achieve this, it is sufficient to demonstrate that Ψ' has the same property. Suppose $z_k \rightharpoonup u$ in E, then $\{z_k\}$ converges uniformly to z on $[0, 4N]$. Hence, for every given $y \in E$, we see that $(\nabla H(t, z_k(t)), y(t))$ converges to $(\nabla H(t, z(t)), y(t))$ in measure on $[0, 4N]$. Moreover, by (3.8), one has

$$\left| \left(\nabla H(t, z_k(t)), y(t) \right) \right| \leq \left(\varepsilon \|z_k\|_\infty + \bar{\beta} \|z_k\|_\infty + M \right) \|y\|_\infty \leq c_5$$

for all k and $t \in [0, 4N]$, where $\bar{\beta} = \max_{t \in [0, 2N]} \{\beta(t)\}$ and $\| \cdot \|_\infty$ denotes the natural norm of $C(S^1, \mathbf{R}^{2N})$. Thus, the Vitali theorem is applicable and

$$\left\langle \Psi'(z_k), y \right\rangle = \int_0^{4N} \left(\nabla H(t, z_k), y \right) dt \to \int_0^{4N} \left(\nabla H(t, z), y \right) dt = \left\langle \Psi'(z), y \right\rangle$$

for any $y \in E$. So Φ satisfies (Φ_0).

Step 2. Φ satisfies (Φ_1).

By (3.1) and (3.2), for any $\varepsilon > 0$, there exist $C_\varepsilon > 0$ and $p > 2$ such that

$$H(t, z) \leq \varepsilon |z|^2 + C_\varepsilon |z|^p, \quad \text{for all } (t, z) \in [0, 4N] \times \mathbf{R}^{2N}.$$

Hence, for $z \in Y$ and small ε, we have

$$
\begin{aligned}
\Phi(z) &= -\frac{1}{2}\langle Lz, z\rangle - \int_0^{4N} H(t, z)\, dt \\
&\geq -\frac{1}{2}\langle Lz, z\rangle - \int_0^{4N} \left(\varepsilon|z|^2 + C_\varepsilon |z|^p\right) dt \\
&\geq \frac{\sigma}{2}\|z\|^2 - \varepsilon\|z\|_{L^2}^2 - C_\varepsilon \|z\|_{L^p}^p \\
&\geq \frac{\sigma}{2}\|z\|^2 - \varepsilon\xi_1\|z\|^2 - C_\varepsilon\xi_2\|z\|^p \\
&\geq \frac{\sigma}{4}\|z\|^2 - c_5\|z\|^p,
\end{aligned}
$$

where $\xi_1, \xi_2 > 0$ satisfy $\|z\|_{L^2} \leq \xi_1\|z\|$, $\|z\|_{L^p} \leq \xi_2\|z\|$; see [4], Proposition 6.6.

Since $p > 2$, there exists a small $\rho > 0$ such that $\frac{1}{8}\rho^2 \geq c_5\rho^p$. Therefore,

$$
\kappa := \inf \Phi(B_\rho \cap Y) \geq \frac{1}{8}\rho^2 > 0,
$$

and hence (Φ_1) holds.

Step 3. Φ satisfies (Φ_2).

Let

$$
\begin{aligned}
Y_0 = \operatorname{span}\Bigg\{ & a_j \cos\left(\frac{\pi}{2N}(2j-1)t\right), b_j \sin\left(\frac{\pi}{2N}(2j-1)t\right): \\
& \beta(t) > \frac{\pi(2j-1)}{2N}\tan\frac{\pi(2j-1)}{4N} > 0, \text{for all } t \in [0, 2N]\Bigg\},
\end{aligned}
$$

where a_j and b_j are defined in Lemma 2.1.

Obviously, $Y_0 \subset Y$ and $\dim Y_0 = 2m$. In order to obtain the desired conclusion, it is sufficient to prove that $\Phi(z) \to -\infty$ as $\|z\| \to \infty$ on $E_0 := X \oplus Y_0$.

Let

$$
m_1 = \max\left\{ j \in \mathbf{Z}^+ : \beta(t) > \frac{\pi(2j-1)}{2N}\tan\frac{\pi(2j-1)}{4N} > 0, \text{for all } t \in [0, 2N]\right\}
$$

and

$$
\Gamma = -\frac{\pi(2m_1-1)}{2N}\tan\frac{\pi(2m_1-1)}{4N}.
$$

By the definition of Γ, there exists a constant $0 < \delta < \sigma$; σ is defined in (2.8), such that

$$
\beta(t) \geq -\Gamma + \delta \tag{3.16}
$$

for $t \in [0, 2N]$. Let

$$
J = \left\{ j \in \mathbf{Z}^+ : \beta(t) > \frac{\pi(2j-1)}{2N}\tan\frac{\pi(2j-1)}{4N} > 0, \text{for all } t \in [0, 2N]\right\}.
$$

For any $y \in Y_0$, by (2.7),

$$\langle Ly, y \rangle = \sum_{j \in J} \left(-\frac{\pi(2j-1)}{2N} \tan \frac{\pi(2j-1)}{4N} \big[(a_j, a_j) + (b_j, b_j) \big] \right)$$

$$\geq \Gamma \|y\|_{L^2}^2. \tag{3.17}$$

Let $\widetilde{H}(t,z) = H(t,z) - \frac{1}{2}(\beta(t)z, z)$. We claim that, for $0 \neq z \in E$,

$$\|z\|^{-2} \int_0^{4N} \widetilde{H}(t,z) \, dt \to 0 \quad \text{as } \|z\| \to \infty. \tag{3.18}$$

Indeed, for $0 \neq z \in E$, by (3.8), one has

$$\|z\|^{-2} \left| \int_0^{4N} \widetilde{H}(t,z) \, dt \right|$$

$$= \|z\|^{-2} \left| \int_0^{4N} \int_0^1 \left(\nabla H(t, sz) - \beta(t)sz, z \right) ds \, dt \right|$$

$$\leq \|z\|^{-2} \int_0^{4N} \left(\varepsilon |z| + M \right) |z| \, dt$$

$$\leq \|z\|^{-2} \left(\varepsilon \|z\|_{L^2}^2 + M \|z\|_{L^2} \right)$$

$$\leq \varepsilon + \frac{M}{\|z\|},$$

which implies that (3.18) is true by the arbitrariness of ε. Then, for $z = z^+ + z^0 + z^- \in E_0$, by (3.16)-(3.17) and (2.8), one has

$$\Phi(z) = -\frac{1}{2} \langle Lz^-, z^- \rangle - \frac{1}{2} \langle Lz^+, z^+ \rangle - \int_0^{4N} H(t,z) \, dt$$

$$\leq -\frac{\Gamma}{2} \|z^-\|_{L^2}^2 - \frac{\delta}{2} \|z^+\|^2 - \frac{1}{2} \int_0^{4N} \left(\beta(t)z, z \right) dt - \int_0^{4N} \widetilde{H}(t,z) \, dt$$

$$\leq -\frac{\Gamma}{2} \|z^-\|_{L^2}^2 - \frac{\delta}{2} \|z^+\|^2 - \frac{1}{2}(-\Gamma + \delta)\|z\|_{L^2}^2 - \int_0^{4N} \widetilde{H}(t,z) \, dt$$

$$\leq -\frac{\Gamma}{2} \|z^-\|_{L^2}^2 - \frac{\delta}{2} \|z^+\|^2 - \frac{1}{2}(-\Gamma + \delta)\left(\|z^-\|_{L^2}^2 + \|z^0\|_{L^2}^2 \right)$$

$$\quad - \int_0^{4N} \widetilde{H}(t,z) \, dt$$

$$\leq -\frac{\delta}{2} \left(\|z^-\|_{L^2}^2 + \|z^+\|^2 + \|z^0\|_{L^2}^2 \right) - \int_0^{4N} \widetilde{H}(t,z) \, dt.$$

Since M^0 and Y_0 are finitely dimensional, (3.18) and the above estimate imply that $\Phi(z) \to -\infty$ as $\|z\| \to \infty$, $z \in E_0 := X \oplus Y_0$. Hence ($\Phi_2$) holds.

The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2 Let $X = M^-(L) \oplus M^0(L)$, $Y = M^+(L)$, $E = X \oplus Y$, and

$$\Phi(z) = \varphi(z), \qquad \Psi(z) = -\psi(z)$$

and

$$Y_0 = \text{span}\left\{ a_j \cos\left(\frac{\pi}{2N}(2j-1)t\right), b_j \sin\left(\frac{\pi}{2N}(2j-1)t\right) : \right.$$
$$\left. \beta(t) < \frac{\pi(2j-1)}{2N} \tan\frac{\pi(2j-1)}{4N} < 0, \text{for } t \in [0,2N] \right\},$$

where a_j and b_j are defined in Lemma 2.1.

Then the conclusion will be obtained by the same argument as in the proof of Theorem 1.1. The proof of Theorem 1.2 is complete. □

Consider the nonautonomous delay equation

$$x'(t) = -f\big(t, x(t-1)\big). \tag{3.19}$$

Example 3.1 Consider the nonautonomous delay equation (3.19), where

$$f(t,x) = a(t)x - b(t)\left[\frac{2x}{\ln(e+x^2)} - \frac{2x^3}{(e+x^2)\ln^2(e+x^2)}\right],$$

and $a(t) = \pi(2 + \tan(\frac{\pi t}{4} - \frac{\pi}{4}))$, $b(t) = \frac{3\pi}{2}$ for $t \in [0,2]$.

It is easy to see that $\alpha(t) = a(t) - 2b(t) \le 0$, $\beta(t) = a(t) \ge \pi$ for $t \in [0,2]$, and $F(t,x) = \int_0^x f(t,s)\,ds = \frac{a(t)x^2}{2} - \frac{b(t)x^2}{\ln(e+x^2)}$. By Theorem 1.1, we get $m = 1$. Thus (3.19) possesses at least two pairs of 4-periodic solutions with $x(t) = -x(t-2)$.

Example 3.2 Consider the nonautonomous delay equation (3.19), where

$$f(t,x) = a(t)x - b(t)\left[\frac{2x}{\ln(e+x^2)} - \frac{2x^3}{(e+x^2)\ln^2(e+x^2)}\right],$$

and $a(t) = 4\pi(-2 + \tan(\frac{\pi t}{4} - \frac{\pi}{4}))$, $b(t) = -6\pi$ for $t \in [0,2]$.

It is easy to see that $\alpha(t) = a(t) - 2b(t) \ge 0$, $\beta(t) = a(t) \le -4\pi$ for $t \in [0,2]$. By Theorem 1.2, we get $m = 2$. Thus (3.19) possesses at least four pairs of 4-periodic solutions with $x(t) = -x(t-2)$.

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