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Homoclinic solutions for a class of nonlinear difference systems with classical (ϕ_1, ϕ_2)-Laplacian

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Abstract
In this paper, we consider the existence of homoclinic solutions for a class of nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian. First, we improve some inequalities in known literature. Then, by using the variational method, some new existence results are obtained. Finally, some examples are given to verify our results.

MSC: 37J45; 58E50; 34C25

Keywords: difference systems; classical (ϕ_1, ϕ_2)-Laplacian; homoclinic solutions; variational method

1 Introduction and main results

Let $\mathbb{R}$ denote the real numbers and $\mathbb{Z}$ the integers. Given $a < b$ in $\mathbb{Z}$. Let $\mathbb{Z}[a,b] = \{a, a+1, \ldots, b\}$. Let $T > 1$ and N be fixed positive integers.

In this paper, we investigate the existence of homoclinic solutions for the following nonlinear difference systems involving classical (ϕ_1, ϕ_2)-Laplacian:

$$\begin{cases} \Delta\phi_1(\Delta u_1(t-1)) + \nabla_{u_1} V(t, u_1(t), u_2(t)) = f_1(t), \\ \Delta\phi_2(\Delta u_2(t-1)) + \nabla_{u_2} V(t, u_1(t), u_2(t)) = f_2(t), \end{cases} \tag{1.1}$$

where $t \in \mathbb{Z}$, $u_m(t) \in \mathbb{R}^N$, $m = 1, 2$, $V(t, x_1, x_2) = -K(t, x_1, x_2) + W(t, x_1, x_2)$, $K, W : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and ϕ_m, $m = 1, 2$, satisfy the following condition:

($\mathcal{A}0$) ϕ_m is a homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ such that $\phi_m(0) = 0$, $\phi_m = \nabla\Phi_m$, with $\Phi_m \in C^1(\mathbb{R}^N, [0, +\infty])$ strictly convex and $\Phi_m(0) = 0$, $m = 1, 2$.

Remark 1.1 Assumption ($\mathcal{A}0$) is given in [1], which is used to characterize the classical homeomorphism. If, furthermore, $\Phi_m : \mathbb{R}^N \to \mathbb{R}$ is coercive (*i.e.*, $\Phi_m(x) \to +\infty$ as $|x| \to \infty$), there exists $\delta_m > 0$ such that

$$\Phi_m(x) \geq \delta_m\big(|x| - 1\big), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $\delta_m = \min_{|x|=1} \Phi_m(x)$, $m = 1, 2$ (see [1]).

We call $u = (u_1, u_2)$ a nontrivial homoclinic solution of system (1.1) if u satisfies system (1.1), $u \neq 0$ and $u(t) \to 0$ as $t \to \infty$.

It is well known that the variational method has become an important tool to study the existence and multiplicity of solutions for various difference systems. Lots of contributions have been obtained (for example, see [1–20]). It is remarkable that, to the best of our knowledge, few people investigated system (1.1). Recently, in [1] and [2], by using the variational approach, J Mawhin investigated the following second order nonlinear difference systems with ϕ-Laplacian:

$$\Delta\phi\bigl[\Delta u(n-1)\bigr] = \nabla_u F\bigl[n, u(n)\bigr] + h(n) \quad (n \in \mathbb{Z}), \tag{1.3}$$

where $\phi = \nabla\Phi$, Φ strictly convex, is a homeomorphism of $\mathbb{R}^N$ onto the ball $B_a \subset \mathbb{R}^N$ or of B_a onto $\mathbb{R}^N$. By using the variational approach, under different conditions, the author obtained that system (1.3) has at least one or $N+1$ geometrically distinct T-periodic solutions. It is interesting that J Mawhin considered three kinds of ϕ: (1) $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a classical homeomorphism, for example, $\phi(x) = |x|^{p-1}x$ for some $p > 1$ and all $x \in \mathbb{R}^N$; (2) $\phi : \mathbb{R}^N \to B_a$ ($a < +\infty$) is a bounded homeomorphism, for example, $\phi(x) = \frac{x}{\sqrt{1+|x|^2}} \in B_1$ for all $x \in \mathbb{R}^N$; (3) $\phi : B_a \subset \mathbb{R}^N \to \mathbb{R}^N$ is a singular homeomorphism, for example, $\phi(x) = \frac{x}{\sqrt{1-|x|^2}}$ for all $x \in B_1$. Recently, in [17], we generalized some results in [2] for classical homeomorphism and bounded homeomorphism to system (1.1), which seem to be the first results for system (1.1).

In 2011, He and Chen [16] investigated the existence of homoclinic solutions for the following discrete p-Laplacian systems:

$$\Delta\bigl(\bigl|\Delta u(t-1)\bigr|^{p-2}\Delta u(t-1)\bigr) = \nabla F\bigl(t, u(t)\bigr) + f(t), \quad t \in \mathbb{Z}, u \in \mathbb{R}^N, \tag{1.4}$$

where $p > 1$. They obtained homoclinic orbits as the limit of the subharmonics for system (1.4).

In this paper, motivated by [1, 2, 15, 16] and [17], we first improve some inequalities in [16] and then investigate the existence of homoclinic solutions for system (1.1) with classical homeomorphism. Next we make the following assumption:

($\mathcal{A}1$) *Let $p > 1$. Assume that there exist positive constants d_1, d_2, d_3, d_4 such that*

$$d_1|x|^p \le \Phi_1(x) \le d_3|x|^p, \qquad d_2|y|^p \le \Phi_2(y) \le d_4|y|^p, \quad \forall x, y \in \mathbb{R}^N$$

and

$$\bigl(\phi_1(x), x\bigr) \le p\Phi_1(x), \qquad \bigl(\phi_2(y), y\bigr) \le p\Phi_2(y), \quad \forall x, y \in \mathbb{R}^N.$$

For every $s \in \mathbb{N}$, define

$$l^s = \left\{ g : \mathbb{Z} \to \mathbb{R}^N, \sum_{t=-\infty}^{+\infty} \bigl|g(t)\bigr|^s < \infty \right\}$$

with the norm

$$\|g\|_{l^s} = \left(\sum_{t=-\infty}^{+\infty} |g(t)|^s \right)^{1/s}.$$

Let $p' > 1$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$ and

$$C_* = \left((2T)^{-p'/p} + \min\left\{ \frac{(T+1)^{p'+1} + T^{p'+1} - 2}{(2T)^{p'}(p'+1)}, \frac{T}{2^{p'/p}} \right\} \right)^{1/p'}.$$

Next, we present our main results.

Theorem 1.1 *Assume that* $(\mathcal{A}1)$ *holds*, $f_i \neq 0$, $i = 1,2$, W *and* K *satisfy the following conditions*:

($\mathcal{V}$) $V(t,x_1,x_2) = -K(t,x_1,x_2) + W(t,x_1,x_2)$, *where* $K, W : \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $K(t,x_1,x_2)$ *and* $W(t,x_1,x_2)$ *are* T*-periodic and for every* $t \in \mathbb{Z}$, $K, W \in C^1(\mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N, \mathbb{R})$;

(H1) *there exist* $\gamma \in (1,p)$ *and* $a_1, a_2 > 0$ *such that*

$$K(t,x_1,x_2) \geq a_1|x_1|^\gamma + a_2|x_2|^\gamma \quad \text{for all } (t,x_1,x_2) \in \mathbb{Z}[0,T-1] \times \mathbb{R}^N \times \mathbb{R}^N;$$

(H2) $K(t,0,0) \equiv 0$ *and*

$$\begin{aligned} &\big(x_1, \nabla_{x_1} K(t,x_1,x_2)\big) + \big(x_2, \nabla_{x_2} K(t,x_1,x_2)\big) \\ &\quad \leq pK(t,x_1,x_2) \quad \text{for all } (t,x_1,x_2) \in \mathbb{Z}[0,T-1] \times \mathbb{R}^N \times \mathbb{R}^N; \end{aligned}$$

(H3) (i) *there exist* $r \in (0,1]$, $0 < b_1 < a_1 C_*^{\gamma-p}$, *and* $0 < b_2 < a_2 C_*^{\gamma-p}$ *such that*

$$\begin{aligned} &W(t,x_1,x_2) \leq b_1|x_1|^p + b_2|x_2|^p, \\ &\quad \forall t \in \mathbb{Z}[0,T-1], |x_1| \leq rC_*, |x_2| \leq rC_*; \end{aligned} \tag{1.5}$$

(ii) *there exist* $r > 1$, $0 < b_1 < a_1(C_* r)^{\gamma-p}$, *and* $0 < b_2 < a_2(C_* r)^{\gamma-p}$ *such that* (1.5) *holds*;

(H4)

$$\lim_{|x_1|+|x_2|\to+\infty} \frac{W(t,x_1,x_2)}{|x_1|^p + |x_2|^p} > d_3 + d_4 + 2^{p-1}A_0 \quad \text{for all } t \in \mathbb{Z}[0,T-1],$$

where

$$A_0 = \max_{|x_1|\leq 1, |x_2|\leq 1, t\in\mathbb{Z}[0,T-1]} K(t,x_1,x_2);$$

(H5) *there exist positive constants* ξ, η_1, η_2 *and* $\nu \in [0, \gamma-1)$ *such that*

$$\begin{aligned} 0 &\leq \left(p + \frac{1}{\xi + \eta_1|x_1|^\nu + \eta_2|x_2|^\nu} \right) W(t,x_1,x_2) \\ &\leq \big(\nabla_{x_1} W(t,x_1,x_2), x_1\big) + \big(\nabla_{x_2} W(t,x_1,x_2), x_2\big) \end{aligned}$$

for all $(t,x_1,x_2) \in \mathbb{Z}[0,T-1] \times \mathbb{R}^N \times \mathbb{R}^N$;

(H6) $f_1, f_2 \in l^{p'} \cap l^{\frac{p-\nu}{p-\nu-1}}$ *and*

(i) *when* $r \in (0,1]$,

$$\max\{\|f_1\|_{l^{p'}}, \|f_2\|_{l^{p'}}\} < \frac{1}{2^{p-1}} \min\{d_1, d_2, a_1 C_*^{\gamma-p} - b_1, a_2 C_*^{\gamma-p} - b_2\} r^{p-1};$$

(ii) *when* $r \in (1,+\infty)$,

$$\max\{\|f_1\|_{l^{p'}}, \|f_2\|_{l^{p'}}\} < \frac{1}{2^{p-1}} \min\{d_1, d_2, a_1 (C_* r)^{\gamma-p} - b_1, a_2 (C_* r)^{\gamma-p} - b_2\} r^{p-1}.$$

Then system (1.1) *possesses a nontrivial homoclinic solution.*

Theorem 1.2 *Assume that* ($\mathcal{A}1$) *holds*, $f_i \neq 0$, $i = 1, 2$, W *and* K *satisfy* ($\mathcal{V}$), (H1)-(H5) *and the following conditions*:

(H6)′ $f_1, f_2 \in l^1$ *and*

(i) *when* $r \in (0,1]$,

$$\max\{\|f_1\|_{l^1}, \|f_2\|_{l^1}\} < \frac{1}{2^{p-1} C_*} \min\{d_1, d_2, a_1 C_*^{\gamma-p} - b_1, a_2 C_*^{\gamma-p} - b_2\} r^{p-1};$$

(ii) *when* $r \in (1,+\infty)$,

$$\max\{\|f_1\|_{l^1}, \|f_2\|_{l^1}\} < \frac{1}{2^{p-1} C_*} \min\{d_1, d_2, a_1 (C_* r)^{\gamma-p} - b_1, a_2 (C_* r)^{\gamma-p} - b_2\} r^{p-1}.$$

Then system (1.1) *possesses a nontrivial homoclinic solution.*

Theorem 1.3 *Assume that* ($\mathcal{A}1$) *holds*, $f_i \neq 0$, $i = 1, 2$, W *and* K *satisfy* ($\mathcal{V}$), (H2), (H4), (H5) *and the following conditions*:

(H1)′ *there exist* $a_1, a_2 > 0$ *such that*

$$K(t, x_1, x_2) \geq a_1 |x_1|^p + a_2 |x_2|^p \quad \text{for all } (t, x_1, x_2) \in \mathbb{Z}[0, T-1] \times \mathbb{R}^N \times \mathbb{R}^N;$$

(H3)′ *there exist* $r > 0$ *and* $0 < b_1 < a_1$, $0 < b_2 < a_2$ *such that*

$$W(t, x_1, x_2) \leq b_1 |x_1|^p + b_2 |x_2|^p, \quad \forall |x_1| \leq r C_*, |x_2| \leq r C_*;$$

(H6)″ $f_1, f_2 \in l^{p'} \cap l^{\frac{p-\nu}{p-\nu-1}}$ *and*

$$\max\{\|f_1\|_{l^{p'}}, \|f_2\|_{l^{p'}}\} < \frac{1}{2^{p-1}} \min\{d_1, d_2, a_1 - b_1, a_2 - b_2\} r^{p-1}.$$

Then system (1.1) *possesses a nontrivial homoclinic solution.*

Theorem 1.4 *Assume that* ($\mathcal{A}1$) *holds*, $f_i \neq 0$, $i = 1,2$, W *and* K *satisfy* ($\mathcal{V}$), (H1)$'$, (H2), (H3)$'$, (H4), (H5) *and the following condition*:

(H6)$'''$ $f_1, f_2 \in l^1$ *and*

$$\max\{\|f_1\|_{l^1}, \|f_2\|_{l^1}\} < \frac{1}{2^{p-1}C_*}\min\{d_1, d_2, a_1 - b_1, a_2 - b_2\}r^{p-1}.$$

Then system (1.1) *possesses a nontrivial homoclinic solution*.

Remark 1.2 Theorem 1.3 and Theorem 1.4 show that f_1, f_2 can be large when r is large.

2 Preliminaries

Similar to [15] and [16], we will obtain the homoclinic orbit of system (1.1) as a limit of solutions of a sequence of difference systems:

$$\begin{cases} \Delta\phi_1(\Delta u_1(t-1)) + \nabla_{u_1}V(t, u_1(t), u_2(t)) = f_{1,k}(t), \\ \Delta\phi_2(\Delta u_2(t-1)) + \nabla_{u_2}V(t, u_1(t), u_2(t)) = f_{2,k}(t), \end{cases} \tag{2.1}$$

where $f_{m,k} : \mathbb{Z} \to \mathbb{R}^N$ is a $2kT$-periodic extension of restriction of f_m to the interval $\mathbb{Z}[-kT, kT-1]$, $k \in \mathbb{N}$, $m = 1,2$.

Next, we present some basic notations. We use $|\cdot|$ to denote the usual Euclidean norm in $\mathbb{R}^N$. Define

$$\begin{aligned} \mathcal{V} = \{&u = (u_1, u_2)^\tau = \{u(t)\} | u(t) = (u_1(t), u_2(t))^\tau \in \mathbb{R}^{2N}, \\ &u_m = \{u_m(t)\}, u_m(t) \in \mathbb{R}^N, m = 1,2, t \in \mathbb{Z}\}. \end{aligned}$$

$\mathcal{H}$ is defined as a subspace of $\mathcal{V}$ by

$$\mathcal{H}_k = \{u = \{u(t)\} \in \mathcal{V} | u(t + 2kT) = u(t), t \in \mathbb{Z}\}.$$

Define

$$\mathcal{H}_{m,k} = \{u_m = \{u_m(t)\} | u_m(t + 2kT) = u_m(t), u_m(t) \in \mathbb{R}^N, t \in \mathbb{Z}\}, \quad m = 1,2.$$

Then $\mathcal{H}_k = \mathcal{H}_{1,k} \times \mathcal{H}_{2,k}$. For $u_m \in \mathcal{H}_{m,k}$, set

$$\|u_m\|_{s,k} = \left(\sum_{t=-kT}^{kT-1} |u_m(t)|^s\right)^{1/s}, \quad m = 1,2, s > 1.$$

Moreover, l^∞_{2kT} denote the space of all bounded real functions on $\mathbb{Z}[-kT, kT-1]$ endowed with the norm

$$\|u_m\|_{l^\infty_{2kT}} = \max_{t \in \mathbb{Z}[-kT, kT-1]} |u_m(t)|, \quad m = 1,2.$$

For $1 < p < +\infty$, on $\mathcal{H}_{m,k}$, we define

$$\|u_m\|_{\mathcal{H}_{m,k}} = \left(\sum_{t=-kT}^{kT-1} |\Delta u_m(t)|^p + \sum_{t=-kT}^{kT-1} |u_m(t)|^p\right)^{1/p}, \quad m = 1,2.$$

For $u = (u_1, u_2)^\tau \in \mathcal{H}_k$, define

$$\|u\|_{\mathcal{H}_k} = \|u_1\|_{\mathcal{H}_{1,k}} + \|u_2\|_{\mathcal{H}_{2,k}}.$$

Then $(\mathcal{H}_k, \|u\|_{\mathcal{H}_k})$, $(\mathcal{H}_{1,k}, \|u\|_{\mathcal{H}_{1,k}})$ and $(\mathcal{H}_{2,k}, \|u\|_{\mathcal{H}_{2,k}})$ are reflexive Banach spaces.

Lemma 2.1 *Let $a, b \in \mathbb{Z}$, $a \geq 1$, $b \geq 0$, $\varrho > 1$, $u_m \in \mathcal{H}_{m,k}$, $m = 1, 2$. Then, for every $t \in \mathbb{Z}$,*

$$\begin{aligned} |u_m(t)| \leq{}& (a+b+1)^{-1/\varrho}\left(\sum_{s=t-a}^{t+b}|u_m(s)|^\varrho\right)^{1/\varrho} \\ &+ \min\left\{\frac{[(a+1)^{p'+1}+(b+1)^{p'+1}-2]^{1/p'}}{(a+b+1)^{p'}(p'+1)^{1/p'}}, \frac{\max\{a,b\}}{(a+b+1)^{1/p}}\right\} \\ &\cdot\left(\sum_{s=t-a}^{t+b}|\Delta u_m(s)|^p\right)^{1/p}, \end{aligned} \tag{2.2}$$

where $m = 1, 2$.

Proof Fix $t \in \mathbb{Z}$. For every $\tau \in \mathbb{Z}[t-a, t-1]$, we have

$$u_m(t) = u_m(\tau) + \sum_{s=\tau}^{t-1}\Delta u_m(s) \tag{2.3}$$

and for every $\tau \in \mathbb{Z}[t, t+b]$,

$$u_m(t) = u_m(\tau) - \sum_{s=t}^{\tau-1}\Delta u_m(s). \tag{2.4}$$

Summing (2.3) over $\mathbb{Z}[t-a, t-1]$ and (2.4) over $\mathbb{Z}[t, t+b]$, we have

$$\begin{aligned} au_m(t) &= \sum_{\tau=t-a}^{t-1}u_m(\tau) + \sum_{\tau=t-a}^{t-1}\sum_{s=\tau}^{t-1}\Delta u_m(s) \\ &= \sum_{\tau=t-a}^{t-1}u_m(\tau) + \sum_{s=t-a}^{t-1}(s-t+a+1)\Delta u_m(s) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} (b+1)u_m(t) &= \sum_{\tau=t}^{t+b}u_m(\tau) - \sum_{\tau=t}^{t+b}\sum_{s=t}^{\tau-1}\Delta u_m(s) \\ &= \sum_{\tau=t}^{t+b}u_m(\tau) - \sum_{s=t}^{t+b-1}(t+b-s)\Delta u_m(s). \end{aligned} \tag{2.6}$$

Set

$$\phi(s) = \begin{cases} s-t+a+1, & t-a \leq s \leq t-1, \\ t+b-s, & t \leq s \leq t+b. \end{cases}$$

Combining (2.5) with (2.6) and using Hölder's inequality, we obtain

$$\begin{aligned}
&(a+b+1)\left|u_m(t)\right| \\
&= \left|\sum_{\tau=t-a}^{t+b} u_m(\tau) + \sum_{s=t-a}^{t-1}(s-t+a+1)\Delta u_m(s) - \sum_{s=t}^{t+b-1}(t+b-s)\Delta u_m(s)\right| \\
&\le \sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right| + \sum_{s=t-a}^{t-1}(s-t+a+1)\left|\Delta u_m(s)\right| + \sum_{s=t}^{t+b-1}(t+b-s)\left|\Delta u_m(s)\right| \\
&= \sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right| + \sum_{s=t-a}^{t+b-1}\phi(s)\left|\Delta u_m(s)\right| = \sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right| + \sum_{s=t-a}^{t+b}\phi(s)\left|\Delta u_m(s)\right| \\
&\le (a+b+1)^{(\varrho-1)/\varrho}\left(\sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right|^{\varrho}\right)^{1/\varrho} + \left(\sum_{s=t-a}^{t+b}\left[\phi(s)\right]^{p'}\right)^{1/p'}\left(\sum_{s=t-a}^{t+b}\left|\Delta u_m(s)\right|^{p}\right)^{1/p} \\
&= (a+b+1)^{(\varrho-1)/\varrho}\left(\sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right|^{\varrho}\right)^{1/\varrho} \\
&\quad + \left(\sum_{s=t-a}^{t-1}(s-t+a+1)^{p'} + \sum_{s=t}^{t+b}(t+b-s)^{p'}\right)^{1/p'}\left(\sum_{s=t-a}^{t+b}\left|\Delta u_m(s)\right|^{q}\right)^{1/q}.
\end{aligned} \tag{2.7}$$

Since

$$\begin{aligned}
&\sum_{s=t-a}^{t-1}(s-t+a+1)^{p'} = \sum_{s=1}^{a} s^{p'} < \frac{(a+1)^{p'+1}-1}{p'+1}, \\
&\sum_{s=t}^{t+b}(t+b-s)^{p'} = \sum_{k=1}^{b} k^{p'} < \frac{(b+1)^{p'+1}-1}{p'+1}
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
&\sum_{s=t-a}^{t-1}(s-t+a+1)^{p'} + \sum_{s=t}^{t+b}(t+b-s)^{p'} \\
&\quad \le \sum_{s=t-a}^{t-1} a^{p'} + \sum_{s=t}^{t+b} b^{p'} \le \sum_{s=t-a}^{t+b}\max\left\{a^{p'}, b^{p'}\right\} = \max\left\{a^{p'}, b^{p'}\right\}(b+a+1).
\end{aligned} \tag{2.9}$$

Equation (2.7) implies that

$$\begin{aligned}
&(a+b+1)\left|u_m(t)\right| \\
&\le (a+b+1)^{(\varrho-1)/\varrho}\left(\sum_{\tau=t-a}^{t+b}\left|u_m(\tau)\right|^{\varrho}\right)^{1/\varrho} \\
&\quad + \left(\min\left\{\frac{(a+1)^{p'+1}+(b+1)^{p'+1}-2}{p'+1}, \max\left\{a^{p'}, b^{p'}\right\}(b+a+1)\right\}\right)^{1/p'} \\
&\quad \cdot \left(\sum_{s=t-a}^{t+b}\left|\Delta u_m(s)\right|^{p}\right)^{1/p},
\end{aligned}$$

which implies that (2.2) holds. Thus the proof is complete. □

Corollary 2.1 *Let $u_m \in \mathcal{H}_{m,k}$, $m = 1, 2$. Then*

$$\|u_m\|_{l^\infty_{2kT}} \le (2T)^{-1/\varrho}\left(\sum_{s=-kT}^{kT-1}\left|u_m(s)\right|^\varrho\right)^{1/\varrho} + \min\left\{\frac{[(T+1)^{p'+1}+T^{p'+1}-2]^{1/p'}}{(2T)^{p'}(p'+1)^{1/p'}}, \frac{T^{1/p'}}{2^{1/p}}\right\}\left(\sum_{s=-kT}^{kT-1}\left|\Delta u_m(s)\right|^p\right)^{1/p}, \quad (2.10)$$

where $m = 1, 2$.

Proof Obviously, there exists $t^* \in \mathbb{Z}[-kT, kT-1]$ such that

$$\left|u_m(t^*)\right| = \|u_m\|_{l^\infty_{2kT}} = \max_{t\in\mathbb{Z}[-kT,kT-1]}\left|u_m(s)\right|.$$

In Lemma 2.1, let $a = T$ and $b = T - 1$,

$$\begin{aligned}
\left|u_m(t^*)\right| &\le (2T)^{-1/\varrho}\left(\sum_{s=t^*-T}^{t^*+T-1}\left|u_m(s)\right|^\varrho\right)^{1/\varrho} \\
&\quad + \min\left\{\frac{[(T+1)^{p'+1}+T^{p'+1}-2]^{1/p'}}{(2T)^{p'}(p'+1)^{1/p'}}, \frac{T}{(2T)^{1/p}}\right\}\left(\sum_{s=t^*-T}^{t^*+T-1}\left|\Delta u_m(s)\right|^p\right)^{1/p} \\
&\le (2T)^{-1/\varrho}\left(\sum_{s=t^*-kT}^{t^*+kT-1}\left|u_m(s)\right|^\varrho\right)^{1/\varrho} \\
&\quad + \min\left\{\frac{[(T+1)^{p'+1}+T^{p'+1}-2]^{1/p'}}{(2T)^{p'}(p'+1)^{1/p'}}, \frac{T}{(2T)^{1/p}}\right\}\left(\sum_{s=t^*-kT}^{t^*+kT-1}\left|\Delta u_m(s)\right|^p\right)^{1/p} \\
&= (2T)^{-1/\varrho}\left(\sum_{s=-kT}^{kT-1}\left|u_m(s)\right|^\varrho\right)^{1/\varrho} \\
&\quad + \min\left\{\frac{[(T+1)^{p'+1}+T^{p'+1}-2]^{1/p'}}{(2T)^{p'}(p'+1)^{1/p'}}, \frac{T^{1/p'}}{2^{1/p}}\right\}\left(\sum_{s=-kT}^{kT-1}\left|\Delta u_m(s)\right|^p\right)^{1/p}.
\end{aligned}$$

The proof is complete. □

Corollary 2.2 *Let $u_m \in \mathcal{H}_{m,k}$, $m = 1, 2$. Then*

$$\|u_m\|_{l^\infty_{2kT}} \le \left((2T)^{-p'/p} + \min\left\{\frac{(T+1)^{p'+1}+T^{p'+1}-2}{(2T)^{p'^2}(p'+1)^{1/p'}}, \frac{T}{2^{p'/p}}\right\}\right)^{1/p'} \cdot \left(\sum_{s=-kT}^{kT-1}\left|u_m(s)\right|^p + \sum_{s=-kT}^{kT-1}\left|\Delta u_m(s)\right|^p\right)^{1/p}, \quad (2.11)$$

where $m = 1, 2$.

Proof In Corollary 2.1, let $\varrho = p$ and then use Hölder's inequality. Then the proof is completed easily. □

Remark 2.1 As $a \geq 1$, Lemma 2.1, Corollary 2.1, and Corollary 2.2 improve Lemma 3.1, Corollary 3.1, and Corollary 3.2 in [16], respectively.

By Lemma 2.3 and Lemma 2.4 in [17], we have the following two lemmas.

Lemma 2.2 (see [17]) *For any $u = (u_1, u_2), v = (v_1, v_2) \in \mathcal{H}_k$, the following two equalities hold*:

$$-\sum_{t=-kT}^{kT-1} \bigl(\Delta\phi_1\bigl(\Delta u_1(t-1)\bigr), v_1(t)\bigr) = \sum_{t=-kT}^{kT-1} \bigl(\Delta\phi_1\bigl(\Delta u_1(t)\bigr), \Delta v_1(t)\bigr), \tag{2.12}$$

$$-\sum_{t=-kT}^{kT-1} \bigl(\Delta\phi_2\bigl(\Delta u_2(t-1)\bigr), v_2(t)\bigr) = \sum_{t=-kT}^{kT-1} \bigl(\Delta\phi_2\bigl(\Delta u_2(t)\bigr), \Delta v_2(t)\bigr). \tag{2.13}$$

Lemma 2.3 (see [17]) *Let $L : \mathbb{Z}[-kT, kT-1] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}, (t, x_1, x_2, y_1, y_2) \to L(t, x_1, x_2, y_1, y_2)$ and assume that L is continuously differential in (x_1, x_2, y_1, y_2) for all $t \in \mathbb{Z}[-kT, kT-1]$. Then the function $\varphi_k : \mathcal{H}_k \to \mathbb{R}$ defined by*

$$\varphi_k(u) = \varphi_k(u_1, u_2) = \sum_{t=-kT}^{kT-1} L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr)$$

is continuously differentiable on $\mathcal{H}_k$ and

$$\begin{aligned}
\bigl\langle\varphi_k'(u), v\bigr\rangle &= \bigl\langle\varphi_k'(u_1, u_2), (v_1, v_2)\bigr\rangle \\
&= \sum_{t=-kT}^{kT-1} \bigl[\bigl(D_{x_1}L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), v_1(t)\bigr) \\
&\quad + \bigl(D_{y_1}L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), \Delta v_1(t)\bigr) \\
&\quad + \bigl(D_{x_2}L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), v_2(t)\bigr) \\
&\quad + \bigl(D_{y_2}L\bigl(t, u_1(t), u_2(t), \Delta u_1(t), \Delta u_2(t)\bigr), \Delta v_2(t)\bigr)\bigr],
\end{aligned}$$

where $u, v \in \mathcal{H}_k$.

Let

$$L(t, x_1, x_2, y_1, y_2) = \Phi_1(y_1) + \Phi_2(y_2) + K(t, x_1, x_2) - W(t, x_1, x_2) + \bigl(f_{1,k}(t), x_1\bigr) + \bigl(f_{2,k}(t), x_2\bigr)$$

and define $\eta_k : \mathcal{H}_k \to [0, +\infty)$ by

$$\eta_k(u) = \eta_k(u_1, u_2) = \sum_{t=-kT}^{kT-1} \bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + K\bigl(t, u_1(t), u_2(t)\bigr)\bigr].$$

Then

$$\begin{aligned}
\varphi_k(u) &= \varphi_k(u_1, u_2) \\
&= \sum_{t=-kT}^{kT-1} \bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + K\bigl(t, u_1(t), u_2(t)\bigr)
\end{aligned}$$

$$
\begin{aligned}
&\quad - W\big(t, u_1(t), u_2(t)\big) + \big(f_{1,k}(t), u_1(t)\big) + \big(f_{2,k}(t), u_2(t)\big)\big] \\
&= \eta_k(u) + \sum_{t=-kT}^{kT-1} \big[-W\big(t, u_1(t), u_2(t)\big) + \big(f_{1,k}(t), u_1(t)\big) + \big(f_{2,k}(t), u_2(t)\big)\big]. \qquad (2.14)
\end{aligned}
$$

It follows from $(\mathcal{A}0)$, $(\mathcal{V})$ and Lemma 2.3 that

$$
\begin{aligned}
\big\langle \varphi_k'(u), v\big\rangle &= \big\langle \varphi_k'(u_1, u_2), (v_1, v_2)\big\rangle \\
&= \sum_{t=-kT}^{kT-1} \big[\big(\phi_1\big(\Delta u_1(t)\big), \Delta v_1(t)\big) + \big(\phi_2\big(\Delta u_2(t)\big), \Delta v_2(t)\big) \\
&\quad + \big(\nabla_{u_1} K\big(t, u_1(t), u_2(t)\big), v_1(t)\big) + \big(\nabla_{u_2} K\big(t, u_1(t), u_2(t)\big), v_2(t)\big) \\
&\quad - \big(\nabla_{u_1} W\big(t, u_1(t), u_2(t)\big), v_1(t)\big) - \big(\nabla_{u_2} W\big(t, u_1(t), u_2(t)\big), v_2(t)\big) \\
&\quad + \big(f_{1,k}(t), v_1(t)\big) + \big(f_{2,k}(t), v_2(t)\big)\big], \quad \forall u, v \in \mathcal{H}_k. \qquad (2.15)
\end{aligned}
$$

By Lemma 2.2, it is easy to see that critical points of φ_k in $\mathcal{H}_k$ are $2kT$-periodic solutions of system (2.1).

We shall use one linking method in [21] to obtain the critical points of φ (the details can be seen in [21]). Let $(E, \|\cdot\|)$ be a Banach space. Define a continuous map $\Gamma : [0,1] \times E \to E$ by $\Gamma(t,x) = \Gamma(t)x$, where $\Gamma(t)$ satisfies the following conditions:

(1) $\Gamma(0) = I$, the identity map.
(2) For each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of E onto E and $\Gamma^{-1}(t) \in C(E \times [0,1), E)$.
(3) $\Gamma(1)E$ is a single point in E and $\Gamma(t)A$ converges uniformly to $\Gamma(1)E$ as $t \to 1$ for each bounded set $A \subset E$.
(4) For each $t_0 \in [0,1)$ and each bounded set $A \subset E$,

$$
\sup_{\substack{0 \le t \le t_0 \\ u \in A}} \big\{\big\|\Gamma(t)u\big\| + \big\|\Gamma^{-1}(t)u\big\|\big\} < \infty.
$$

Let Φ be the set of all continuous maps Γ as defined above.

Definition 2.1 (see [21], Definition 3.2) We say that A links B[hm] if A and B are subsets of E such that $A \cap B = \emptyset$, and for each $\Gamma \in \Phi$, there is $t' \in (0,1]$ such that $\Gamma(t')A \cap B \neq \emptyset$.

Example 1 (see [21], p.21) Let B be an open set in E, and let A consist of two points e_1, e_2 with $e_1 \in B$ and $e_2 \notin \bar{B}$. Then A links ∂B[hm].

We use the following theorem to prove our main results.

Theorem 2.1 (see [21], Theorem 3.4 and Theorem 2.12) *Let E be a Banach space, $\varphi \in C^1(E, \mathbb{R})$ and A and B be two subsets of E such that A links B*[hm]. *Assume that*

$$
\sup_A \varphi \le \inf_B \varphi
$$

and

$$
c := \inf_{\Gamma \in \Phi} \sup_{\substack{s \in [0,1] \\ u \in A}} \varphi\big(\Gamma(s)u\big) < \infty.
$$

Let $\psi(t)$ be a positive, nonincreasing, locally Lipschitz continuous function on $[0,\infty)$ satisfying $\int_0^\infty \psi(r)\,dr = \infty$. Then there exists a sequence $\{u_n\} \subset E$ such that $\varphi(u_n) \to c$ and $\varphi'(u_n)/\psi(\|u_n\|) \to 0$, as $n \to \infty$. Moreover, if $c = \sup_A \varphi$, then there is a sequence $\{u_n\} \subset E$ satisfying $\varphi(u_n) \to c$, $\varphi'(u_n) \to 0$, and $d(u_n, B) \to 0$, as $n \to \infty$.

Remark 2.2 Since A links B, by Definition 2.1, it is easy to know that $c \geq \inf_B \varphi$. By [21], if we let $\psi(r) = \frac{1}{1+r}$, the sequence $\{u_n\}$ is the Cerami sequence that is $\{u_n\}$ satisfying

$$\varphi(u_n) \to c, \qquad \bigl(1 + \|u_n\|\bigr)\bigl\|\varphi'(u_n)\bigr\| \to 0, \quad \text{as } n \to \infty.$$

3 Proofs

Lemma 3.1 *Suppose that* (H2) *holds. Then*

$$K(t,x_1,x_2) \leq 2^{p-1} K\left(t, \frac{x_1}{|x|}, \frac{x_2}{|x|}\right)\bigl(|x_1|^p + |x_2|^p\bigr) \quad \textit{for all } t \in \mathbb{Z}[0,T-1], |x| \geq 1;$$

$$K(t,x_1,x_2) \geq \frac{1}{2} K\left(t, \frac{x_1}{|x|}, \frac{x_2}{|x|}\right)\bigl(|x_1|^p + |x_2|^p\bigr) \quad \textit{for all } t \in \mathbb{Z}[0,T-1], |x| \leq 1.$$

Proof Define the function $\xi \in (0,+\infty) \to K(t,\xi^{-1}x_1,\xi^{-1}x_2)(\xi^p + \xi^p)$. Then we have

$$\begin{aligned}
&\bigl(K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr)\bigl(\xi^p + \xi^p\bigr)\bigr)'_\xi \\
&\quad = -\bigl(\nabla_{x_1} K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr), \xi^{-2}x_1\bigr)\bigl(\xi^p + \xi^p\bigr) \\
&\qquad - \bigl(\nabla_{x_2} K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr), \xi^{-2}x_2\bigr)\bigl(\xi^p + \xi^p\bigr) + K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr)\bigl(p\xi^{p-1} + p\xi^{p-1}\bigr) \\
&\quad \geq -\bigl(\nabla_{x_1} K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr), \xi^{-1}x_1\bigr)\bigl(\xi^{p-1} + \xi^{p-1}\bigr) \\
&\qquad - \bigl(\nabla_{x_2} K\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr), \xi^{-1}x_2\bigr)\bigl(\xi^{p-1} + \xi^{p-1}\bigr) + pK\bigl(t,\xi^{-1}x_1,\xi^{-1}x_2\bigr)\bigl(\xi^{p-1} + \xi^{p-1}\bigr) \\
&\quad \geq 0.
\end{aligned}$$

Hence the function $\xi \in (0,+\infty) \to K(t,\xi^{-1}x_1,\xi^{-1}x_2)(\xi^p + \xi^p)$ is nondecreasing. Moreover, note that

$$\frac{|x_1|^p + |x_2|^p}{2} \leq |x|^p \leq \bigl(|x_1| + |x_2|\bigr)^p \leq 2^{p-1}\bigl(|x_1|^p + |x_2|^p\bigr).$$

Then the proof can be completed easily. □

Lemma 3.2 *Suppose that* (H1) *holds. Then, for any $u \in \mathcal{H}_k$,*

$$\begin{aligned}
\eta_k(u) &\geq \min\bigl\{d_1\|u_1\|^p_{\mathcal{H}_{1,k}}, a_1 C_*^{\gamma-p}\|u_1\|^\gamma_{\mathcal{H}_{1,k}}\bigr\} \\
&\quad + \min\bigl\{d_2\|u_2\|^p_{\mathcal{H}_{1,k}}, a_2 C_*^{\gamma-p}\|u_2\|^\gamma_{\mathcal{H}_{2,k}}\bigr\}, \quad \forall k \in \mathbb{N}.
\end{aligned}$$

Proof It follows from ($\mathcal{A}1$), (H1), $\gamma \in (1,p)$ and Corollary 2.2 that

$$\begin{aligned}
\eta_k(u) &= \sum_{t=-kT}^{kT-1} \bigl[\Phi_1\bigl(\Delta u_1(t)\bigr) + \Phi_2\bigl(\Delta u_2(t)\bigr) + K\bigl(t,u_1(t),u_2(t)\bigr)\bigr] \\
&\geq \sum_{t=-kT}^{kT-1} \bigl[d_1\bigl|\Delta u_1(t)\bigr|^p + d_2\bigl|\Delta u_2(t)\bigr|^p + a_1\bigl|u_1(t)\bigr|^\gamma + a_2\bigl|u_2(t)\bigr|^\gamma\bigr]
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{t=-kT}^{kT-1}\left[d_1\left|\Delta u_1(t)\right|^p + d_2\left|\Delta u_2(t)\right|^p + a_1\|u_1\|_{l^\infty_{2kT}}^{\gamma-p}\left|u_1(t)\right|^p + a_2\|u_2\|_{l^\infty_{2kT}}^{\gamma-p}\left|u_2(t)\right|^p\right] \\
&\geq \sum_{t=-kT}^{kT-1}\left[d_1\left|\Delta u_1(t)\right|^p + d_2\left|\Delta u_2(t)\right|^p + a_1\left(C_*\|u_1\|_{\mathcal{H}_{1,k}}\right)^{\gamma-p}\left|u_1(t)\right|^p\right. \\
&\quad \left. + a_2\left(C_*\|u_2\|_{\mathcal{H}_{2,k}}\right)^{\gamma-p}\left|u_2(t)\right|^p\right] \\
&\geq \min\left\{d_1, a_1\left(C_*\|u_1\|_{\mathcal{H}_{1,k}}\right)^{\gamma-p}\right\}\|u_1\|_{\mathcal{H}_{1,k}}^p + \min\left\{d_2, a_2\left(C_*\|u_2\|_{\mathcal{H}_{1,k}}\right)^{\gamma-p}\right\}\|u_2\|_{\mathcal{H}_{2,k}}^p \\
&= \min\left\{d_1\|u_1\|_{\mathcal{H}_{1,k}}^p, a_1C_*^{\gamma-p}\|u_1\|_{\mathcal{H}_{1,k}}^\gamma\right\} + \min\left\{d_2\|u_2\|_{\mathcal{H}_{2,k}}^p, a_2C_*^{\gamma-p}\|u_2\|_{\mathcal{H}_{2,k}}^\gamma\right\}. \qquad \square
\end{aligned}$$

Proof of Theorem 1.1 We divide the proof into the following Lemmas 3.3-3.5.

Lemma 3.3 *Under the assumptions of Theorem* 1.1, *for every* $k \in \mathbb{N}$, *system* (2.1) *has a nontrivial solution* u_k *in* $\mathcal{H}_k$.

Proof We first construct A and B which satisfy the assumptions in Theorem 2.1.

(i) When $r \in (0,1]$, by Corollary 2.2, (H1), (H3)(i), Hölder's inequality and $\gamma < p$, for $u \in \mathcal{H}_k$ with $\|u\|_{\mathcal{H}_k} = r$, we have $\|u_1\|_{l^\infty_{2kT}} \leq C_*\|u_1\|_{\mathcal{H}_{1,k}} \leq rC_*$ and $\|u_2\|_{l^\infty_{2kT}} \leq C_*\|u_2\|_{\mathcal{H}_{2,k}} \leq rC_*$,

$$\begin{aligned}
\varphi_k(u) &\geq \eta_k(u) - b_1\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p - b_2\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p \\
&\quad - \left(\sum_{t=-kT}^{kT-1}\left|f_{1,k}(t)\right|^{p'}\right)^{1/p'}\left(\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p\right)^{1/p} \\
&\quad - \left(\sum_{t=-kT}^{kT-1}\left|f_{2,k}(t)\right|^{p'}\right)^{1/p'}\left(\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p\right)^{1/p} \\
&\geq \sum_{t=-kT}^{kT-1}\left[d_1\left|\Delta u_1(t)\right|^p + d_2\left|\Delta u_2(t)\right|^p + a_1\left|u_1(t)\right|^\gamma + a_2\left|u_2(t)\right|^\gamma\right] - b_1\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p \\
&\quad - b_2\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p - \left(\sum_{t=-kT}^{kT-1}\left|f_{1,k}(t)\right|^{p'}\right)^{1/p'}\left(\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p\right)^{1/p} \\
&\quad - \left(\sum_{t=-kT}^{kT-1}\left|f_{2,k}(t)\right|^{p'}\right)^{1/p'}\left(\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p\right)^{1/p} \\
&\geq \sum_{t=-kT}^{kT-1}d_1\left|\Delta u_1(t)\right|^p + \sum_{t=-kT}^{kT-1}d_2\left|\Delta u_2(t)\right|^p \\
&\quad + a_1\left(C_*\|u_1\|_{\mathcal{H}_{1,k}}\right)^{\gamma-p}\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p - b_1\sum_{t=-kT}^{kT-1}\left|u_1(t)\right|^p \\
&\quad - b_2\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p + a_2\left(C_*\|u_2\|_{\mathcal{H}_{2,k}}\right)^{\gamma-p}\sum_{t=-kT}^{kT-1}\left|u_2(t)\right|^p \\
&\quad - \|f_1\|_{l^{p'}}\|u_1\|_{\mathcal{H}_{1,k}} - \|f_2\|_{l^{p'}}\|u_2\|_{\mathcal{H}_{2,k}} \\
&\geq \min\left\{d_1, a_1(C_*r)^{\gamma-p} - b_1\right\}\|u_1\|_{\mathcal{H}_{1,k}}^p + \min\left\{d_2, a_2(C_*r)^{\gamma-p} - b_2\right\}\|u_2\|_{\mathcal{H}_{2,k}}^p
\end{aligned}$$

$$
\begin{aligned}
&\quad - \|f_1\|_{l^{p'}} \|u_1\|_{\mathcal{H}_{1,k}} - \|f_2\|_{l^{p'}} \|u_2\|_{\mathcal{H}_{2,k}} \\
&\geq \min\{d_1, a_1 C_*^{\gamma-p} - b_1\} \|u_1\|_{\mathcal{H}_{1,k}}^p + \min\{d_2, a_2 C_*^{\gamma-p} - b_2\} \|u_2\|_{\mathcal{H}_{2,k}}^p \\
&\quad - \|f_1\|_{l^{p'}} \|u_1\|_{\mathcal{H}_{1,k}} - \|f_2\|_{l^{p'}} \|u_2\|_{\mathcal{H}_{2,k}} \\
&\geq \min\{d_1, d_2, a_1 C_*^{\gamma-p} - b_1, a_2 C_*^{\gamma-p} - b_2\} \frac{1}{2^{p-1}} \left(\|u_1\|_{\mathcal{H}_{1,k}} + \|u_2\|_{\mathcal{H}_{2,k}}\right)^p \\
&\quad - \max\{\|f_1\|_{l^{p'}}, \|f_2\|_{l^{p'}}\} \left(\|u_1\|_{\mathcal{H}_{1,k}} + \|u_2\|_{\mathcal{H}_{2,k}}\right).
\end{aligned} \tag{3.1}
$$

(H6)(i) implies that there exists $\alpha > 0$ such that

$$\varphi_k(u) \geq \alpha > 0 \quad \text{for all } u \in \mathcal{H}_k \text{ with } \|u\|_{\mathcal{H}_k} = r, \forall k \in \mathbb{N}.$$

(ii) When $r \in (1, +\infty)$, by Corollary 2.2, (H1), (H3)(ii), Hölder's inequality and $\gamma < p$, for $u \in \mathcal{H}_k$ with $\|u\|_{\mathcal{H}_k} = r$, we have

$$
\begin{aligned}
\varphi_k(u) &\geq \min\{d_1, a_1 (C_* r)^{\gamma-p} - b_1\} \|u_1\|_{\mathcal{H}_{1,k}}^p + \min\{d_2, a_2 (C_* r)^{\gamma-p} - b_2\} \|u_2\|_{\mathcal{H}_{2,k}}^p \\
&\quad - \|f_1\|_{l^{p'}} \|u_1\|_{\mathcal{H}_{1,k}} - \|f_2\|_{l^{q'}} \|u_2\|_{\mathcal{H}_{2,k}} \\
&\geq \min\{d_1, d_2, a_1 (C_* r)^{\gamma-p} - b_1, a_2 (C_* r)^{\gamma-p} - b_2\} \frac{1}{2^{p-1}} \left(\|u_1\|_{\mathcal{H}_{1,k}} + \|u_2\|_{\mathcal{H}_{2,k}}\right)^p \\
&\quad - \max\{\|f_1\|_{l^{p'}}, \|f_2\|_{l^{p'}}\} \left(\|u_1\|_{\mathcal{H}_{1,k}} + \|u_2\|_{\mathcal{H}_{2,k}}\right).
\end{aligned} \tag{3.2}
$$

(H6)(ii) implies that there exists $\alpha > 0$ such that

$$\varphi_k(u) \geq \alpha > 0 \quad \text{for all } u \in \mathcal{H}_k \text{ with } \|u\|_{\mathcal{H}_k} = r, \forall k \in \mathbb{N}.$$

By Lemma 3.1 and the T-periodicity of K, there exists a constant $B_0 > 0$ such that

$$K(t, x_1, x_2) \leq 2^{p-1} A_0 \left(|x_1|^p + |x_2|^p\right) + B_0 \quad \text{for all } (t, x_1, x_2) \in \mathbb{Z} \times \mathbb{R}^N \times \mathbb{R}^N, \tag{3.3}$$

where

$$A_0 = \max_{|x_1| \leq 1, |x_2| \leq 1, t \in \mathbb{Z}[0, T-1]} K(t, x_1, x_2).$$

By (H4), we know that there exist $\varepsilon_0 > 0$ and $L > 0$ such that

$$
\begin{aligned}
&W(t, x_1, x_2) \geq \left(d_3 + d_4 + 2^{p-1} A_0 + \varepsilon_0\right) \left(|x_1|^p + |x_2|^p\right) \\
&\quad \text{for all } t \in \mathbb{Z}[0, T-1] \text{ and } \forall |x| \geq L.
\end{aligned} \tag{3.4}
$$

By (3.4) and the T-periodicity of W, there exists a constant $B_1 > 0$ such that

$$W(t, x_1, x_2) \geq \left(d_3 + d_4 + 2^{p-1} A_0 + \varepsilon_0\right) \left(|x_1|^p + |x_2|^p\right) - B_1 \tag{3.5}$$

for all $(t, x_1, x_2) \in \mathbb{Z}[0, T-1] \times \mathbb{R}^N \times \mathbb{R}^N$. For any $k \in \mathbb{N}$, define $w^{(k)} \in \mathcal{H}_k$ by

$$w^{(k)}(t) = \left(w_1^{(k)}(t), w_2^{(k)}(t)\right) = \begin{cases} (1, 0, \ldots, 0, 1, 0, \ldots, 0) & \text{if } t = 0, \\ 0 & \text{if } t \in \mathbb{Z}[-kT, kT-1]/\{0\}, \end{cases}$$

where

$$w_i^{(k)}(t) = \begin{cases} (1,0,\dots,0) & \text{if } t=0, \\ 0 & \text{if } t \in \mathbb{Z}[-kT,kT-1]/\{0\}, \end{cases} \qquad i=1,2.$$

Since $K(t,0,0)\equiv 0$ and $W(t,0,0)\equiv 0$, which are implied by (H2) and (H5), then by (3.3) and (3.5) we have

$$\begin{aligned}
\varphi_k\bigl(\xi w^{(k)}\bigr) &= \sum_{t=-kT}^{kT-1}\bigl[\Phi_1\bigl(\xi\Delta w_1^{(k)}(t)\bigr)+\Phi_2\bigl(\xi\Delta w_2^{(k)}(t)\bigr)+K\bigl(t,\xi w_1^{(k)}(t),\xi w_2^{(k)}(t)\bigr) \\
&\quad - W\bigl(t,\xi w_1^{(k)}(t),\xi w_2^{(k)}(t)\bigr)+\xi\bigl(f_{1,k}(t),w_1^{(k)}(t)\bigr)+\xi\bigl(f_{2,k}(t),w_2^{(k)}(t)\bigr)\bigr] \\
&\le d_3|\xi|^p\sum_{t=-kT}^{kT-1}\bigl|\Delta w_1^{(k)}(t)\bigr|^p + d_4|\xi|^p\sum_{t=-kT}^{kT-1}\bigl|\Delta w_2^{(k)}(t)\bigr|^p \\
&\quad + K\bigl(0,\xi w_1^{(k)}(0),\xi w_2^{(k)}(0)\bigr) - W\bigl(0,\xi w_1^{(k)}(0),\xi w_2^{(k)}(0)\bigr) \\
&\quad + \xi\bigl(f_{1,k}(0),w_1^{(k)}(0)\bigr)+\xi\bigl(f_{2,k}(0),w_2^{(k)}(0)\bigr) \\
&= d_3|\xi|^p\bigl(\bigl|\Delta w_1^{(k)}(-1)\bigr|^p+\bigl|\Delta w_1^{(k)}(0)\bigr|^p\bigr)+d_4|\xi|^p\bigl(\bigl|\Delta w_2^{(k)}(-1)\bigr|^p+\bigl|\Delta w_2^{(k)}(0)\bigr|^p\bigr) \\
&\quad + K\bigl(0,\xi w_1^{(k)}(0),\xi w_2^{(k)}(0)\bigr) - W\bigl(0,\xi w_1^{(k)}(0),\xi w_2^{(k)}(0)\bigr) \\
&\quad + \xi\bigl(f_{1,k}(0),w_1^{(k)}(0)\bigr)+\xi\bigl(f_{2,k}(0),w_2^{(k)}(0)\bigr) \\
&\le 2d_3|\xi|^p+2d_4|\xi|^p+2^{p-1}A_0|\xi|^p+2^{p-1}A_0|\xi|^p \\
&\quad + B_0-\bigl(d_3+d_4+2^{p-1}A_0+\varepsilon_0\bigr)\bigl(|\xi|^p+|\xi|^p\bigr) \\
&\quad + B_1+|\xi|\bigl|f_{1,k}(0)\bigr|+|\xi|\bigl|f_{2,k}(0)\bigr| \\
&\le -2\varepsilon_0|\xi|^p+|\xi|\bigl|f_{1,k}(0)\bigr|+|\xi|\bigl|f_{2,k}(0)\bigr|+B_0+B_1.
\end{aligned} \tag{3.6}$$

So there exists $\xi_0\in\mathbb{R}$ such that $\|\xi_0 w^{(k)}\|>r$ and $\varphi_k(\xi_0 w^{(k)})<0$. Moreover, it is clear that $\varphi_k(0)=0$. Let $e_1=\xi_0 w^{(k)}$ and

$$A=\{0,e_1\}, \qquad B=\bigl\{u\in\mathcal{H}_k:\|u\|_{\mathcal{H}_k}<r\bigr\}.$$

Then $0\in B$ and $e_1\notin\bar{B}$. So by Example 1 in Section 2, we know that A links ∂B[hm]. So by Theorem 2.1 and Remark 2.2, we have

$$c_k=\inf_{\Gamma\in\Phi}\sup_{\substack{s\in[0,1]\\ u\in A}}\varphi_k\bigl(\Gamma(s)u\bigr)\ge\inf_{\partial B}\varphi_k>\alpha>0, \tag{3.7}$$

and there exists a sequence $\{u_n=(u_1^{(n)},u_2^{(n)})\}_{n=1}^{\infty}\subset\mathcal{H}_k$ such that

$$\varphi_k(u_n)\to c_k, \qquad \bigl(1+\|u_n\|_{\mathcal{H}_k}\bigr)\bigl\|\varphi_k'(u_n)\bigr\|\to 0.$$

Then there exists a constant $C_{1k}>0$ such that

$$\bigl|\varphi_k(u_n)\bigr|\le C_{1k}, \qquad \bigl(1+\|u_n\|_{\mathcal{H}_k}\bigr)\bigl\|\varphi_k'(u_n)\bigr\|\le C_{1k} \quad \text{for all } n\in\mathbb{N}. \tag{3.8}$$

It follows from (H5) and the T-periodicity and continuity of W, $\nabla_{x_1} W$ and $\nabla_{x_2} W$ that

$$\begin{aligned}&\big[\big(\nabla_{x_1} W(t,x_1,x_2),x_1\big)+\big(\nabla_{x_2} W(t,x_1,x_2),x_2\big)-pW(t,x_1,x_2)\big]\big(\zeta+\eta_1|x_1|^\nu+\eta_2|x_2|^\nu\big)\\&\quad\geq W(t,x_1,x_2)\geq 0,\quad \forall(t,x_1,x_2)\in\mathbb{Z}\times\mathbb{R}^N\times\mathbb{R}^N.\end{aligned}\tag{3.9}$$

So by (3.5) and $p-\nu>1$, there exists $C_2>0$ such that

$$\begin{aligned}&\big[\big(\nabla_{x_1} W(t,x_1,x_2),x_1\big)+\big(\nabla_{x_2} W(t,x_1,x_2),x_2\big)-pW(t,x_1,x_2)\big]\\&\quad\geq\frac{W(t,x_1,x_2)}{\zeta+\eta_1|x_1|^\nu+\eta_2|x_2|^\nu}\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)(|x_1|^p+|x_2|^p)-B_1}{\zeta+\eta_1|x_1|^\nu+\eta_2|x_2|^\nu}\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)\frac{1}{2^{p-1}}(|x_1|+|x_2|)^p-B_1}{\zeta+2\max\{\eta_1,\eta_2\}(|x_1|+|x_2|)^\nu}\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)\frac{1}{2^{p-1}}}{4\max\{\eta_1,\eta_2\}}\big(|x_1|+|x_2|\big)^{p-\nu}-C_2\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)\frac{1}{2^{p-1}}}{4\max\{\eta_1,\eta_2\}}\big(|x_1|^{p-\nu}+|x_2|^{p-\nu}\big)-C_2,\quad\forall x\in\mathbb{R}^N.\end{aligned}\tag{3.10}$$

Hence, it follows from (H2), (3.8) and (3.10) that

$$\begin{aligned}&pC_{1k}+C_{1k}\\&\quad\geq p\varphi_k(u_n)-\langle\varphi_k'(u_n),u_n\rangle\\&\quad=p\varphi_k\big(u_1^{(n)},u_2^{(n)}\big)-\big\langle\varphi_k'\big(u_1^{(n)},u_2^{(n)}\big),\big(u_1^{(n)},u_2^{(n)}\big)\big\rangle\\&\quad\geq\sum_{t=-kT}^{kT-1}\big[\big(\nabla_{u_1}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_1^{(n)}(t)\big)+\big(\nabla_{u_2}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_2^{(n)}(t)\big)\\&\qquad-pW\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big)\big]\\&\qquad+(p-1)\sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_1^{(n)}(t)\big)+(p-1)\sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_2^{(n)}(t)\big)\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)\frac{1}{2^{p-1}}}{4\max\{\eta_1,\eta_2\}}\sum_{t=-kT}^{kT-1}\big(\big|u_1^{(n)}(t)\big|^{p-\nu}+\big|u_2^{(n)}(t)\big|^{p-\nu}\big)-2kTC_2\\&\qquad-(p-1)\sum_{t=-kT}^{kT-1}\big|f_{1,k}(t)\big|\big|u_1^{(n)}(t)\big|-(p-1)\sum_{t=-kT}^{kT-1}\big|f_{2,k}(t)\big|\big|u_2^{(n)}(t)\big|\\&\quad\geq\frac{(d_3+d_4+2^{p-1}A_0+\varepsilon_0)\frac{1}{2^{p-1}}}{4\max\{\eta_1,\eta_2\}}\sum_{t=-kT}^{kT-1}\big(\big|u_1^{(n)}(t)\big|^{p-\nu}+\big|u_2^{(n)}(t)\big|^{p-\nu}\big)-2kTC_2\\&\qquad-(p-1)\left(\sum_{t=-kT}^{kT-1}\big|f_{1,k}(t)\big|^{\frac{p-\nu}{p-\nu-1}}\right)^{\frac{p-\nu-1}{p-\nu}}\left(\sum_{t=-kT}^{kT-1}\big|u_1^{(n)}(t)\big|^{p-\nu}\right)^{1/(p-\nu)}\\&\qquad-(p-1)\left(\sum_{t=-kT}^{kT-1}\big|f_{2,k}(t)\big|^{\frac{p-\nu}{p-\nu-1}}\right)^{\frac{p-\nu-1}{p-\nu}}\left(\sum_{t=-kT}^{kT-1}\big|u_2^{(n)}(t)\big|^{p-\nu}\right)^{1/(p-\nu)}.\end{aligned}\tag{3.11}$$

The fact $p-\nu>1$ and the above inequality show that $\sum_{t=-kT}^{kT-1}|u_1^{(n)}(t)|^{p-\nu}$ and $\sum_{t=-kT}^{kT-1}|u_2^{(n)}(t)|^{p-\nu}$ are bounded. By ($\mathcal{A}1$), (H1), (H6), (3.8), (3.9), (3.11), Hölder's inequality and Corollary 2.2, we have

$$\begin{aligned}
&d_1\|u_1^{(n)}\|_{\mathcal{H}_{1,k}}^p+d_2\|u_2^{(n)}\|_{\mathcal{H}_{2,k}}^p\\
&\quad=d_1\sum_{t=-kT}^{kT-1}\left|\Delta u_1^{(n)}(t)\right|^p+d_1\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^p+d_2\sum_{t=-kT}^{kT-1}\left|\Delta u_2^{(n)}(t)\right|^p+d_2\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^p\\
&\quad\le\varphi_k\big(u^{(n)}\big)-\sum_{t=-kT}^{kT-1}K\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big)\\
&\qquad+\sum_{t=-kT}^{kT-1}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big)+d_1\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^p\\
&\qquad+d_2\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^p-\sum_{t=-kT}^{kT-1}\big(f_{1k}(t),u_1^{(n)}(t)\big)-\sum_{t=-kT}^{kT-1}\big(f_{2k}(t),u_2^{(n)}(t)\big)\\
&\quad\le\varphi_k\big(u^{(n)}\big)\\
&\qquad+\sum_{t=-kT}^{kT-1}\Big[\big(\nabla_{u_1}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_1^{(n)}(t)\big)+\big(\nabla_{u_2}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_2^{(n)}(t)\big)\\
&\qquad-pW\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big)\Big]\big(\zeta+\eta_1\left|u_1^{(n)}(t)\right|^\nu+\eta_2\left|u_2^{(n)}(t)\right|^\nu\big)\\
&\qquad+d_1\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^p+d_2\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^p+\left(\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^p\right)^{\frac{1}{p}}\left(\sum_{t\in\mathbb{Z}}\left|f_1(t)\right|^{p'}\right)^{\frac{1}{p'}}\\
&\qquad+\left(\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^p\right)^{\frac{1}{p}}\left(\sum_{t\in\mathbb{Z}}\left|f_2(t)\right|^{p'}\right)^{\frac{1}{p'}}\\
&\quad\le C_{1k}+d_1\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^p+d_2\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^p+\|u_1^{(n)}\|_{\mathcal{H}_{1,k}}\left(\sum_{t\in\mathbb{Z}}\left|f_1(t)\right|^{p'}\right)^{\frac{1}{p'}}\\
&\qquad+\|u_2^{(n)}\|_{\mathcal{H}_{2,k}}\left(\sum_{t\in\mathbb{Z}}\left|f_2(t)\right|^{p'}\right)^{\frac{1}{p'}}\\
&\qquad+\big(\zeta+\eta_1\|u_1^{(n)}\|_{l_{2kT}^\infty}^\nu+\eta_2\|u_2^{(n)}\|_{l_{2kT}^\infty}^\nu\big)\\
&\qquad\cdot\sum_{t=-kT}^{kT-1}\Big[\big(\nabla_{u_1}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_1^{(n)}(t)\big)+\big(\nabla_{u_2}W\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big),u_2^{(n)}(t)\big)\\
&\qquad-pW\big(t,u_1^{(n)}(t),u_2^{(n)}(t)\big)\Big]\\
&\quad\le C_{1k}+d_1\|u_1^{(n)}\|_{l_{2kT}^\infty}^\nu\sum_{t=-kT}^{kT-1}\left|u_1^{(n)}(t)\right|^{p-\nu}+d_2\|u_2^{(n)}\|_{l_{2kT}^\infty}^\nu\sum_{t=-kT}^{kT-1}\left|u_2^{(n)}(t)\right|^{p-\nu}\\
&\qquad+\|u_1^{(n)}\|_{\mathcal{H}_{1,k}}\left(\sum_{t\in\mathbb{Z}}\left|f_1(t)\right|^{p'}\right)^{\frac{1}{p'}}+\|u_2^{(n)}\|_{\mathcal{H}_{2,k}}\left(\sum_{t\in\mathbb{Z}}\left|f_2(t)\right|^{p'}\right)^{\frac{1}{p'}}\\
&\qquad+\big(\zeta+\eta_1\|u_1^{(n)}\|_{l_{2kT}^\infty}^\nu+\eta_2\|u_2^{(n)}\|_{l_{2kT}^\infty}^\nu\big)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \Bigg[(p+1)C_{1k} + (p-1)\big\|u_1^{(n)}\big\|_{\mathcal{H}_{1,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_1(t)\big|^{p'}\bigg)^{\frac{1}{p'}} \\
&\quad + (p-1)\big\|u_2^{(n)}\big\|_{\mathcal{H}_{2,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_2(t)\big|^{p'}\bigg)^{\frac{1}{p'}}\Bigg] \\
&\le C_{1k} + d_1 C_*^{\nu} \big\|u_1^{(n)}\big\|_{\mathcal{H}_{1,k}}^{\nu} \sum_{t=-kT}^{kT-1} \big|u_1^{(n)}(t)\big|^{p-\nu} + d_2 C_*^{\nu} \big\|u_2^{(n)}\big\|_{\mathcal{H}_{2,k}}^{\nu} \sum_{t=-kT}^{kT-1} \big|u_2^{(n)}(t)\big|^{p-\nu} \\
&\quad + \big\|u_1^{(n)}\big\|_{\mathcal{H}_{1,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_1(t)\big|^{p'}\bigg)^{\frac{1}{p'}} + \big\|u_2^{(n)}\big\|_{\mathcal{H}_{2,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_2(t)\big|^{p'}\bigg)^{\frac{1}{p'}} \\
&\quad + \big(\zeta + \eta_1 C_*^{\nu} \big\|u_1^{(n)}\big\|_{\mathcal{H}_{1,k}}^{\nu} + \eta_2 C_*^{\nu} \big\|u_2^{(n)}\big\|_{\mathcal{H}_{2,k}}^{\nu}\big) \\
&\quad \cdot \Bigg[(p+1)C_{1k} + (p-1)\big\|u_1^{(n)}\big\|_{\mathcal{H}_{1,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_1(t)\big|^{p'}\bigg)^{\frac{1}{p'}} \\
&\quad + (p-1)\big\|u_2^{(n)}\big\|_{\mathcal{H}_{2,k}} \bigg(\sum_{t\in\mathbb{Z}} \big|f_2(t)\big|^{p'}\bigg)^{\frac{1}{p'}}\Bigg].
\end{aligned} \tag{3.12}
$$

Since $\nu < p-1$, (3.12) and the boundedness of $\sum_{t=-kT}^{kT-1} |u_1^{(n)}(t)|^{p-\nu}$ and $\sum_{t=-kT}^{kT-1} |u_2^{(n)}(t)|^{p-\nu}$ imply that $\|u_1^{(n)}\|_{\mathcal{H}_{1,k}}$ and $\|u_2^{(n)}\|_{\mathcal{H}_{2,k}}$ are bounded. Since $\mathcal{H}$ is a finite-dimensional space, $\{u^{(n)} = (u_1^{(n)}, u_2^{(n)})\}$ has a convergence subsequence, still denoted by $\{u^{(n)} = (u_1^{(n)}, u_2^{(n)})\}$, such that $u^{(n)} = (u_1^{(n)}, u_2^{(n)}) \to u_k = (u_{1k}, u_{2k})$ as $n \to \infty$. Moreover, by the continuity of φ_k and φ_k', we obtain $\varphi_k'(u_k) = 0$ and $\varphi_k(u_k) = c_k > 0$. It is clear that $u_k \neq 0$ and so u_k is a desired nontrivial solution of system (2.1). The proof is complete. □

Lemma 3.4 *Let $\{u_k = (u_{1k}, u_{2k})\}_{k\in\mathbb{N}}$ be the solutions of system* (2.1). *Then there exists $M_1 > 0$ such that $\|u_{1k}\|_{l_{2kT}^{\infty}} \le M_1$ and $\|u_{2k}\|_{l_{2kT}^{\infty}} \le M_1$.*

Proof First, we prove that the sequence $\{c_k\}_{k\in\mathbb{N}}$ is bounded. For every $k \in \mathbb{N}$, define $\Gamma_k : [0,1] \times \mathcal{H}_k \to \mathcal{H}_k$ by

$$
\Gamma_k(s)v = (1-s)v, \quad v \in \mathcal{H}_k.
$$

Then $\Gamma \in \Phi$. Note that the set $A = \{0, e_1\}$. So (3.7) and the argument of (3.6) imply that

$$
\begin{aligned}
\varphi_k(u_k) = c_k &\le \sup_{s\in[0,1] u\in A} \varphi_k\big((1-s)u\big) \\
&= \sup_{s\in[0,1]} \varphi_k\big((1-s)e_1\big) \\
&= \sup_{s\in[0,1]} \varphi_k\big((1-s)e_1\big) \\
&= \sup_{s\in[0,1]} \Bigg\{ \sum_{t=-kT}^{kT-1} \big[\Phi_1\big((1-s)\Delta w_1^{(k)}(t)\big) + \Phi_2\big((1-s)\Delta w_2^{(k)}(t)\big) \\
&\quad + K\big(t, (1-s)w_1^{(k)}(t), (1-s)w_2^{(k)}(t)\big) - W\big(t, (1-s)w_1^{(k)}(t), (1-s)w_2^{(k)}(t)\big) \\
&\quad + (1-s)\big(f_{1,k}(t), w_1^{(k)}(t)\big) + (1-s)\big(f_{2,k}(t), w_2^{(k)}(t)\big)\big] \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sup_{s\in[0,1]}\left\{-2\varepsilon_0|1-s|^p+|1-s|\left|f_{1,k}(0)\right|+|1-s|\left|f_{2,k}(0)\right|+B_0+B_1\right\} \\
&\leq \left|f_1(0)\right|+\left|f_2(0)\right|+B_0+B_1:=M_2,
\end{aligned}
\tag{3.13}
$$

where M_2 is independent of $k\in\mathbb{N}$, which implies that the sequence $\{c_k\}_{k\in\mathbb{N}}$ is bounded. Moreover, $\varphi_k'(u_k)=0$. Then it follows from ($\mathcal{A}1$), (H2), and (3.10) that

$$
\begin{aligned}
pM_2\geq pc_k &= p\varphi_k(u_k)-\left\langle\varphi_k'(u_k),u_k\right\rangle \\
&= p\varphi_k(u_{1k},u_{2k})-\left\langle\varphi_k'(u_{1k},u_{2k}),(u_{1k},u_{2k})\right\rangle \\
&\geq \sum_{t=-kT}^{kT-1}\Big[\big(\nabla_{u_1}W\big(t,u_{1k}(t),u_{2k}(t)\big),u_{1k}(t)\big)+\big(\nabla_{u_2}W\big(t,u_{1k}(t),u_{2k}(t)\big),u_{2k}(t)\big) \\
&\quad -pW\big(t,u_{1k}(t),u_{2k}(t)\big)\Big] \\
&\quad +(p-1)\sum_{t=-kT}^{kT-1}\big(f_{1k}(t),u_{1k}(t)\big)+(p-1)\sum_{t=-kT}^{kT-1}\big(f_{2k}(t),u_{2k}(t)\big) \\
&\geq \sum_{t=-kT}^{kT-1}\frac{W(t,u_{1k}(t),u_{2k}(t))}{\xi+\eta_1|u_{1k}(t)|^\nu+\eta_2|u_{2k}(t)|^\nu}+(p-1)\sum_{t=-kT}^{kT-1}\big(f_{1k}(t),u_{1k}(t)\big) \\
&\quad +(p-1)\sum_{t=-kT}^{kT-1}\big(f_{2k}(t),u_{2k}(t)\big).
\end{aligned}
$$

So

$$
\begin{aligned}
&\sum_{t=-kT}^{kT-1}\frac{W(t,u_{1k}(t),u_{2k}(t))}{\xi+\eta_1|u_{1k}(t)|^\nu+\eta_2|u_{2k}(t)|^\nu} \\
&\quad\leq pM_2-(p-1)\sum_{t=-kT}^{kT-1}\big(f_{1k}(t),u_{1k}(t)\big)-(p-1)\sum_{t=-kT}^{kT-1}\big(f_{2k}(t),u_{2k}(t)\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
\eta_k(u_k) &= \varphi_k(u_k)+\sum_{t=-kT}^{kT-1}\frac{W(t,u_{1k}(t),u_{2k}(t))}{\xi+\eta_1|u_{1k}(t)|^\nu+\eta_2|u_{2k}(t)|^\nu}\left(\xi+\eta_1\left|u_{1k}(t)\right|^\nu+\eta_2\left|u_{2k}(t)\right|^\nu\right) \\
&\quad -\sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big)-\sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\leq \varphi_k(u_k)+\left(\xi+\eta_1\|u_{1k}\|_{l_{2kT}^\infty}^\nu+\eta_2\|u_{2k}\|_{l_{2kT}^\infty}^\nu\right)\sum_{t=-kT}^{kT-1}\frac{W(t,u_{1k}(t),u_{2k}(t))}{\xi+\eta_1|u_{1k}(t)|^\nu+\eta_2|u_{2k}(t)|^\nu} \\
&\quad -\sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big)-\sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\leq \varphi_k(u_k)+\left(\xi+\eta_1C_*^\nu\|u_{1k}\|_{\mathcal{H}_{1,k}}^\nu+\eta_2C_*^\nu\|u_{2k}\|_{\mathcal{H}_{2,k}}^\nu\right) \\
&\quad \cdot\left[pM_2-(p-1)\sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big)-(p-1)\sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big)\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big) - \sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&= M_2 + p\xi M_2 + p\eta_1 C_*^{\nu} M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} + p\eta_2 C_*^{\nu} M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad - (p-1)\xi \sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big) - (p-1)\xi \sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\quad - (p-1)\eta_1 C_*^{\nu}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} \sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big) \\
&\quad - (p-1)\eta_2 C_*^{\nu}\|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\quad - (p-1)\eta_1 C_*^{\nu}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} \sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\quad - (p-1)\eta_2 C_*^{\nu}\|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big) \\
&\quad - \sum_{t=-kT}^{kT-1}\big(f_{1,k}(t),u_{1k}(t)\big) - \sum_{t=-kT}^{kT-1}\big(f_{2,k}(t),u_{2k}(t)\big) \\
&\le (1+p\xi)M_2 + \big[(p-1)\xi+1\big]\bigg(\sum_{t\in\mathbb{Z}}\big|f_1(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{1k}(t)\big|^{p}\bigg)^{1/p} \\
&\quad + \big[(p-1)\xi+1\big]\bigg(\sum_{t\in\mathbb{Z}}\big|f_2(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{2k}(t)\big|^{p}\bigg)^{1/p} \\
&\quad + p\eta_1 C_*^{\nu} M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} + p\eta_2 C_*^{\nu} M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad + (p-1)\eta_1 C_*^{\nu}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu}\Bigg(\bigg(\sum_{t\in\mathbb{Z}}\big|f_1(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{1k}(t)\big|^{p}\bigg)^{1/p} \\
&\quad + \bigg(\sum_{t\in\mathbb{Z}}\big|f_2(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{2k}(t)\big|^{p}\bigg)^{1/p}\Bigg) \\
&\quad + (p-1)\eta_2 C_*^{\nu}\|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu}\Bigg(\bigg(\sum_{t\in\mathbb{Z}}\big|f_1(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{1k}(t)\big|^{p}\bigg)^{1/p} \\
&\quad + \bigg(\sum_{t\in\mathbb{Z}}\big|f_2(t)\big|^{p'}\bigg)^{1/p'}\bigg(\sum_{t=-kT}^{kT-1}\big|u_{2k}(t)\big|^{p}\bigg)^{1/p}\Bigg) \\
&\le (1+p\xi)M_2 + \big[(p-1)\xi+1\big]\bigg(\sum_{t\in\mathbb{Z}}\big|f_1(t)\big|^{p'}\bigg)^{1/p'}\|u_{1k}\|_{\mathcal{H}_{1,k}} \\
&\quad + \big[(p-1)\xi+1\big]\bigg(\sum_{t\in\mathbb{Z}}\big|f_2(t)\big|^{p'}\bigg)^{1/p'}\|u_{2k}\|_{\mathcal{H}_{2,k}} \\
&\quad + p\eta_1 C_*^{\nu} M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} + p\eta_2 C_*^{\nu} M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad + (p-1)\eta_1 C_*^{\nu}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \cdot \left(\left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \|u_{1k}\|_{\mathcal{H}_{1,k}} + \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \|u_{2k}\|_{\mathcal{H}_{2,k}} \right) \\
&\quad + (p-1)\eta_2 C_*^{\nu} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad \cdot \left(\left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \|u_{1k}\|_{\mathcal{H}_{1,k}} + \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \|u_{2k}\|_{\mathcal{H}_{2,k}} \right) \\
&\le (1+p\xi)M_2 + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \|u_{1k}\|_{\mathcal{H}_{1,k}} \\
&\quad + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \|u_{2k}\|_{\mathcal{H}_{2,k}} \\
&\quad + p\eta_1 C_*^{\nu} M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} + p\eta_2 C_*^{\nu} M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad + (p-1)\eta_1 C_*^{\nu} \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_1 C_*^{\nu} \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} \|u_{2k}\|_{\mathcal{H}_{2,k}} \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_2 C_*^{\nu} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_2 C_*^{\nu} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \|u_{1k}\|_{\mathcal{H}_{1,k}} \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \\
&\le (1+p\xi)M_2 + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \|u_{1k}\|_{\mathcal{H}_{1,k}} \\
&\quad + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \|u_{2k}\|_{\mathcal{H}_{2,k}} \\
&\quad + p\eta_1 C_*^{\nu} M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu} + p\eta_2 C_*^{\nu} M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu} \\
&\quad + (p-1)\eta_1 C_*^{\nu} \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_1 C_*^{\nu} \left(\frac{\nu}{\nu+1} \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} + \frac{1}{\nu+1} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} \right) \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_2 C_*^{\nu} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \\
&\quad + (p-1)\eta_2 C_*^{\nu} \left(\frac{\nu}{\nu+1} \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} + \frac{1}{\nu+1} \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} \right) \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'}. \qquad (3.14)
\end{aligned}
$$

Thus (3.14) and Lemma 3.2 imply that

$$
\begin{aligned}
&(1+p\xi)M_2 + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_1(t) \right|^{p'} \right)^{1/p'} \|u_{1k}\|_{\mathcal{H}_{1,k}} \\
&\quad + \left[(p-1)\xi + 1\right] \left(\sum_{t\in\mathbb{Z}} \left| f_2(t) \right|^{p'} \right)^{1/p'} \|u_{2k}\|_{\mathcal{H}_{2,k}}
\end{aligned}
$$

$$
\begin{aligned}
&+ p\eta_1 C_*^\nu M_2 \|u_{1k}\|_{\mathcal{H}_{1,k}}^\nu + p\eta_2 C_*^\nu M_2 \|u_{2k}\|_{\mathcal{H}_{2,k}}^\nu \\
&+ (p-1)\eta_1 C_*^\nu \|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left|f_1(t)\right|^{p'}\right)^{1/p'} \\
&+ (p-1)\eta_1 C_*^\nu \left(\frac{\nu}{\nu+1}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1} + \frac{1}{\nu+1}\|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1}\right)\left(\sum_{t\in\mathbb{Z}} \left|f_2(t)\right|^{p'}\right)^{1/p'} \\
&+ (p-1)\eta_2 C_*^\nu \|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} \left(\sum_{t\in\mathbb{Z}} \left|f_2(t)\right|^{p'}\right)^{1/p'} \\
&+ (p-1)\eta_2 C_*^\nu \left(\frac{\nu}{\nu+1}\|u_{2k}\|_{\mathcal{H}_{2,k}}^{\nu+1} + \frac{1}{\nu+1}\|u_{1k}\|_{\mathcal{H}_{1,k}}^{\nu+1}\right)\left(\sum_{t\in\mathbb{Z}} \left|f_1(t)\right|^{p'}\right)^{1/p'} \\
\geq{}& \min\left\{d_1\|u_1\|_{\mathcal{H}_{1,k}}^p, a_1 C_*^{\gamma-p}\|u_1\|_{\mathcal{H}_{1,k}}^\gamma\right\} + \min\left\{d_2\|u_2\|_{\mathcal{H}_{2,k}}^p, a_2 C_*^{\gamma-p}\|u_2\|_{\mathcal{H}_{2,k}}^\gamma\right\}.
\end{aligned}
$$

Note that $p > \gamma > \nu + 1$. So (H6) implies there exists $M_3 > 0$ (independent of k) such that

$$\|u_{1k}\|_{\mathcal{H}_{1,k}} \leq M_3, \qquad \|u_{2k}\|_{\mathcal{H}_{2,k}} \leq M_3 \quad \text{for every } k \in \mathbb{N}.$$

By Corollary 2.1,

$$\|u_{1k}\|_{l_{2kT}^\infty} \leq C_* M_3, \qquad \|u_{2k}\|_{l_{2kT}^\infty} \leq C_* M_3 \quad \text{for every } k \in \mathbb{N}.$$

Let $M_1 = \max\{C_{1*}M_3, C_{2*}M_3\}$. Thus the proof is complete. □

Lemma 3.5 *Let* $\{u_k\}$ *be determined by Lemma* 3.4. *Then there exists a subsequence* $\{u_{k_j} = (u_{1k_j}, u_{2k_j})\}$ *of* $\{u_k\}_{k\in\mathbb{N}}$ *convergent to a certain function* $u_\infty = (u_{1\infty}, u_{2\infty})$ *and when* $f_1 \neq 0$ *and* $f_2 \neq 0$, u_∞ *is a nontrivial solution of system* (1.1) *such that* $u_\infty(t) \to 0$ *and* $\Delta u_\infty(t-1) \to 0$ *as* $t \to \pm\infty$.

Proof Note that

$$\|u_{1k}\|_{\mathcal{H}_{1,k}} \leq M_1, \qquad \|u_{2k}\|_{\mathcal{H}_{2,k}} \leq M_1 \quad \text{for every } k \in \mathbb{N}.$$

Then, similar to the argument in [15] or [16], one can prove that $\{u_{mk}\}_{k\in\mathbb{N}}$ has a convergent subsequence $\{u_{mk_j}\}$ such that $u_{mk_j} \to u_{m\infty}$ and $u_{m\infty}(t) \to 0$ and $\Delta u_{m\infty}(t-1) \to 0$ as $t \to \pm\infty$, where $m = 1, 2$. Let $u_\infty = (u_{1\infty}, u_{2\infty})$. By (3.13) and the continuity of Φ_m, $K(t,\cdot,\cdot)$, $W(t,\cdot,\cdot)$ and φ_k', similar to the argument in [15] or [16], the proof is easy to be completed. □

□

Proof of Theorem 1.2 The proof is easy to be completed by replacing

$$
\begin{aligned}
\sum_{t=-kT}^{kT-1} \left(f_m(t), u_m(t)\right) &\leq \left(\sum_{t=-kT}^{kT-1} \left|f_m(t)\right|^{p'}\right)^{1/p'} \left(\sum_{t=-kT}^{kT-1} \left|u_m(t)\right|^p\right)^{1/p} \\
&\leq \|u_m\|_{\mathcal{H}_{m,k}} \left(\sum_{t\in\mathbb{Z}} \left|f_m(t)\right|^{p'}\right)^{1/p'}
\end{aligned}
$$

with

$$\sum_{t=-kT}^{kT-1}\big(f_m(t),u_m(t)\big)\le\|u_m\|_{l^\infty_{2kT}}\sum_{t=-kT}^{kT-1}\big|f_m(t)\big|\le C_*\|u_m\|_{\mathcal{H}_{m,k}}\sum_{t\in\mathbb{Z}}\big|f_m(t)\big|,\quad m=1,2,$$

in the proofs of Lemma 3.3 and Lemma 3.4. □

Proofs of Theorem 1.3 *and Theorem* 1.4 We only note that in the proof of Lemma 3.3, when $\gamma=p$, we do not need to consider the case that $r\in(0,1]$ alone and it is sufficient that $r>0$. Other proofs are the same as those of Theorem 1.1 and Theorem 1.2, respectively. □

4 Examples

We first give two examples about Φ which satisfy assumption $(\mathcal{A}1)$.

(I) An example with $N=1$. Define $\Phi_m:\mathbb{R}\to\mathbb{R}^N$, $m=1,2$, by

$$\Phi_1(x)=\begin{cases}\alpha_1|x|^p, & x\ge 0,\\ \alpha_2|x|^p, & x<0,\end{cases}\qquad \Phi_2(y)=\begin{cases}\beta_1|y|^p, & y\ge 0,\\ \beta_2|y|^p, & y<0,\end{cases}$$

where $\alpha_1,\alpha_2\in[d_1,d_3]$, $\beta_1,\beta_2\in[d_2,d_4]$. Then it is easy to verify that Φ_m, $m=1,2$, satisfies $(\mathcal{A}1)$.

(II) As described in [1], the following classical case with p-Laplacian also satisfies the assumption $(\mathcal{A}1)$. Define $\Phi_m:\mathbb{R}^N\to\mathbb{R}^N$, $m=1,2$, by

$$\Phi_1(x)=\alpha|x|^p,\qquad \Phi_2(y)=\beta|y|^p,$$

where $\alpha\in[d_1,d_3]$, $\beta\in[d_2,d_4]$.

Next, we present some examples of K and W which satisfy those assumptions in Theorem 1.1. There are lots of examples of K. For example, let

$$K(t,x_1,x_2)=a_1(t)|x_1|^\gamma+a_2(t)|x_2|^\gamma,\quad (t,x_1,x_2)\in\mathbb{Z}[0,T-1]\times\mathbb{R}^N\times\mathbb{R}^N,$$

where $\gamma\in(1,p)$, $a_i, i=1,2:\mathbb{Z}\to\mathbb{R}^+$ are T-periodic. Let $a_i=\min_{t\in\mathbb{Z}[0,T-1]}a_i(t)$. Then it is easy to see that K satisfies (H1) and (H2).

For W, we assume that

$$W(t,x_1,x_2)=b(t)\big(|x_1|^p+|x_2|^p\big)\ln\big(|x_1|^p+|x_2|^p+1\big),\quad (t,x_1,x_2)\in\mathbb{Z}[0,T-1]\times\mathbb{R}^N\times\mathbb{R}^N,$$

where $b:\mathbb{Z}\to\mathbb{R}^+$ is T-periodic. Let $b^+=\max_{t\in\mathbb{Z}[0,T-1]}\{b(t)\}$. Then

$$\begin{aligned}&W(t,x_1,x_2)\le b^+\big(|x_1|^p+|x_2|^p\big)\ln\big(|rC_*|^p+|rC_*|^p+1\big)\\ &\quad\text{for all } t\in\mathbb{Z}[0,T-1], |x_1|\le rC_*, |x_2|\le rC_*.\end{aligned}$$

Let $b_1=b_2=b^+\ln(|rC_*|^p+|rC_*|^p+1)$. If r is sufficiently small, then (H3)(i) holds. It is easy to see that

$$\lim_{|x_1|+|x_2|\to+\infty}\frac{W(t,x_1,x_2)}{|x_1|^p+|x_2|^p}=+\infty\quad\text{for all } t\in\mathbb{Z}[0,T-1].$$

So (H4) holds. Let $\nu \in (0, \gamma - 1)$. Note that

$$p\xi\left(|x_1|^p + |x_2|^p\right) \geq \ln\left(|x_1|^p + |x_2|^p + 1\right), \qquad p\left(\eta_1|x_1|^\nu + \eta_2|x_2|^\nu\right) \geq \ln\left(|x_1|^p + |x_2|^p + 1\right)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$, when we choose sufficiently large ξ, η_1 and η_2. Hence

$$\begin{aligned}
&p\xi\left(|x_1|^p + |x_2|^p\right) + p\left(|x_1|^p + |x_2|^p\right)\left(\eta_1|x_1|^\nu + \eta_2|x_2|^\nu\right) \\
&\quad \geq \ln\left(|x_1|^p + |x_2|^p + 1\right) + \ln\left(|x_1|^p + |x_2|^p + 1\right)\left(|x_1|^p + |x_2|^p\right) \\
&\quad \Longleftrightarrow \quad p\left(\xi + \eta_1|x_1|^\nu + \eta_2|x_2|^\nu\right)\left(|x_1|^p + |x_2|^p\right) \\
&\qquad\qquad \geq \ln\left(|x_1|^p + |x_2|^p + 1\right)\left(|x_1|^p + |x_2|^p + 1\right) \\
&\quad \Longleftrightarrow \quad p\left(\xi + \eta_1|x_1|^\nu + \eta_2|x_2|^\nu\right)\left(|x_1|^p + |x_2|^p\right)^2 \\
&\qquad\qquad \geq \left(|x_1|^p + |x_2|^p\right)\ln\left(|x_1|^p + |x_2|^p + 1\right)\left(|x_1|^p + |x_2|^p + 1\right) \\
&\quad \Longleftrightarrow \quad \frac{p(|x_1|^p + |x_2|^p)^2}{|x_1|^p + |x_2|^p + 1} \geq \frac{(|x_1|^p + |x_2|^p)\ln(|x_1|^p + |x_2|^p + 1)}{\xi + \eta_1|x_1|^\nu + \eta_2|x_2|^\nu} \\
&\quad \Longleftrightarrow \quad \left(\nabla_{x_1} W(t, x_1, x_2), x_1\right) + \left(\nabla_{x_2} W(t, x_1, x_2), x_2\right) - pW(t, x_1, x_2) \\
&\qquad\qquad \geq \frac{W(t, x_1, x_2)}{\xi + \eta_1|x_1|^\nu + \eta_2|x_2|^\nu}
\end{aligned}$$

for all $(t, x_1, x_2) \in \mathbb{Z}[0, T-1] \times \mathbb{R}^N \times \mathbb{R}^N$, which implies (H5) holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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