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# On Malmquist type theorem of systems of complex difference equations

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**Abstract**

The main purpose of this paper is to give the Malmquist type result of the meromorphic solutions of a system of complex difference equations of the following form:

$$\begin{cases} \sum_{\lambda_1\in I_1,\mu_1\in J_1}\alpha_{\lambda_1,\mu_1}(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda_1,\nu}}\prod_{\nu=1}^{n}g(z+c_\nu)^{m_{\mu_1,\nu}}) = \frac{\sum_{i=0}^{p}a_i(z)g(z)^i}{\sum_{j=0}^{q}b_j(z)g(z)^j}, \\ \sum_{\lambda_2\in I_2,\mu_2\in J_2}\beta_{\lambda_2,\mu_2}(z)(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda_2,\nu}}\prod_{\nu=1}^{n}g(z+c_\nu)^{m_{\mu_2,\nu}}) = \frac{\sum_{k=0}^{s}d_k(z)f(z)^k}{\sum_{l=0}^{t}e_l(z)f(z)^l}, \end{cases}$$

where $c_1, c_2, \ldots, c_n$ are distinct, nonzero complex numbers, the coefficients $\alpha_{\lambda_1,\mu_1}(z)$ $(\lambda_1 \in I_1, \mu_1 \in J_1)$, $\beta_{\lambda_2,\mu_2}(z)$ $(\lambda_2 \in I_2, \mu_2 \in J_2)$, $a_i(z)$ $(i = 0, 1, \ldots, p)$, $b_j(z)$ $(j = 0, 1, \ldots, q)$, $d_k(z)$ $(k = 0, 1, \ldots, s)$, and $e_l(z)$ $(l = 0, 1, \ldots, t)$ are small functions relative to $f(z)$ and $g(z)$, $I_i = \{\lambda_i = (l_{\lambda_i,1}, l_{\lambda_i,2}, \ldots, l_{\lambda_i,n}) | l_{\lambda_i,\nu} \in N \cup \{0\}, \nu = 1, 2, \ldots, n\}$ $(i = 1, 2)$ and $J_j = \{\mu_j = (m_{\mu_j,1}, m_{\mu_j,2}, \ldots, m_{\mu_j,n}) | m_{\mu_j,\nu} \in N \cup \{0\}, \nu = 1, 2, \ldots, n\}$ $(j = 1, 2)$ are finite index sets. The growth of meromorphic solutions of a related system of complex functional equations is also investigated.

**Keywords:** systems of complex difference equations; meromorphic functions; Malmquist type theorem; functional equation

## 1 Introduction and main results

Let $f(z)$ be a meromorphic function in the complex plane $C$. We assume that the reader is familiar with the standard notations and results in Nevanlinna's value distribution theory of meromorphic functions (see *e.g.* [1–3]). We use $\rho(f)$ to denote the growth order of a meromorphic function $f(z)$. The notation $S(r,f)$ denotes any quantity that satisfies the condition $S(r,f) = o(T(r,f))$ as $r \to \infty$ possibly outside an exceptional set of $r$ of finite logarithmic measure. A meromorphic function $a(z)$ is called a small function of $f(z)$ if and only if $T(r,a(z)) = S(r,f)$.

In the last ten years, there has been a great deal of interest in studying the properties of complex difference equations (see *e.g.* [4–20]). Especially, a number of papers (see *e.g.* [4, 6, 10, 11, 15, 17–19]) focusing on a Malmquist type theorem of the complex difference equations emerged. In 2000, Ablowitz *et al.* [4] proved some results on the Malmquist theorem of the complex difference equations by utilizing Nevanlinna theory. They obtained the following two results.

**Theorem A** *If the second-order difference equation*

$$f(z+1)+f(z-1)=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q},$$

*with polynomial coefficients* $a_i$ $(i=1,2,\ldots,p)$ *and* $b_j$ $(j=1,2,\ldots,q)$, *admits a transcendental meromorphic solution of finite order, then* $d=\max\{p,q\}\leq 2$.

**Theorem B** *If the second-order difference equation*

$$f(z+1)f(z-1)=\frac{a_0(z)+a_1(z)f+\cdots+a_p(z)f^p}{b_0(z)+b_1(z)f+\cdots+b_q(z)f^q},$$

*with polynomial coefficients* $a_i$ $(i=1,2,\ldots,p)$ *and* $b_j$ $(j=1,2,\ldots,q)$, *admits a transcendental meromorphic solution of finite order, then* $d=\max\{p,q\}\leq 2$.

Subsequently, Heittokangas *et al.* [10], Laine *et al.* [15] and Huang *et al.* [11], respectively, gave some generalizations of the above two results. In 2010, the first author in this paper and Liao [18] obtained the following more general result.

**Theorem C** *Let* $c_1,c_2,\ldots,c_n$ *be distinct, nonzero complex numbers, and suppose that* $f(z)$ *is a transcendental meromorphic solution of the difference equation*

$$\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda,\nu}}\right)=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_p(z)f(z)^p}{b_0(z)+b_1(z)f(z)+\cdots+b_q(z)f(z)^q}, \tag{1}$$

*with coefficients* $\alpha_\lambda(z)$ $(\lambda\in I)$, $a_i(z)$ $(i=0,1,\ldots,p)$, *and* $b_j(z)$ $(j=0,1,\ldots,q)$, *which are small functions relative to* $f(z)$, *where* $I=\{\lambda=(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})|l_{\lambda,\nu}\in N\cup\{0\},\nu=1,2,\ldots,n\}$ *is a finite index set, and denote*

$$\sigma_\nu=\max_\lambda\{l_{\lambda,\nu}\}\quad(\nu=1,2,\ldots,n),\qquad \sigma=\sum_{\nu=1}^{n}\sigma_\nu.$$

*If the order* $\rho(f)$ *is finite, then* $d=\max\{p,q\}\leq\sigma$.

If all the coefficients in the complex difference equation (1) are rational functions, then in [19], we have the following Malmquist type result, which is reminiscent of the classical Malmquist theorem in complex differential equations.

**Theorem D** *Let* $c_1,c_2,\ldots,c_n$ *be distinct, nonzero complex numbers and suppose that* $f(z)$ *is a transcendental meromorphic solution of the equation*

$$P[z,f]:=\sum_{\lambda\in I}\alpha_\lambda(z)\left(\prod_{\nu=1}^{n}f(z+c_\nu)^{l_{\lambda,\nu}}\right)=R\bigl(z,f(z)\bigr)=\frac{P(z,f(z))}{Q(z,f(z))},$$

*where* $I=\{\lambda=(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})|l_{\lambda,\nu}\in N\cup\{0\},\nu=1,2,\ldots,n\}$ *is a finite index set,* $P$ *and* $Q$ *are relatively prime polynomials in* $f$ *over the field of rational functions, the coefficients* $\alpha_\lambda$ $(\lambda\in I)$ *are rational functions. Denoting the degree of* $P[z,f]$ *by*

$$\gamma_P:=\max_{\lambda\in I}\bigl\{l_{\lambda,1}+l_{\lambda,2}+\cdots+l_{\lambda,n}|\lambda=(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})\bigr\}.$$

*If $f(z)$ is finite order and has at most finitely many poles, then $R(z,f)$ reduces to a polynomial in $f$ of degree $d \le \gamma_P$.*

More recently, people began to study the properties of meromorphic solutions of systems of complex difference equations. In [21], Gao discussed the proximity function and counting function of meromorphic solutions of some classes of systems of complex difference equations. In 2013, Wang *et al.* [16] investigated the growth of meromorphic solutions of systems of complex difference equations.

Now, we give the Malmquist type result of a system of complex difference equations as follows.

**Theorem 1** *Let $c_1, c_2, \ldots, c_n$ be distinct, nonzero complex numbers and suppose that $(f(z), g(z))$ is a transcendental meromorphic solution of a system of complex difference equations of the form*

$$\begin{cases} \sum_{\lambda_1 \in I_1, \mu_1 \in J_1} \alpha_{\lambda_1,\mu_1}(z)(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_1,\nu}} \prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_1,\nu}}) = \frac{\sum_{i=0}^{p} a_i(z)g(z)^i}{\sum_{j=0}^{q} b_j(z)g(z)^j}, \\ \sum_{\lambda_2 \in I_2, \mu_2 \in J_2} \beta_{\lambda_2,\mu_2}(z)(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_2,\nu}} \prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_2,\nu}}) = \frac{\sum_{k=0}^{s} d_k(z)f(z)^k}{\sum_{l=0}^{t} e_l(z)f(z)^l}, \end{cases} \tag{2}$$

*with coefficients $\alpha_{\lambda_1,\mu_1}(z)$ ($\lambda_1 \in I_1$, $\mu_1 \in J_1$), $\beta_{\lambda_2,\mu_2}(z)$ ($\lambda_2 \in I_2$, $\mu_2 \in J_2$), $a_i(z)$ ($i = 0,1,\ldots,p$), $b_j(z)$ ($j = 0,1,\ldots,q$), $d_k(z)$ ($k = 0,1,\ldots,s$), and $e_l(z)$ ($l = 0,1,\ldots,t$) are small functions relative to $f(z)$ and $g(z)$, $a_p(z), b_q(z), d_s(z), e_t(z) \not\equiv 0$, where $I_i = \{\lambda_i = (l_{\lambda_i,1}, l_{\lambda_i,2}, \ldots, l_{\lambda_i,n}) | l_{\lambda_i,\nu} \in N \cup \{0\}, \nu = 1,2,\ldots,n\}$ ($i = 1,2$), and $J_j = \{\mu_j = (m_{\mu_j,1}, m_{\mu_j,2}, \ldots, m_{\mu_j,n}) | m_{\mu_j,\nu} \in N \cup \{0\}, \nu = 1,2,\ldots,n\}$ ($j = 1,2$) are finite index sets, and denote*

$$\xi_{1,\nu} = \max_{\lambda_1 \in I_1}\{l_{\lambda_1,\nu}\}, \qquad \eta_{1,\nu} = \max_{\mu_1 \in J_1}\{m_{\mu_1,\nu}\},$$

$$\xi_{2,\nu} = \max_{\lambda_2 \in I_2}\{l_{\lambda_2,\nu}\}, \qquad \eta_{2,\nu} = \max_{\mu_2 \in J_2}\{m_{\mu_2,\nu}\}$$

*($\nu = 1,2,\ldots,n$), and*

$$\sigma_{11} = \sum_{\nu=1}^{n} \xi_{1,\nu}, \qquad \sigma_{12} = \sum_{\nu=1}^{n} \eta_{1,\nu}, \qquad \sigma_{21} = \sum_{\nu=1}^{n} \xi_{2,\nu}, \qquad \sigma_{22} = \sum_{\nu=1}^{n} \eta_{2,\nu}.$$

*If $\max\{p,q\} > \sigma_{12}$, $\max\{s,t\} > \sigma_{21}$, and $\max\{\rho(f), \rho(g)\} < +\infty$, then $\rho(f) = \rho(g)$ and $(\max\{p,q\} - \sigma_{12}) \cdot (\max\{s,t\} - \sigma_{21}) \le \sigma_{11}\sigma_{22}$.*

**Example 1** It is easy to check that $(f(z), g(z)) = (\tan z, \cot z)$ satisfies the following system of difference equations:

$$\begin{cases} f(z+\frac{\pi}{4})g(z+\frac{\pi}{3})^2 + zg(z+\frac{\pi}{4}) \\ \quad = \frac{(3z+1)g^4+[(2\sqrt{3}-6)z+2-2\sqrt{3}]g^3+(4-4\sqrt{3})(z+1)g^2+[(2\sqrt{3}-2)z+6-2\sqrt{3}]g+z+3}{3g^4+2\sqrt{3}g^3-2g^2-2\sqrt{3}g-1}, \\ f(z+\frac{\pi}{3})^2 g(z+\frac{\pi}{4}) + f(z+\frac{\pi}{3})g(z+\frac{\pi}{4})^2 = \frac{-(\sqrt{3}+1)f^4-2f^3+2f^2-2f+3+\sqrt{3}}{3f^4+(6-2\sqrt{3})f^3+(4-4\sqrt{3})f^2+(2-2\sqrt{3})f+1}. \end{cases}$$

In Example 1, we have $\max\{p,q\} = 4$, $\max\{s,t\} = 4$, $\sigma_{11} = 1$, $\sigma_{12} = 3$, $\sigma_{21} = 2$, $\sigma_{22} = 2$, $\rho(f) = \rho(g) = 1 < +\infty$, and $(\max\{p,q\} - \sigma_{12}) \cdot (\max\{s,t\} - \sigma_{21}) = (4-3)(4-2) = 2 = 1 \times 2 = \sigma_{11}\sigma_{22}$. Therefore, the estimation in Theorem 1 is sharp.

**Remark 1** Obviously, if the condition $\max\{p,q\} > \sigma_{12}$, $\max\{s,t\} > \sigma_{21}$ in Theorem 1 is replaced by $(\max\{p,q\} - \sigma_{12})(\max\{s,t\} - \sigma_{21}) = 0$ or $(\max\{p,q\} - \sigma_{12})(\max\{s,t\} - \sigma_{21}) < 0$, the estimation $(\max\{p,q\} - \sigma_{12}) \cdot (\max\{s,t\} - \sigma_{21}) \le \sigma_{11}\sigma_{22}$ is still correct. If $\sigma_{11} = 0$ or $\sigma_{22} = 0$, then the first or second equation in (2) gets the form of (1). For some results as regards (1), the reader may refer to the paper [18].

**Remark 2** If the condition $\max\{p,q\} > \sigma_{12}$, $\max\{s,t\} > \sigma_{21}$ in Theorem 1 is replaced by $\max\{p,q\} < \sigma_{12}$, $\max\{s,t\} < \sigma_{21}$, then the estimation $(\max\{p,q\} - \sigma_{12}) \cdot (\max\{s,t\} - \sigma_{21}) \le \sigma_{11}\sigma_{22}$ is not true generally. For example, $(f(z), g(z)) = (\tan z, \cot z)$ satisfies the following system of complex difference equations:

$$\begin{cases} f(z+\frac{\pi}{4})^2 g(z-\frac{\pi}{4}) g(z+\frac{\pi}{4})^5 + 2g(z+\frac{\pi}{4})^2 = \frac{g^2-2g+1}{g^2+2g+1}, \\ f(z+\frac{\pi}{4})^4 f(z-\frac{\pi}{4}) g(z+\frac{\pi}{4}) + zf(z+\frac{\pi}{4}) = \frac{-(z+1)f^2-2f+z+1}{f^2-2f+1}, \end{cases} \tag{3}$$

where $\max\{p,q\} = 2$, $\max\{s,t\} = 2$, $\sigma_{11} = 2$, $\sigma_{12} = 6$, $\sigma_{21} = 5$, $\sigma_{22} = 1$, $\max\{p,q\} < \sigma_{12}$, $\max\{s,t\} < \sigma_{21}$. However, $(\max\{p,q\} - \sigma_{12}) \cdot (\max\{s,t\} - \sigma_{21}) = (-4) \times (-3) = 12 > 2 = \sigma_{11}\sigma_{22}$.

If $\sigma_{12} = \sigma_{21} = 0$, then we have the following simpler result.

**Corollary 1** *Let $c_1, c_2, \dots, c_n$ be distinct, nonzero complex numbers, and suppose that $(f(z), g(z))$ is a transcendental meromorphic solution of a system of complex difference equations of the form*

$$\begin{cases} \sum_{\lambda \in I} \alpha_\lambda(z) (\prod_{\nu=1}^n f(z+c_\nu)^{l_{\lambda,\nu}}) = \frac{\sum_{i=0}^p a_i(z) g(z)^i}{\sum_{j=0}^q b_j(z) g(z)^j}, \\ \sum_{\mu \in J} \beta_\mu(z) (\prod_{\nu=1}^n g(z+c_\nu)^{m_{\mu,\nu}}) = \frac{\sum_{k=0}^s d_k(z) f(z)^k}{\sum_{l=0}^t e_l(z) f(z)^l}, \end{cases} \tag{4}$$

*with coefficients $\alpha_\lambda(z)$ ($\lambda \in I$), $\beta_\mu(z)$ ($\mu \in J$), $a_i(z)$ ($i = 0, 1, \dots, p$), $b_j(z)$ ($j = 0, 1, \dots, q$), $d_k(z)$ ($k = 0, 1, \dots, s$), and $e_l(z)$ ($l = 0, 1, \dots, t$) are small functions relative to $f(z)$ and $g(z)$, $a_p(z), b_q(z), d_s(z), e_t(z) \not\equiv 0$, where $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \dots, l_{\lambda,n}) | l_{\lambda,\nu} \in N \cup \{0\}, \nu = 1, 2, \dots, n\}$ and $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \dots, m_{\mu,n}) | m_{\mu,\nu} \in N \cup \{0\}, \nu = 1, 2, \dots, n\}$ are two finite index sets, and denote*

$$\xi_\nu = \max_\lambda \{l_{\lambda,\nu}\} \quad (\nu = 1, 2, \dots, n), \qquad \sigma_1 = \sum_{\nu=1}^n \xi_\nu$$

*and*

$$\eta_\nu = \max_\mu \{m_{\mu,\nu}\} \quad (\nu = 1, 2, \dots, n), \qquad \sigma_2 = \sum_{\nu=1}^n \eta_\nu.$$

*If $\rho(f) < +\infty$ or $\rho(g) < +\infty$, then $\rho(f) = \rho(g)$ and $\max\{p,q\} \cdot \max\{s,t\} \le \sigma_1\sigma_2$.*

**Example 2** Let $c_1 = \arctan 2$, $c_2 = \arctan(-2)$. It is easy to check that $(f(z), g(z)) = (\tan z, \cot z)$ satisfies the following system of difference equations:

$$\begin{cases} f(z+c_1)^2 f(z+c_2) + f(z+c_1) f(z+c_2)^2 = \frac{-40g^3+10g}{g^4-8g^2+16}, \\ g(z+c_1) g(z+c_2) + g(z+c_1)^2 = \frac{-20f^2+10f}{f^3+2f^2-4f-8}. \end{cases}$$

In Example 2, we have $\max\{p,q\} = 4$, $\max\{s,t\} = 3$, $\sigma_1 = 4$, $\sigma_2 = 3$, $\rho(f) = \rho(g) = 1 < +\infty$, and $\max\{p,q\} \cdot \max\{s,t\} = \sigma_1 \cdot \sigma_2 = 12$. Therefore, the estimation in Corollary 1 is sharp.

In [15], Laine *et al.* also considered the growth of meromorphic solutions of some classes of complex difference functional equations and obtained the following result.

**Theorem E** *Suppose that $f$ is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z)\biggl(\prod_{j\in J} f(z+c_j)\biggr) = f\bigl(p(z)\bigr),$$

*where $p(z)$ is a polynomial of degree $k \geq 2$, $\{J\}$ is the collection of all subsets of $\{1,2,\ldots,n\}$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to $f$ and that $n \geq k$. Then*

$$T(r,f) = O\bigl((\log r)^{\alpha+\varepsilon}\bigr),$$

*where $\alpha = \frac{\log n}{\log k}$.*

In 2010, Zhang *et al.* [18] got a more generalized result than Theorem E. Next we give the growth of meromorphic solutions of a system of complex functional equations as follows.

**Theorem 2** *Let $c_1, c_2, \ldots, c_n$ be distinct, nonzero complex numbers and suppose that $(f(z), g(z))$ is a transcendental meromorphic solution of a system of complex functional equations of the form*

$$\begin{cases} \sum_{\lambda_1\in I_1,\mu_1\in J_1} \alpha_{\lambda_1,\mu_1}(z)(\prod_{\nu=1}^n f(z+c_\nu)^{l_{\lambda_1,\nu}} \prod_{\nu=1}^n g(z+c_\nu)^{m_{\mu_1,\nu}}) = f(p(z)), \\ \sum_{\lambda_2\in I_2,\mu_2\in J_2} \beta_{\lambda_2,\mu_2}(z)(\prod_{\nu=1}^n f(z+c_\nu)^{l_{\lambda_2,\nu}} \prod_{\nu=1}^n g(z+c_\nu)^{m_{\mu_2,\nu}}) = g(p(z)), \end{cases} \tag{5}$$

*where $p(z)$ is a polynomial of degree $k \geq 2$, $I_i = \{\lambda_i = (l_{\lambda_i,1}, l_{\lambda_i,2}, \ldots, l_{\lambda_i,n}) | l_{\lambda_i,\nu} \in N \cup \{0\}, \nu = 1,2,\ldots,n\}$ $(i = 1,2)$ and $J_j = \{\mu_j = (m_{\mu_j,1}, m_{\mu_j,2}, \ldots, m_{\mu_j,n}) | m_{\mu_j,\nu} \in N \cup \{0\}, \nu = 1,2,\ldots,n\}$ $(j = 1,2)$ are finite index sets, and denote*

$$\xi_{1,\nu} = \max_{\lambda_1\in I_1}\{l_{\lambda_1,\nu}\}, \qquad \eta_{1,\nu} = \max_{\mu_1\in J_1}\{m_{\mu_1,\nu}\},$$

$$\xi_{2,\nu} = \max_{\lambda_2\in I_2}\{l_{\lambda_2,\nu}\}, \qquad \eta_{2,\nu} = \max_{\mu_2\in J_2}\{m_{\mu_2,\nu}\}$$

$(\nu = 1,2,\ldots,n)$,

$$\sigma_{11} = \sum_{\nu=1}^n \xi_{1,\nu}, \qquad \sigma_{12} = \sum_{\nu=1}^n \eta_{1,\nu}, \qquad \sigma_{21} = \sum_{\nu=1}^n \xi_{2,\nu}, \qquad \sigma_{22} = \sum_{\nu=1}^n \eta_{2,\nu}$$

*and*

$$\sigma = \max\{\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}\}.$$

*Moreover, we assume that the coefficients* $\alpha_{\lambda_1,\mu_1}(z)$ $(\lambda_1 \in I_1,\ \mu_1 \in J_1)$, $\beta_{\lambda_2,\mu_2}(z)$ $(\lambda_2 \in I_2,\ \mu_2 \in J_2)$ *are small functions relative to* $f(z)$ *and* $g(z)$, *and that* $2\sigma \geq k$. *Then*

$$T(r,f) = O\bigl((\log r)^{\alpha+\varepsilon}\bigr), \qquad T(r,g) = O\bigl((\log r)^{\alpha+\varepsilon}\bigr),$$

*where* $\alpha = \frac{\log 2\sigma}{\log k}$.

If $\sigma_{12} = \sigma_{21} = 0$, then we can obtain the following result easily.

**Corollary 2** *Let* $c_1, c_2, \ldots, c_n$ *be distinct, nonzero complex numbers and suppose that* $(f(z), g(z))$ *is a transcendental meromorphic solution of a system of complex functional equations of the form*

$$\begin{cases} \sum_{\lambda\in I} \alpha_\lambda(z)(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda,\nu}}) = g(p(z)), \\ \sum_{\mu\in J} \beta_\mu(z)(\prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu,\nu}}) = f(p(z)), \end{cases} \tag{6}$$

*where* $p(z)$ *is a polynomial of degree* $k \geq 2$, $I = \{\lambda = (l_{\lambda,1}, l_{\lambda,2}, \ldots, l_{\lambda,n}) | l_{\lambda,\nu} \in N \cup \{0\}, \nu = 1, 2, \ldots, n\}$ *and* $J = \{\mu = (m_{\mu,1}, m_{\mu,2}, \ldots, m_{\mu,n}) | m_{\mu,\nu} \in N \cup \{0\}, \nu = 1, 2, \ldots, n\}$ *are two finite index sets, and denote*

$$\xi_\nu = \max_\lambda\{l_{\lambda,\nu}\} \quad (\nu = 1, 2, \ldots, n), \qquad \sigma_1 = \sum_{\nu=1}^{n} \xi_\nu,$$

$$\eta_\nu = \max_\mu\{m_{\mu,\nu}\} \quad (\nu = 1, 2, \ldots, n), \qquad \sigma_2 = \sum_{\nu=1}^{n} \eta_\nu$$

*and*

$$\sigma = \max\{\sigma_1, \sigma_2\}.$$

*Moreover, we assume that the coefficients* $\alpha_\lambda(z)$ $(\lambda \in I)$, $\beta_\mu(z)$ $(\mu \in J)$ *are small functions relative to* $f(z)$ *and* $g(z)$, *and that* $2\sigma \geq k$. *Then*

$$T(r,f) = O\bigl((\log r)^{\alpha+\varepsilon}\bigr), \qquad T(r,g) = O\bigl((\log r)^{\alpha+\varepsilon}\bigr),$$

*where* $\alpha = \frac{\log 2\sigma}{\log k}$.

## 2 Some lemmas

In order to prove our results, we need the following lemmas.

**Lemma 1** (see [3]) *Let* $f(z)$ *be a meromorphic function. Then for all irreducible rational functions in* $f$,

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j},$$

*such that the meromorphic coefficients* $a_i(z)$, $b_j(z)$ *satisfy*

$$\begin{cases} T(r,a_i) = S(r,f), & i = 0, 1, \ldots, p, \\ T(r,b_j) = S(r,f), & j = 0, 1, \ldots, q, \end{cases}$$

*we have*

$$T\big(r,R(z,f)\big)=\max\{p,q\}\cdot T(r,f)+S(r,f).$$

In [22], AZ Mokhon'ko and VD Mokhon'ko gave an estimation of Nevanlinna's characteristic function of

$$F(z)=\frac{P(z)}{Q(z)}=\frac{\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}}{\sum_{\mu\in J}f_1^{m_{\mu,1}}f_2^{m_{\mu,2}}\cdots f_n^{m_{\mu,n}}},$$

where $f_1,f_2,\ldots,f_n$ are distinct meromorphic functions, $I=\{\lambda=(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})|l_{\lambda,\nu}\in N\cup\{0\},\nu=1,2,\ldots,n\}$ and $J=\{\mu=(m_{\mu,1},m_{\mu,2},\ldots,m_{\mu,n})|m_{\mu,\nu}\in N\cup\{0\},\nu=1,2,\ldots,n\}$ are two finite index sets. However, the method of the proof was too complex. For $F(z)$ of the form $\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}$, Zheng *et al.* [20] gave a simpler proof, but the estimation of $T(r,F)$ was not sharp. For completeness, we give the proof of the following lemma.

**Lemma 2** *Let $f_1,f_2,\ldots,f_n$ be distinct meromorphic functions. Then*

$$T\left(r,\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}\right)\le\sum_{j=1}^{n}\sigma_jT(r,f_j)+\log t,$$

*where $I=\{(l_{\lambda,1},l_{\lambda,2},\ldots,l_{\lambda,n})|l_{\lambda,j}\in N\cup\{0\},j=1,2,\ldots,n\}$ is an finite index set consisting of $t$ elements and $\sigma_j=\max_{\lambda\in I}\{l_{\lambda,j}\}$ $(j=1,2,\ldots,n)$.*

*Proof* All the poles of the function $\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}$ are generated by the poles of the functions $f_j$ $(j=1,2,\ldots,n)$, and every pole of multiplicity $k$ of $f_j$ $(j=1,2,\ldots,n)$ has order at most $k\sigma_j$. This implies that

$$n\left(r,\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}\right)\le\sum_{j=1}^{n}\sigma_jn(r,f_j).$$

Thus we obtain

$$N\left(r,\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}\right)\le\sum_{j=1}^{n}\sigma_jN(r,f_j).\tag{7}$$

We next prove that

$$m\left(r,\sum_{\lambda\in I}f_1^{l_{\lambda,1}}f_2^{l_{\lambda,2}}\cdots f_n^{l_{\lambda,n}}\right)\le\sum_{j=1}^{n}\sigma_jm(r,f_j)+\log t,\tag{8}$$

and we define

$$\begin{cases}f_j^*(z)=f_j(z), & |f_j(z)|>1,\\ f_j^*(z)=1, & |f_j(z)|\le1,\end{cases}$$

for $j = 1, 2, \ldots, n$. Thus we have

$$\begin{aligned}\left|\sum_{\lambda\in I} f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}\right| &\le \sum_{\lambda\in I} \left|f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}\right| \\ &\le \sum_{\lambda\in I} \left|f_1^{*l_{\lambda,1}} f_2^{*l_{\lambda,2}} \cdots f_n^{*l_{\lambda,n}}\right| \\ &= \left|f_1^{*\sigma_1} f_2^{*\sigma_2} \cdots f_n^{*\sigma_n}\right| \left(\sum_{\lambda\in I} \frac{|f_1^{*l_{\lambda,1}} f_2^{*l_{\lambda,2}} \cdots f_n^{*l_{\lambda,n}}|}{|f_1^{*\sigma_1} f_2^{*\sigma_2} \cdots f_n^{*\sigma_n}|}\right) \\ &\le t\left|f_1^{*\sigma_1} f_2^{*\sigma_2} \cdots f_n^{*\sigma_n}\right|.\end{aligned}$$

By the definition of $m(r,f)$, we immediately conclude that

$$\begin{aligned} m\left(r, \sum_{\lambda\in I} f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|\sum_{\lambda\in I} f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}\right| d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f_1^{*\sigma_1} f_2^{*\sigma_2} \cdots f_n^{*\sigma_n}\right| d\theta + \log t \\ &= \sum_{j=1}^{n} \sigma_j m(r, f_j) + \log t. \end{aligned}$$

By (7) and (8), the assertion follows. □

**Remark 3** If we suppose that $\alpha_\lambda(z) = o(T(r,f_j))$ $(\lambda \in I)$ hold for all $j \in \{1, 2, \ldots, n\}$, and denote $T(r, a_\lambda) = S(r,f)$ $(\lambda \in I)$, then we have the following estimation:

$$T\left(r, \sum_{\lambda\in I} \alpha_\lambda(z) f_1^{l_{\lambda,1}} f_2^{l_{\lambda,2}} \cdots f_n^{l_{\lambda,n}}\right) \le \sum_{j=1}^{n} \sigma_j T(r, f_j) + S(r,f).$$

**Lemma 3** (see [6]) *Let $f(z)$ be a meromorphic function with order $\rho = \rho(f)$, $\rho < +\infty$, and $c$ be a fixed non zero complex number, then for each $\varepsilon > 0$, we have*

$$T\bigl(r, f(z+c)\bigr) = T(r,f) + O\bigl(r^{\rho-1+\varepsilon}\bigr) + O(\log r).$$

**Lemma 4** (see [3]) *Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set $E$ of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \le h(\alpha r)$ for all $r_0$.*

**Lemma 5** (see [23]) *Let $f$ be a transcendental meromorphic function, and $p(z) = a_k z^k + a_{k-1} z^{k-1} + \cdots + a_1 z + a_0$, $a_k \ne 0$, be a nonconstant polynomial of degree $k$. Given $0 < \delta < |a_k|$, denote $\lambda = |a_k| + \delta$ and $\mu = |a_k| - \delta$. Then given $\varepsilon > 0$ and $a \in C \cup \{\infty\}$, we have*

$$\begin{aligned} &kn\bigl(\mu r^k, a, f\bigr) \le n\bigl(r, a, f\bigl(p(z)\bigr)\bigr) \le kn\bigl(\lambda r^k, a, f\bigr), \\ &N\bigl(\mu r^k, a, f\bigr) + O(\log r) \le N\bigl(r, a, f\bigl(p(z)\bigr)\bigr) \le N\bigl(\lambda r^k, a, f\bigr) + O(\log r), \\ &(1-\varepsilon) T\bigl(\mu r^k, f\bigr) \le T\bigl(r, f\bigl(p(z)\bigr)\bigr) \le (1+\varepsilon) T\bigl(\lambda r^k, f\bigr), \end{aligned}$$

*for all $r$ large enough.*

## 3 Proofs of theorems

*Proof of Theorem* 1 We assume that $(f(z),g(z))$ is a transcendental meromorphic solution of the system of complex difference equations (2). By the first equation in (2), Lemma 1, Lemma 2, and Lemma 3, we have, for each $\varepsilon > 0$,

$$\begin{aligned}
&\max\{p,q\}T(r,g)\\
&\quad= T\Biggl(r, \sum_{\lambda_1\in I_1,\mu_1\in J_1}\alpha_{\lambda_1,\mu_1}(z)\Biggl(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_1,\nu}}\prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_1,\nu}}\Biggr)\Biggr) + S(r,g)\\
&\quad\le \sum_{\nu=1}^{n}\xi_{1,\nu}T\bigl(r,f(z+c_\nu)\bigr) + \sum_{\nu=1}^{n}\eta_{1,\nu}T\bigl(r,g(z+c_\nu)\bigr) + S(r,f) + S(r,g)\\
&\quad= \sum_{\nu=1}^{n}\xi_{1,\nu}T\bigl(r,f(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + \sum_{\nu=1}^{n}\eta_{1,\nu}T\bigl(r,g(z)\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr)\\
&\qquad + O(\log r) + S(r,f) + S(r,g)\\
&\quad= \Biggl(\sum_{\nu=1}^{n}\xi_{1,\nu}\Biggr)T\bigl(r,f(z)\bigr) + \Biggl(\sum_{\nu=1}^{n}\eta_{1,\nu}\Biggr)T\bigl(r,g(z)\bigr)\\
&\qquad + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr) + O(\log r) + S(r,f) + S(r,g)\\
&\quad= \sigma_{11}T\bigl(r,f(z)\bigr) + \sigma_{12}T\bigl(r,g(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr)\\
&\qquad + O(\log r) + S(r,f) + S(r,g).
\end{aligned}\tag{9}$$

By the above inequality, we get, for each $\varepsilon > 0$,

$$\begin{aligned}
&\bigl(\max\{p,q\} - \sigma_{12}\bigr)T(r,g)\\
&\quad\le \sigma_{11}T\bigl(r,f(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr)\\
&\qquad + O(\log r) + S(r,f) + S(r,g).
\end{aligned}\tag{10}$$

Since $\max\{p,q\} > \sigma_{12}$ by the assumption, we have, for each $\varepsilon > 0$,

$$\begin{aligned}
T(r,g) \le{}& \frac{\sigma_{11}}{\max\{p,q\} - \sigma_{12}}T\bigl(r,f(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr)\\
&+ O(\log r) + S(r,f) + S(r,g).
\end{aligned}\tag{11}$$

Similarly, by the second equation in (2), we obtain, for each $\varepsilon > 0$,

$$\begin{aligned}
&\max\{s,t\}T(r,f)\\
&\quad= T\Biggl(r, \sum_{\lambda_2\in I_2,\mu_2\in J_2}\beta_{\lambda_2,\mu_2}(z)\Biggl(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_2,\nu}}\prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_2,\nu}}\Biggr)\Biggr) + S(r,f)\\
&\quad\le \sum_{\nu=1}^{n}\xi_{2,\nu}T\bigl(r,f(z+c_\nu)\bigr) + \sum_{\nu=1}^{n}\eta_{2,\nu}T\bigl(r,g(z+c_\nu)\bigr) + S(r,f) + S(r,g)\\
&\quad= \sum_{\nu=1}^{n}\xi_{2,\nu}T\bigl(r,f(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + \sum_{\nu=1}^{n}\eta_{2,\nu}T\bigl(r,g(z)\bigr)
\end{aligned}$$

$$\begin{aligned}
&+ O\bigl(r^{\rho(g)-1+\varepsilon}\bigr) + O(\log r) + S(r,f) + S(r,g) \\
&= \left(\sum_{\nu=1}^{n} \xi_{2,\nu}\right) T\bigl(r,f(z)\bigr) + \left(\sum_{\nu=1}^{n} \eta_{2,\nu}\right) T\bigl(r,g(z)\bigr) \\
&\quad + O(\log r) + S(r,f) + S(r,g) \\
&= \sigma_{21} T\bigl(r,f(z)\bigr) + \sigma_{22} T\bigl(r,g(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr) \\
&\quad + O(\log r) + S(r,f) + S(r,g).
\end{aligned} \tag{12}$$

By (12) and $\max\{s,t\} > \sigma_{21}$, we have, for each $\varepsilon > 0$,

$$\begin{aligned}
&\bigl(\max\{s,t\} - \sigma_{21}\bigr) T(r,f) \\
&\quad \leq \sigma_{22} T\bigl(r,g(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr) \\
&\qquad + O(\log r) + S(r,f) + S(r,g)
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
T(r,f) \leq{}& \frac{\sigma_{22}}{\max\{s,t\} - \sigma_{21}} T\bigl(r,g(z)\bigr) + O\bigl(r^{\rho(f)-1+\varepsilon}\bigr) + O\bigl(r^{\rho(g)-1+\varepsilon}\bigr) \\
&+ O(\log r) + S(r,f) + S(r,g).
\end{aligned} \tag{14}$$

Using (11), we can obtain $\rho(g) \leq \rho(f)$. Similarly, we can get $\rho(f) \leq \rho(g)$ from (14). Therefore, we have $\rho(f) = \rho(g)$.

It follows from (10) and (13) that

$$\begin{aligned}
&\bigl(\max\{p,q\} - \sigma_{12}\bigr)\bigl(\max\{s,t\} - \sigma_{21}\bigr) T(r,f) T(r,g) \\
&\quad \leq \sigma_{11}\sigma_{22} T\bigl(r,f(z)\bigr) T\bigl(r,g(z)\bigr) + o\bigl(T(r,f)T(r,g)\bigr).
\end{aligned} \tag{15}$$

From (15), we conclude that

$$\bigl(\max\{p,q\} - \sigma_{12}\bigr) \cdot \bigl(\max\{s,t\} - \sigma_{21}\bigr) \leq \sigma_{11}\sigma_{22}.$$

This yields the asserted result. □

*Proof of Theorem* 2 We assume that $(f(z),g(z))$ is a transcendental meromorphic solution of a system of complex functional equations (5). Let $C = \max\{|c_1|, |c_2|, \ldots, |c_n|\}$. According to the first equation in (5), Lemma 2, Lemma 3, and the last assertion of Lemma 5, we get

$$\begin{aligned}
&(1-\varepsilon) T\bigl(\mu r^k, f\bigr) \\
&\quad \leq T\bigl(r, f\bigl(p(z)\bigr)\bigr) \\
&\quad = T\left(r, \sum_{\lambda_1 \in I_1, \mu_1 \in J_1} \alpha_{\lambda_1,\mu_1}(z) \left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_1,\nu}} \prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_1,\nu}}\right)\right) \\
&\quad \leq \sum_{\nu=1}^{n} \xi_{1,\nu} T\bigl(r, f(z+c_\nu)\bigr) + \sum_{\nu=1}^{n} \eta_{1,\nu} T\bigl(r, g(z+c_\nu)\bigr) + S(r,f) + S(r,g)
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{\nu=1}^{n} \xi_{1,\nu} T\bigl(r+C,f(z)\bigr) + \sum_{\nu=1}^{n} \eta_{1,\nu} T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g) \\
&= \left(\sum_{\nu=1}^{n} \xi_{1,\nu}\right) T\bigl(r+C,f(z)\bigr) + \left(\sum_{\nu=1}^{n} \eta_{1,\nu}\right) T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g) \\
&= \sigma_{11} T\bigl(r+C,f(z)\bigr) + \sigma_{12} T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g).
\end{aligned}$$

Since $T(r+C,f) \leq T(\beta r,f)$ and $T(r+C,g) \leq T(\beta r,g)$ hold for $r$ large enough for $\beta > 1$, we may assume $r$ to be large enough to satisfy

$$(1-\varepsilon)T\bigl(\mu r^k,f\bigr) \leq \sigma_{11}(1+\varepsilon)T(\beta r,f) + \sigma_{12}(1+\varepsilon)T(\beta r,g)$$

outside a possible exceptional set of finite linear measure. By Lemma 4, we know that, whenever $\gamma > 1$,

$$(1-\varepsilon)T\bigl(\mu r^k,f\bigr) \leq \sigma_{11}(1+\varepsilon)T(\gamma\beta r,f) + \sigma_{12}(1+\varepsilon)T(\gamma\beta r,g) \tag{16}$$

holds for all $r$ large enough. Let $t = \gamma\beta r$, then the inequality (16) may be written in the form

$$T\left(\frac{\mu}{(\gamma\beta)^k}t^k,f\right) \leq \frac{\sigma_{11}(1+\varepsilon)}{1-\varepsilon}T(t,f) + \frac{\sigma_{12}(1+\varepsilon)}{1-\varepsilon}T(t,g). \tag{17}$$

Similarly, by the second equation in (5), for all $r$ large enough and $\beta > 1$, $\gamma > 1$, we have

$$\begin{aligned}
&(1-\varepsilon)T\bigl(\mu r^k,g\bigr) \\
&\quad\leq T\bigl(r,g\bigl(p(z)\bigr)\bigr) \\
&\quad= T\left(r, \sum_{\lambda_2\in I_2,\mu_2\in J_2} \beta_{\lambda_2,\mu_2}(z)\left(\prod_{\nu=1}^{n} f(z+c_\nu)^{l_{\lambda_2,\nu}} \prod_{\nu=1}^{n} g(z+c_\nu)^{m_{\mu_2,\nu}}\right)\right) \\
&\quad\leq \sum_{\nu=1}^{n} \xi_{2,\nu} T\bigl(r,f(z+c_\nu)\bigr) + \sum_{\nu=1}^{n} \eta_{2,\nu} T\bigl(r,g(z+c_\nu)\bigr) + S(r,f) + S(r,g) \\
&\quad\leq \sum_{\nu=1}^{n} \xi_{2,\nu} T\bigl(r+C,f(z)\bigr) + \sum_{\nu=1}^{n} \eta_{2,\nu} T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g) \\
&\quad= \left(\sum_{\nu=1}^{n} \xi_{2,\nu}\right) T\bigl(r+C,f(z)\bigr) + \left(\sum_{\nu=1}^{n} \eta_{2,\nu}\right) T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g) \\
&\quad= \sigma_{21} T\bigl(r+C,f(z)\bigr) + \sigma_{22} T\bigl(r+C,g(z)\bigr) + S(r,f) + S(r,g) \\
&\quad\leq \sigma_{21}(1+\varepsilon)T(\gamma\beta r,f) + \sigma_{22}(1+\varepsilon)T(\gamma\beta r,g).
\end{aligned}$$

Let $t = \gamma\beta r$, we have

$$T\left(\frac{\mu}{(\gamma\beta)^k}t^k,g\right) \leq \frac{\sigma_{21}(1+\varepsilon)}{1-\varepsilon}T(t,f) + \frac{\sigma_{22}(1+\varepsilon)}{1-\varepsilon}T(t,g). \tag{18}$$

Letting $s = \log t + \frac{\log \frac{\mu}{(\gamma\beta)^k}}{k-1}$, then $t = e^s(\frac{\mu}{(\gamma\beta)^k})^{\frac{1}{1-k}}$ and

$$\begin{aligned} ks &= k\log t + \frac{k}{k-1}\log\frac{\mu}{(\gamma\beta)^k} \\ &= k\log t + \log\frac{\mu}{(\gamma\beta)^k} + \frac{\log\frac{\mu}{(\gamma\beta)^k}}{k-1} \\ &= \log\frac{\mu}{(\gamma\beta)^k}t^k + \log\left(\frac{\mu}{(\gamma\beta)^k}\right)^{\frac{1}{k-1}}. \end{aligned}$$

So

$$\frac{\mu}{(\gamma\beta)^k}t^k = e^{ks}\left(\frac{\mu}{(\gamma\beta)^k}\right)^{\frac{1}{1-k}}. \tag{19}$$

Let $T(t,f) = T(e^s(\frac{\mu}{(\gamma\beta)^k})^{\frac{1}{1-k}},f) = \Phi(s,f)$, $T(t,g) = T(e^s(\frac{\mu}{(\gamma\beta)^k})^{\frac{1}{1-k}},g) = \Phi(s,g)$, $M = \max\{\frac{\sigma_{11}(1+\varepsilon)}{1-\varepsilon}, \frac{\sigma_{12}(1+\varepsilon)}{1-\varepsilon}, \frac{\sigma_{21}(1+\varepsilon)}{1-\varepsilon}, \frac{\sigma_{22}(1+\varepsilon)}{1-\varepsilon}\}$, then from (17) and (19), we have

$$\begin{aligned} \Phi(ks,f) &= T\left(e^{ks}\left(\frac{\mu}{(\gamma\beta)^k}\right)^{\frac{1}{1-k}},f\right) \\ &= T\left(\frac{\mu}{(\gamma\beta)^k}t^k,f\right) \\ &\le \frac{\sigma_{11}(1+\varepsilon)}{1-\varepsilon}T(t,f) + \frac{\sigma_{12}(1+\varepsilon)}{1-\varepsilon}T(t,g) \\ &\le M\Phi(s,f) + M\Phi(s,g). \end{aligned} \tag{20}$$

Similarly, from (18), we can get

$$\Phi(ks,g) \le M\Phi(s,f) + M\Phi(s,g). \tag{21}$$

The inequalities (20) and (21) hold for all $s$ large enough.

Letting now $\alpha = \frac{\log 2M}{\log k}$, namely, $2M = k^\alpha$. Write $\Psi(s,f) = \frac{\Phi(s,f)}{s^\alpha}$ and $\Psi(s,g) = \frac{\Phi(s,g)}{s^\alpha}$, thus we have

$$\Psi(ks,f) \le \frac{1}{2}\Psi(s,f) + \frac{1}{2}\Psi(s,g) \tag{22}$$

and

$$\Psi(ks,g) \le \frac{1}{2}\Psi(s,f) + \frac{1}{2}\Psi(s,g). \tag{23}$$

The inequalities (22) and (23) hold for all $s$ large enough, we may assume that (22) and (23) hold for all $s \ge s_0$.

Let $M_1 = \sup_{s_0\le s\le ks_0}\Psi(s,f)$ and $M_2 = \sup_{s_0\le s\le ks_0}\Psi(s,g)$, then by (22) and (23) we can obtain

$$\begin{aligned} \sup_{s\in[ks_0,k^2s_0]}\Psi(s,f) &= \sup_{s\in[s_0,ks_0]}\Psi(ks,f) \le \frac{1}{2}\sup_{s\in[s_0,ks_0]}\Psi(s,f) + \frac{1}{2}\sup_{s\in[s_0,ks_0]}\Psi(s,g) \le \frac{M_1}{2} + \frac{M_2}{2}, \\ \sup_{s\in[ks_0,k^2s_0]}\Psi(s,g) &= \sup_{s\in[s_0,ks_0]}\Psi(ks,g) \le \frac{1}{2}\sup_{s\in[s_0,ks_0]}\Psi(s,f) + \frac{1}{2}\sup_{s\in[s_0,ks_0]}\Psi(s,g) \le \frac{M_1}{2} + \frac{M_2}{2}. \end{aligned}$$

Similarly, we have

$$\sup_{s\in[k^2s_0,k^3s_0]}\Psi(s,f)=\sup_{s\in[ks_0,k^2s_0]}\Psi(ks,f)\le\frac{1}{2}M_1+\frac{1}{2}M_2,$$
$$\sup_{s\in[k^2s_0,k^3s_0]}\Psi(s,g)=\sup_{s\in[ks_0,k^2s_0]}\Psi(ks,g)\le\frac{1}{2}M_1+\frac{1}{2}M_2,$$
$$\ldots.$$

Thus, we deduce that

$$\sup_{s\ge ks_0}\Psi(s,f)\le\frac{1}{2}M_1+\frac{1}{2}M_2<+\infty,$$
$$\sup_{s\ge ks_0}\Psi(s,g)\le\frac{1}{2}M_1+\frac{1}{2}M_2<+\infty.$$

Therefore, $\Psi(s,f)$ and $\Psi(s,g)$ are bounded for all $s\ge s_0$. There exist some constants $K_1$, $K_2$, $K_3$, $K_4$, such that, for any $\varepsilon>0$,

$$T(t,f)=\Phi(s,f)=\Psi(s,f)s^\alpha\le K_1s^\alpha=K_1\left(\log t+\frac{\log\frac{\mu}{(\gamma\beta)^k}}{k-1}\right)^\alpha\le K_2(\log t)^{\alpha+\varepsilon}$$

and

$$T(t,g)=\Phi(s,g)=\Psi(s,g)s^\alpha\le K_3s^\alpha=K_3\left(\log t+\frac{\log\frac{\mu}{(\gamma\beta)^k}}{k-1}\right)^\alpha\le K_4(\log t)^{\alpha+\varepsilon}.$$

Therefore, we have

$$T(r,f)=O\bigl((\log r)^{\alpha+\varepsilon}\bigr)$$

and

$$T(r,g)=O\bigl((\log r)^{\alpha+\varepsilon}\bigr),$$

where

$$\alpha=\frac{\log 2M}{\log k}=\frac{\log 2\sigma}{\log k}+o(1).$$

Letting now $\alpha=\frac{\log 2\sigma}{\log k}$, we obtain the required form. Theorem 2 is proved. □

**Competing interests**
The author declares that there is no conflict of interests regarding the publication of the article.

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**References**

1. Cherry, W, Ye, Z: Nevanlinna's Theory of Value Distribution. Monographs in Math. Springer, Berlin (2001)
2. Hayman, WK: Meromorphic Functions. Clarendon, Oxford (1964)
3. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
4. Ablowitz, MJ, Halburd, R, Herbst, B: On the extension of the Painlevé property to difference equations. Nonlinearity **13**(3), 889-905 (2000)
5. Chen, ZX, Shon, KH: On zeros and fixed points of differences of meromorphic functions. J. Math. Anal. Appl. **344**, 373-383 (2008)
6. Chiang, YM, Feng, SJ: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**(1), 105-129 (2008)
7. Chiang, YM, Feng, SJ: On the growth of logarithmic differences, difference quotients and logarithmic derivatives of meromorphic functions. Trans. Am. Math. Soc. **361**(7), 3767-3791 (2009)
8. Halburd, RG, Korhonen, RJ: Meromorphic solutions of difference equations, integrability and the discrete Painlevé equations. J. Phys. A, Math. Theor. **40**, R1-R38 (2007)
9. Halburd, RG, Korhonen, RJ: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477-487 (2006)
10. Heittokangas, J, Korhonen, R, Laine, I, Rieppo, J, Tohge, K: Complex difference equations of Malmquist type. Comput. Methods Funct. Theory **1**(1), 27-39 (2001)
11. Huang, ZB, Chen, ZX: Meromorphic solutions of some complex difference equations. Adv. Differ. Equ. **2009**, Article ID 982681 (2009)
12. Ishizaki, K, Yanagihara, N: Wiman-Valiron method for difference equations. Nagoya Math. J. **175**, 75-102 (2004)
13. Laine, I, Yang, CC: Clunie theorems for difference and $q$-difference polynomials. J. Lond. Math. Soc. **76**(2), 556-566 (2007)
14. Laine, I, Yang, CC: Value distribution of difference polynomials. Proc. Jpn. Acad., Ser. A, Math. Sci. **83**(8), 148-151 (2007)
15. Laine, I, Rieppo, J, Silvennoinen, H: Remarks on complex difference equations. Comput. Methods Funct. Theory **5**(1), 77-88 (2005)
16. Wang, H, Xu, HY, Liu, BX: The poles and growth of solutions of systems of complex difference equations. Adv. Differ. Equ. **2013**, 75 (2013)
17. Zhang, J, Zhang, JJ: Meromorphic solutions to complex difference and $q$-difference equations of Malmquist type. Electron. J. Differ. Equ. **2014**, 16 (2014)
18. Zhang, JJ, Liao, LW: Further extending results of some classes of complex difference and functional equations. Adv. Differ. Equ. **2010**, Article ID 404582 (2010)
19. Zhang, JJ, Liao, LW: On Malmquist type theorem of complex difference equations. Houst. J. Math. **39**(3), 969-981 (2013)
20. Zheng, XM, Chen, ZX: Some properties of meromorphic solutions of $q$-difference equations. J. Math. Anal. Appl. **361**, 472-480 (2010)
21. Gao, LY: Estimates of $N$-function and $m$-function of meromorphic solutions of systems of complex difference equations. Acta Math. Sci. Ser. B **32**(4), 1495-1502 (2012)
22. Mokhon'ko, AZ, Mokhon'ko, VD: Estimates for the Nevanlinna characteristics of some classes of meromorphic functions and their applications to differential equations. Sib. Mat. Zh. **15**(6), 1305-1322 (1974)
23. Goldstein, R: Some results on factorisation of meromorphic functions. J. Lond. Math. Soc. **4**(2), 357-364 (1971)