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Global behavior of a plant-herbivore model

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Abstract

The present work deals with an analysis of the local asymptotic stability and global behavior of the unique positive equilibrium point of the following discrete-time plant-herbivore model: $x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}}$, $y_{n+1} = \gamma(x_n + 1)y_n$, where $\alpha \in (1, \infty)$, $\beta \in (0, \infty)$, and $\gamma \in (0, 1)$ with $\alpha + \beta > 1 + \frac{\beta}{\gamma}$ and initial conditions x_0, y_0 are positive real numbers. Moreover, the rate of convergence of positive solutions that converge to the unique positive equilibrium point of this model is also discussed. In particular, our results solve an open problem and a conjecture proposed by Kulenović and Ladas in their monograph (Dynamics of Second Order Rational Difference Equations: With Open Problems and Conjectures, 2002). Some numerical examples are given to verify our theoretical results.

MSC: 39A10; 40A05

Keywords: plant-herbivore system; steady-states; local stability; global behavior; rate of convergence

1 Introduction and preliminaries

In mathematical biology, the model of the plant-herbivore has attracted many researchers during the last few decades. There are several theoretical results for the mutual dependence between plant and herbivore populations. We consider here an open problem proposed in [1] related to the global character of solutions of the plant-herbivore model

$$x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}}, \qquad y_{n+1} = \gamma(x_n + 1)y_n, \tag{1}$$

where x_n and y_n are the population biomasses of the plant and the herbivore in successive generations n and $n+1$, respectively. Moreover, $\alpha \in (1, \infty)$, $\beta \in (0, \infty)$, and $\gamma \in (0, 1)$ with $\alpha + \beta > 1 + \frac{\beta}{\gamma}$ and initial conditions x_0, y_0 are positive real numbers. Furthermore, (1) has a unique positive equilibrium point under the following inequalities:

$$0 < \gamma < 1, \quad \alpha + \beta > 1 + \frac{\beta}{\gamma}.$$

It is pointed out in [2] that the system (1) has a very complex behavior, which has been investigated mainly numerically and visually. Moreover, there is a range of the values of parameters for which the equilibrium points on the x-axis are globally asymptotically stable. The computer simulations indicate that for some parameter values there is chaotic behavior, but proofs are yet to be completed.

In 2002, Kulenović and Ladas [1] proposed the following open problem:

Open Problem 6.10.13 ([1], p.128) (a plant-herbivore system) Assume that $\alpha \in (1,\infty)$, $\beta \in (0,\infty)$, and $\gamma \in (0,1)$ with $\alpha + \beta > 1 + \frac{\beta}{\gamma}$. Obtain conditions for the global asymptotic stability of the positive equilibrium of the system

$$x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}}, \qquad y_{n+1} = \gamma(x_n + 1)y_n.$$

The above model is for the interaction of the apple twig borer (an insect pest of the grape vine) and grapes in the Texas High Plains was developed and studied by Allen *et al.* [3]. See also Allen *et al.* [4] and Allen *et al.* [5].

In this paper, our aim is to investigate the local asymptotic stability, the global asymptotic character of the unique positive equilibrium point, and the rate of convergence of positive solutions of the system (1). For some interesting results related to the qualitative behavior of discrete dynamical systems, we refer to [6–14].

Let $(\bar{x}, \bar{y})$ be an equilibrium point of the system (1), then

$$\bar{x} = \frac{\alpha \bar{x}}{\beta \bar{x} + e^{\bar{y}}}, \qquad \bar{y} = \gamma(\bar{x} + 1)\bar{y}.$$

Hence, $P_0 = (0,0)$, $P_1 = (\frac{\alpha-1}{\beta}, 0)$ and $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ are equilibrium points of the system (1). Furthermore, $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ is a unique positive equilibrium point of the system (1) if and only if

$$0 < \gamma < 1, \quad \alpha + \beta > 1 + \frac{\beta}{\gamma}.$$

The Jacobian matrix of linearized system of (1) about the fixed point $(\bar{x}, \bar{y})$ is given by

$$F_J(\bar{x}, \bar{y}) = \begin{bmatrix} \frac{\alpha e^{\bar{y}}}{(\beta\bar{x}+e^{\bar{y}})^2} & -\frac{\alpha\bar{x}e^{\bar{y}}}{(\beta\bar{x}+e^{\bar{y}})^2} \\ \gamma\bar{y} & \gamma\bar{x}+\gamma \end{bmatrix}.$$

Lemma 1.1 (Jury condition) *Consider the second-degree polynomial equation*

$$\lambda^2 + p\lambda + q = 0, \tag{2}$$

where p and q are real numbers. Then the necessary and sufficient condition for both roots of (2) *to lie inside the open disk* $|\lambda| < 1$ *is*

$$|p| < 1 + q < 2.$$

2 Boundedness

The following theorem shows that every positive solution of the system (1) is bounded.

Theorem 2.1 *Assume that* $x_n \leq x_{n+1}$ *for all* $n = 0, 1, \ldots$, *then every positive solution* $\{(x_n, y_n)\}$ *of the system* (1) *is bounded.*

Proof Assume that the initial conditions x_0, y_0 of (1) are positive, then every solution of (1) is positive. Assume that $\{(x_n, y_n)\}$ is an arbitrary positive solution of the system (1). Then, from the first part of the system (1), one has

$$x_{n+1} = \frac{\alpha x_n}{\beta x_n + e^{y_n}} \leq \frac{\alpha}{\beta},$$

for all $n = 0, 1, 2, \ldots$. Assume that $x_n \leq x_{n+1}$ for all $n = 0, 1, \ldots$. Moreover, from the first equation of the system (1), we obtain

$$\begin{aligned} y_n &= \ln\left(\frac{x_n(\alpha - \beta x_{n+1})}{x_{n+1}}\right) \\ &\leq \ln(\alpha - \beta x_{n+1}). \end{aligned} \tag{3}$$

From (3) and the second part of the system (1), we have

$$\begin{aligned} y_{n+1} &= \gamma(x_n + 1)y_n \\ &= \gamma(x_n + 1)\ln\left(\frac{\alpha x_n - \beta x_n x_{n+1}}{x_{n+1}}\right) \\ &\leq \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln\left(\frac{\alpha x_n - \beta x_n x_{n+1}}{x_{n+1}}\right) \\ &\leq \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln(\alpha - \beta x_{n+1}), \end{aligned}$$

for all $n = 0, 1, \ldots$. Finally, we let $F(x) = \ln(\alpha - \beta x)$ such that $0 \leq x < \frac{\alpha}{\beta}$. Then it follows that $F'(x) = -\frac{\beta}{\alpha - \beta x} < 0$ for all $0 \leq x < \frac{\alpha}{\beta}$. Hence, $F(x) \leq F(0)$ for all $0 \leq x < \frac{\alpha}{\beta}$, *i.e.*, $\ln(\alpha - \beta x) \leq \ln(\alpha)$ for all $0 \leq x < \frac{\alpha}{\beta}$. Thus for any positive solution $\{(x_n, y_n)\}$ of the system (1), we have

$$0 \leq x_n \leq \frac{\alpha}{\beta}, \qquad 0 \leq y_n \leq \gamma\left(\frac{\alpha}{\beta} + 1\right)\ln(\alpha),$$

for all $n = 1, 2, \ldots$. The proof is therefore completed. □

3 Linearized stability

Theorem 3.1 *Assume that* $\alpha \in (1, \infty)$ *and* $\gamma \in (0, 1)$, *then the following statements are true*:

(i) *The equilibrium point* $P_0 = (0, 0)$ *of the system* (1) *is a saddle point.*
(ii) *The equilibrium point* $P_1 = (\frac{\alpha-1}{\beta}, 0)$ *of the system* (1) *is locally asymptotically stable if and only if* $\beta > \frac{\gamma - \alpha\gamma}{\gamma - 1}$.

Proof (i) The proof follows from the fact that the Jacobian matrix of linearized system of (1) about the equilibrium point $(0, 0)$ is given by

$$F_J(0, 0) = \begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}.$$

(ii) The Jacobian matrix of linearized system of (1) about the fixed point $(\frac{\alpha-1}{\beta}, 0)$ is given by

$$F_J\left(\frac{\alpha - 1}{\beta}, 0\right) = \begin{bmatrix} \frac{1}{\alpha} & \frac{1-\alpha}{\alpha\beta} \\ 0 & \frac{\gamma(\alpha+\beta-1)}{\beta} \end{bmatrix}.$$

It is obvious that the roots of the characteristic polynomial of $F_J(\frac{\alpha-1}{\beta},0)$ about $(\frac{\alpha-1}{\beta},0)$ are $\lambda_1 = \frac{1}{\alpha} < 1$ for $\alpha \in (1,\infty)$ and $\lambda_2 = \frac{\alpha\gamma+\beta\gamma-\gamma}{\beta} < 1$ if and only if $\beta > \frac{\gamma-\alpha\gamma}{\gamma-1}$. □

The following theorem shows a necessary and sufficient condition for local asymptotic stability of unique positive equilibrium point of the system (1).

Theorem 3.2 *The unique positive equilibrium point* $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ *of the system* (1) *is locally asymptotically stable if and only if the following holds*:

$$\ln\left(\alpha + \beta - \frac{\beta}{\gamma}\right) < \frac{\beta}{\gamma(\alpha+\beta)-\beta}. \tag{4}$$

Proof The Jacobian matrix of linearized system of (1) about the positive equilibrium point $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ is given by

$$F_J(P_2) = \begin{bmatrix} \frac{\alpha\gamma+\beta(\gamma-1)}{\alpha\gamma} & \frac{(\gamma-1)(\alpha\gamma+\beta(\gamma-1))}{\alpha\gamma^2} \\ \gamma\ln(\alpha-\frac{\beta}{\gamma}+\beta) & 1 \end{bmatrix}.$$

The characteristic polynomial of $F_J(P_2)$ about positive equilibrium point $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ is given by

$$P(\lambda) = \lambda^2 + \left(\frac{\beta}{\alpha\gamma} - \frac{\beta}{\alpha} - 2\right)\lambda + 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha\gamma} + \frac{\delta}{\alpha\gamma}, \tag{5}$$

where $\delta = (1-\gamma)(\beta(\gamma-1)+\alpha\gamma)\ln(\alpha+\beta-\frac{\beta}{\gamma})$. Set

$$p = \frac{\beta}{\alpha\gamma} - \frac{\beta}{\alpha} - 2, \qquad q = 1 + \frac{\beta}{\alpha} - \frac{\beta}{\alpha\gamma} + \frac{\delta}{\alpha\gamma}.$$

Then (5) can be written as

$$P(\lambda) = \lambda^2 + p\lambda + q. \tag{6}$$

Then it follows from Lemma 1.1 that both eigenvalues of $F_J(P_2)$ lie inside the unit disk $|\lambda| < 1$, if and only if

$$|p| < 1 + q < 2. \tag{7}$$

Inequality (7) is equivalent to the following three inequalities:

$$\frac{\beta}{\alpha} - \frac{\beta}{\alpha\gamma} + \frac{\delta}{\alpha\gamma} < 0, \tag{8}$$

$$p < 1 + q, \tag{9}$$

$$-1 - q < p. \tag{10}$$

First, we will prove (8). This inequality is equivalent to

$$\beta(\gamma-1) + \delta = (1-\gamma)\left[-\beta + \big((\alpha+\beta)\gamma - \beta\big)\ln\left(\alpha+\beta-\frac{\beta}{\gamma}\right)\right] < 0.$$

Assume that $\ln(\alpha + \beta - \frac{\beta}{\gamma}) < \frac{\beta}{\gamma(\alpha+\beta)-\beta}$, then it follows that $\beta(\gamma - 1) + \delta < 0$. Next, we consider (9). This inequality is equivalent to

$$2\beta(1-\gamma) - 4\alpha\gamma - \delta < 0.$$

It is easy to see that under condition (4), one has $2\beta(1-\gamma) - 4\alpha\gamma - \delta < 0$. Finally, (10) is equivalent to the following inequality:

$$(1-\gamma)\big(\beta(\gamma-1) + \alpha\gamma\big)\ln\left(\alpha + \beta - \frac{\beta}{\gamma}\right) > 0,$$

and this obviously holds due to the assumption that $\alpha + \beta > 1 + \frac{\beta}{\gamma}$. Then it follows from Lemma 1.1 that the unique positive equilibrium point $P_2 = (\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}))$ of the system (1) is locally asymptotically stable if and only if $\ln(\alpha + \beta - \frac{\beta}{\gamma}) < \frac{\beta}{\gamma(\alpha+\beta)-\beta}$. □

4 Global behavior

Lemma 4.1 [7] *Let $I = [a,b]$ and $J = [c,d]$ be real intervals, and let $f : I \times J \to I$ and $g : I \times J \to J$ be continuous functions. Consider the following system*:

$$\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \quad n = 0, 1, \ldots, \end{aligned} \tag{11}$$

with initial conditions $(x_0, y_0) \in I \times J$. Suppose that the following statements are true:

(i) *$f(x,y)$ is non-decreasing in x and non-increasing in y.*

(ii) *$g(x,y)$ is non-decreasing in both arguments.*

(iii) *If $(m_1, M_1, m_2, M_2) \in I^2 \times J^2$ is a solution of the system*

$$\begin{aligned} m_1 &= f(m_1, M_2), & M_1 &= f(M_1, m_2), \\ m_2 &= g(m_1, m_2), & M_2 &= g(M_1, M_2), \end{aligned}$$

such that $m_1 = M_1$ and $m_2 = M_2$. Then there exists exactly one positive equilibrium point $(\bar{x}, \bar{y})$ of the system (11) *such that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$.*

Theorem 4.1 *The positive equilibrium point P_2 of the system* (1) *is a global attractor.*

Proof Let $f(x,y) = \frac{\alpha x}{\beta x + e^y}$ and $g(x,y) = \gamma(x+1)y$. Then it is easy to see that $f(x,y)$ is non-decreasing in x and non-increasing in y. Moreover, $g(x,y)$ is non-decreasing in both x and y. Let (m_1, M_1, m_2, M_2) be a positive solution of the system

$$\begin{aligned} m_1 &= f(m_1, M_2), & M_1 &= f(M_1, m_2), \\ m_2 &= g(m_1, m_2), & M_2 &= g(M_1, M_2). \end{aligned}$$

Then one has

$$m_1 = \frac{\alpha m_1}{\beta m_1 + e^{M_2}}, \qquad M_1 = \frac{\alpha M_1}{\beta M_1 + e^{m_2}}, \tag{12}$$

and

$$m_2 = \gamma(m_1+1)m_2, \qquad M_2 = \gamma(M_1+1)M_2. \tag{13}$$

Furthermore, we make the assumption as in the proof of Theorem 1.16 of [15]; it suffices to suppose that

$$0 < m_1 \le M_1, \qquad 0 < m_2 \le M_2.$$

From (12), one has

$$\beta m_1 + e^{M_2} = \alpha, \qquad \beta M_1 + e^{m_2} = \alpha. \tag{14}$$

From (13), one has

$$\gamma(m_1+1) = 1, \qquad \gamma(M_1+1) = 1. \tag{15}$$

It follows from (15) that

$$m_1 = \frac{1-\gamma}{\gamma} = M_1. \tag{16}$$

Moreover, from (14) and (16), we obtain

$$m_2 = \ln\left(\frac{\alpha\gamma+\beta\gamma-\beta}{\gamma}\right) = M_2.$$

Hence, from Lemma 4.1 the equilibrium point P_2 is a global attractor. □

Lemma 4.2 *Under the condition* (4) *the unique positive equilibrium point* P_2 *of the system* (1) *is globally asymptotically stable.*

Proof The proof follows from Theorem 3.2 and Theorem 4.1. □

5 Conjecture

Now we consider the following conjecture related to special form of the system (1).

Conjecture 6.10.6 ([1], p.128) Assume that $x_0, y_0 \in (0,\infty)$ and that

$$\alpha \in (2,3), \qquad \beta = 1, \qquad \gamma = 0.5.$$

Show that the positive equilibrium of the system (1) is globally asymptotically stable.

In this conjecture, we modify the interval of α for which the unique positive equilibrium point of the system (1) is globally asymptotically stable. In this case the system (1) can be written as

$$x_{n+1} = \frac{\alpha x_n}{x_n + e^{y_n}}, \qquad y_{n+1} = 0.5(x_n+1)y_n, \quad n = 0,1,2,\ldots, \tag{17}$$

where $\alpha \in (2, R_0)$ such that R_0 is a root of the function defined by

$$F(\alpha) := \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha - 1)}{\alpha} - \frac{1}{\alpha}. \tag{18}$$

Theorem 5.1 *The unique positive equilibrium point* $(1, \ln(\alpha - 1))$ *of the system* (17) *is globally asymptotically stable if and only if* $2 < \alpha < R_0$, *where* R_0 *is a root of the function defined in* (18) *and* $R_0 \approx 3.3457507549227654$.

Proof The characteristic polynomial of $F_J(\bar{x}, \bar{y})$ about positive equilibrium point $(\bar{x}, \bar{y}) = (1, \ln(\alpha - 1))$ for the system (17) is given by

$$P(\lambda) = \lambda^2 + \left(\frac{1}{\alpha} - 2\right)\lambda + 1 - \frac{1}{\alpha} + \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha - 1)}{\alpha}. \tag{19}$$

Then it follows from Lemma 1.1 that the unique positive equilibrium point $(1, \ln(\alpha - 1))$ of the system (17) is locally asymptotically stable if and only if

$$\left|\frac{1}{\alpha} - 2\right| < 2 - \frac{1}{\alpha} + \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha - 1)}{\alpha} < 2.$$

It is easy to see that $|\frac{1}{\alpha} - 2| < 2 - \frac{1}{\alpha} + \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha-1)}{\alpha}$ for all $\alpha > 2$. Hence, it is enough to show that $2 - \frac{1}{\alpha} + \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha-1)}{\alpha} < 2$, or equivalently

$$F(\alpha) = \frac{(\frac{\alpha}{2} - \frac{1}{2})\ln(\alpha - 1)}{\alpha} - \frac{1}{\alpha} < 0.$$

Moreover, $F(2) = -\frac{1}{2}$, $F(3.4) = 0.0148713$, and $F'(\alpha) = \frac{\alpha + \ln(\alpha-1) + 2}{2\alpha^2} > 0$ for all $\alpha > 2$. From this it follows that $F(\alpha)$ has the unique root R_0 in $(2, 3.4)$. With the help of Mathematica this unique root of $F(\alpha)$ is approximated by $R_0 \approx 3.3457507549227654$. It is easy to see that $F(\alpha) < 0$ if and only if $2 < \alpha < R_0$. Hence, the unique positive equilibrium point $(1, \ln(\alpha - 1))$ of the system (17) is locally asymptotically stable if and only if $\alpha \in (2, R_0)$. Moreover, Theorem 4.1 shows that the unique positive equilibrium point $(1, \ln(\alpha - 1))$ of the system (17) is a global attractor. Hence, the proof is completed. □

6 Rate of convergence

In this section we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of the system (1).

The following result gives the rate of convergence of solutions of a system of difference equations:

$$X_{n+1} = \big(A + B(n)\big)X_n, \tag{20}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\|B(n)\| \to 0 \tag{21}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm which is associated with the vector norm

$$\|(x,y)\| = \sqrt{x^2 + y^2}.$$

Proposition 6.1 (Perron's theorem) [16] *Suppose that condition* (21) *holds. If X_n is a solution of* (20), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \left(\|X_n\|\right)^{1/n} \tag{22}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Proposition 6.2 [16] *Suppose that condition* (21) *holds. If X_n is a solution of* (20), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{23}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Let $\{(x_n, y_n)\}$ be any solution of the system (1) such that $\lim_{n\to\infty} x_n = \bar{x}$ and $\lim_{n\to\infty} y_n = \bar{y}$. To find the error terms, one has the following system:

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\alpha x_n}{\beta x_n + e^{y_n}} - \frac{\alpha\bar{x}}{\beta\bar{x} + e^{\bar{y}}} \\ &= \frac{\alpha e^{\bar{y}}}{(\beta x_n + e^{y_n})(\beta\bar{x} + e^{\bar{y}})}(x_n - \bar{x}) - \frac{\alpha\bar{x}(e^{y_n} - e^{\bar{y}})}{(\beta x_n + e^{y_n})(\beta\bar{x} + e^{\bar{y}})(y_n - \bar{y})}(y_n - \bar{y}) \end{aligned}$$

and

$$\begin{aligned} y_{n+1} - \bar{y} &= \gamma(x_n + 1)y_n - \gamma(\bar{x} + 1)\bar{y} \\ &= \gamma y_n(x_n - \bar{x}) + (\gamma\bar{x} + \gamma)(y_n - \bar{y}). \end{aligned}$$

Let $e_n^1 = x_n - \bar{x}$ and $e_n^2 = y_n - \bar{y}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2,$$

and

$$e_{n+1}^2 = c_n e_n^1 + d_n e_n^2,$$

where

$$\begin{aligned} &a_n = \frac{\alpha e^{\bar{y}}}{(\beta x_n + e^{y_n})(\beta\bar{x} + e^{\bar{y}})}, \qquad b_n = -\frac{\alpha\bar{x}(e^{y_n} - e^{\bar{y}})}{(\beta x_n + e^{y_n})(\beta\bar{x} + e^{\bar{y}})(y_n - \bar{y})}, \\ &c_n = \gamma y_n, \qquad d_n = \gamma\bar{x} + \gamma. \end{aligned}$$

Moreover,

$$\lim_{n\to\infty} a_n = \frac{\alpha e^{\bar{y}}}{(\beta\bar{x}+e^{\bar{y}})^2}, \qquad \lim_{n\to\infty} b_n = -\frac{\alpha e^{\bar{y}}\bar{x}}{(\beta\bar{x}+e^{\bar{y}})^2},$$

$$\lim_{n\to\infty} c_n = \gamma\bar{y}, \qquad \lim_{n\to\infty} d_n = \gamma\bar{x}+\gamma.$$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e^1_{n+1} \\ e^2_{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\alpha e^{\bar{y}}}{(\beta\bar{x}+e^{\bar{y}})^2} & -\frac{\alpha\bar{x}e^{\bar{y}}}{(\beta\bar{x}+e^{\bar{y}})^2} \\ \gamma\bar{y} & \gamma\bar{x}+\gamma \end{bmatrix} \begin{bmatrix} e^1_n \\ e^2_n \end{bmatrix},$$

which is similar to the linearized system of (1) about the equilibrium point P_2.

Using Proposition 6.1, one has the following result.

Theorem 6.1 *Assume that* $\{(x_n,y_n)\}$ *is a positive solution of the system* (1) *such that* $\lim_{n\to\infty} x_n = \bar{x}$ *and* $\lim_{n\to\infty} y_n = \bar{y}$, *where*

$$(\bar{x},\bar{y}) = \left(\frac{1-\gamma}{\gamma}, \ln\left(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma}\right)\right).$$

Then the error vector $e_n = \binom{e^1_n}{e^2_n}$ *of every solution of* (1) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty} \left(\|e_n\|\right)^{\frac{1}{n}} = \left|\lambda_{1,2}F_J(\bar{x},\bar{y})\right|,$$

$$\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = \left|\lambda_{1,2}F_J(\bar{x},\bar{y})\right|,$$

where $\lambda_{1,2}F_J(\bar{x},\bar{y})$ *are the characteristic roots of Jacobian matrix* $F_J(\bar{x},\bar{y})$.

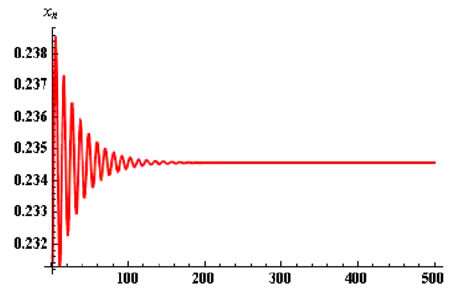
7 Examples

In order to verify our theoretical results we consider some interesting numerical examples in this section. These examples represent different types of qualitative behavior of the system (1). First two examples show that positive equilibrium point of the system (1) is locally asymptotically stable, *i.e.*, condition (4) of Theorem 3.2 is satisfied. Meanwhile, the third example shows that the positive equilibrium point of the system (1) is unstable, *i.e.*, condition (4) of Theorem 3.2 does not hold. All plots in this section are drawn with Mathematica.

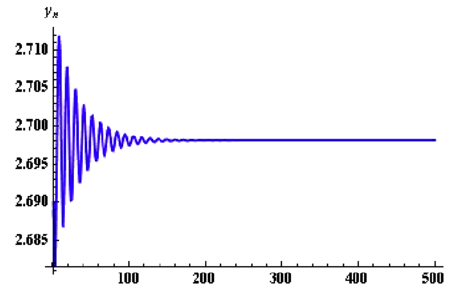
Example 7.1 Let $\alpha = 24$, $\beta = 39$, $\gamma = 0.81$. Then the system (1) can be written as

$$x_{n+1} = \frac{24x_n}{39x_n+e^{y_n}}, \qquad y_{n+1} = 0.81(x_n+1)y_n, \quad n = 0,1,\ldots, \tag{24}$$

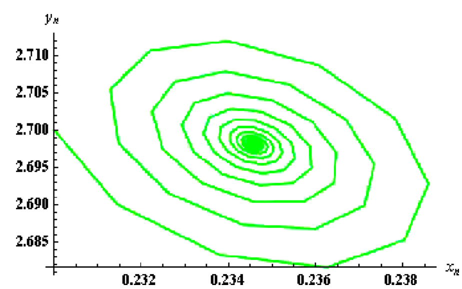
with initial conditions $x_0 = 0.23$, $y_0 = 2.7$. In Figure 1, a plot of x_n is shown in Figure 1(a), a plot of y_n is shown in Figure 1(b), and an attractor of the system (24) is shown in Figure 1(c). In this case, $\ln(\alpha+\beta-\frac{\beta}{\gamma}) = 2.69812$ and $\frac{\beta}{\gamma(\alpha+\beta)-\beta} = 3.2419$. Hence, condition (4) of Theorem 3.2 is satisfied, *i.e.*, $\ln(\alpha+\beta-\frac{\beta}{\gamma}) < \frac{\beta}{\gamma(\alpha+\beta)-\beta}$. The unique positive equilibrium point of the system (24) is given by $(\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma})) = (0.234568, 2.69812)$.



(a) Plot of x_n for the system (24)

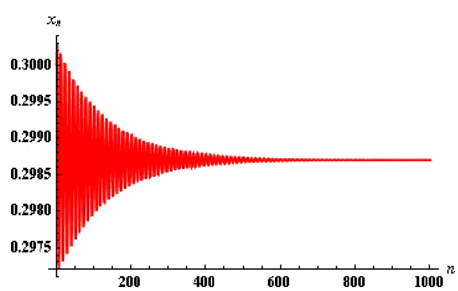


(b) Plot of y_n for the system (24)

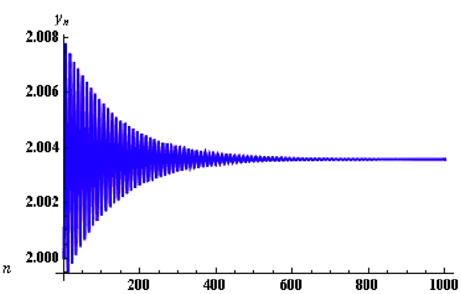


(c) An attractor of the system (24)

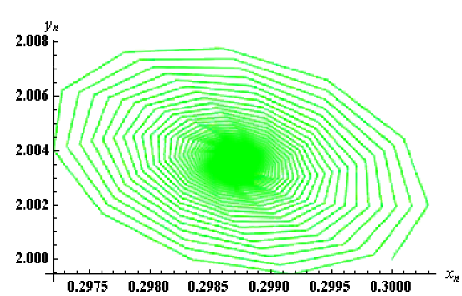
Figure 1 Plots for the system (24).



(a) Plot of x_n for the system (25)

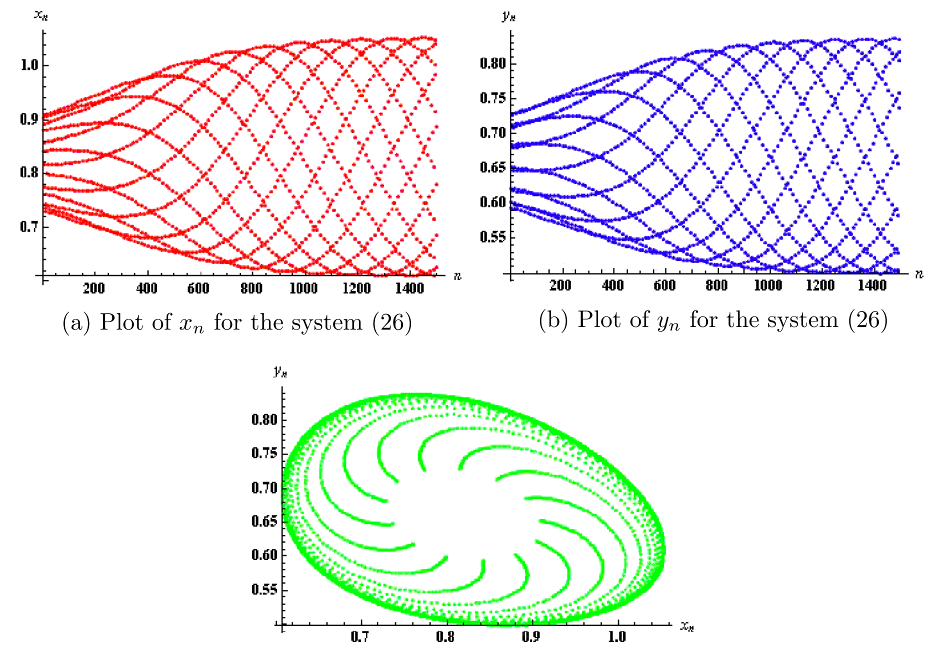


(b) Plot of y_n for the system (25)



(c) An attractor of the system (25)

Figure 2 Plots for the system (25).



(a) Plot of x_n for the system (26)

(b) Plot of y_n for the system (26)

(c) Phase portrait of the system (26)

Figure 3 Plots for the system (26).

Example 7.2 Let $\alpha = 11$, $\beta = 12$, $\gamma = 0.77$. Then the system (1) can be written as

$$x_{n+1} = \frac{11x_n}{12x_n + e^{y_n}}, \qquad y_{n+1} = 0.77(x_n + 1)y_n, \quad n = 0, 1, \dots, \tag{25}$$

with initial conditions $x_0 = 0.3$, $y_0 = 2$. In Figure 2, a plot of x_n is shown in Figure 2(a), a plot of y_n is shown in Figure 2(b), and an attractor of the system (25) is shown in Figure 2(c). In this case, $\ln(\alpha + \beta - \frac{\beta}{\gamma}) = 2.00358$ and $\frac{\beta}{\gamma(\alpha+\beta)-\beta} = 2.10158$. Hence, condition (4) of Theorem 3.2 is satisfied, *i.e.*, $\ln(\alpha + \beta - \frac{\beta}{\gamma}) < \frac{\beta}{\gamma(\alpha+\beta)-\beta}$. The unique positive equilibrium point of the system (25) is given by $(\frac{1-\gamma}{\gamma}, \ln(\frac{\alpha\gamma-\beta+\beta\gamma}{\gamma})) = (0.298701, 2.00358)$.

Example 7.3 Let $\alpha = 2.5$, $\beta = 0.69$, $\gamma = 0.55$. Then the system (1) can be written as

$$x_{n+1} = \frac{2.5x_n}{0.69x_n + e^{y_n}}, \qquad y_{n+1} = 0.55(x_n + 1)y_n, \quad n = 0, 1, \dots, \tag{26}$$

with initial conditions $x_0 = 0.8$, $y_0 = 0.6$. In Figure 3, a plot of x_n is shown in Figure 3(a), a plot of y_n is shown in Figure 3(b), and a phase portrait of the system (26) is shown in Figure 3(c). In this case, $\ln(\alpha + \beta - \frac{\beta}{\gamma}) = 0.660342$ and $\frac{\beta}{\gamma(\alpha+\beta)-\beta} = 0.648192$. Hence, condition (4) of Theorem 3.2 does not hold, *i.e.*, $\ln(\alpha + \beta - \frac{\beta}{\gamma}) > \frac{\beta}{\gamma(\alpha+\beta)-\beta}$.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author carried out the proof of the main results and approved the final manuscript.

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