Lu and Luo *Advances in Difference Equations* (2015) 2015:137
DOI 10.1186/s13662-015-0480-0

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Some unified formulas and representations for the Apostol-type polynomials

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Abstract

Recently, a family of the Apostol-type polynomials was introduced by Luo and Srivastava (Appl. Math. Comput. 217:5702-5728 (2011)). In this paper, we further investigate the Apostol-type polynomials and obtain their unified multiplication formula and explicit representations in terms of the Gaussian hypergeometric function and the generalized Hurwitz zeta function. We also show some special cases, which include the corresponding results of Luo, Garg, Srivastava, Ozden, and Özarslan *etc.*

MSC: Primary 11B68; secondary 11M35; 11B73; 33C05

Keywords: (generalized) Apostol-type polynomials; Apostol-Euler polynomials; Apostol-Bernoulli polynomials; Genocchi polynomials of high order; multiplication formula; Gaussian hypergeometric function; (generalized) Hurwitz zeta function

1 Introduction, definitions, and motivation

Throughout this paper, we always make use of the following notations: $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of natural numbers, $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \dots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

The symbol $(a)_k$ denotes the shifted factorial (or the Pochhammer symbol), defined, $a \in \mathbb{C}$, by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1, & k = 0, \\ a(a+1)\cdots(a+k-1), & k \in \mathbb{N}. \end{cases} \tag{1.1}$$

The symbol $\{n\}_k$ denotes the falling factorial, defined, $a \in \mathbb{C}$, by

$$\{a\}_k = \begin{cases} 1, & k = 0, \\ a(a-1)\cdots(a-k+1) = \frac{\Gamma(a+1)}{\Gamma(a-k+1)}, & k \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\Gamma(x)$ is the usual gamma function.

The classical Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, and Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$, and $G_n^{(\alpha)}(x)$ of order α, are usually defined by means of the following generating functions (see, for details, [1], pp.532-533 and [2]):

$$\left(\frac{z}{e^z-1}\right)^\alpha e^{xz}=\sum_{n=0}^\infty B_n^{(\alpha)}(x)\frac{z^n}{n!}\quad\bigl(|z|<2\pi\bigr),\tag{1.3}$$

$$\left(\frac{2}{e^z+1}\right)^\alpha e^{xz}=\sum_{n=0}^\infty E_n^{(\alpha)}(x)\frac{z^n}{n!}\quad\bigl(|z|<\pi\bigr)\tag{1.4}$$

and

$$\left(\frac{2z}{e^z+1}\right)^\alpha e^{xz}=\sum_{n=0}^\infty G_n^{(\alpha)}(x)\frac{z^n}{n!}\quad\bigl(|z|<\pi\bigr).\tag{1.5}$$

Thus, the Bernoulli polynomials $B_n(x)$, Euler polynomials $E_n(x)$, and Genocchi polynomials $G_n(x)$ are given, respectively, by

$$B_n(x):=B_n^{(1)}(x),\qquad E_n(x):=E_n^{(1)}(x)\quad\text{and}\quad G_n(x):=G_n^{(1)}(x)\quad(n\in\mathbb{N}_0).\tag{1.6}$$

The Bernoulli numbers B_n, Euler numbers E_n, and Genocchi numbers G_n are, respectively,

$$B_n:=B_n(0)=B_n^{(1)}(0),\qquad E_n:=E_n(0)=E_n^{(1)}(0)\quad\text{and}\quad G_n:=G_n(0)=G_n^{(1)}(0).\tag{1.7}$$

Some interesting analogs of the classical Bernoulli polynomials and numbers were first investigated by Apostol (see [3], p.165, Eq. (3.1)) and (more recently) by Srivastava (see [4], pp.83-84). We begin by recalling here Apostol's definitions as follows.

Definition 1.1 (Apostol [3]; see also Srivastava [4]) The Apostol-Bernoulli polynomials $\mathcal{B}_n(x;\lambda)$ ($\lambda\in\mathbb{C}$) are defined by means of the following generating function:

$$\frac{ze^{xz}}{\lambda e^z-1}=\sum_{n=0}^\infty\mathcal{B}_n(x;\lambda)\frac{z^n}{n!}\quad\bigl(|z|<2\pi\text{ when }\lambda=1;|z|<|\log\lambda|\text{ when }\lambda\neq1\bigr)\tag{1.8}$$

with, of course,

$$B_n(x)=\mathcal{B}_n(x;1)\quad\text{and}\quad\mathcal{B}_n(\lambda):=\mathcal{B}_n(0;\lambda),\tag{1.9}$$

where $\mathcal{B}_n(\lambda)$ denotes the so-called Apostol-Bernoulli numbers.

Recently, Luo and Srivastava [5] further extended the Apostol-Bernoulli polynomials as the so-called Apostol-Bernoulli polynomials of order α.

Definition 1.2 (Luo and Srivastava [5]) The Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ ($\lambda\in\mathbb{C}$) of order α ($\alpha\in\mathbb{N}$) are defined by means of the following generating function:

$$\left(\frac{z}{\lambda e^z-1}\right)^\alpha\cdot e^{xz}=\sum_{n=0}^\infty\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}$$
$$\bigl(|z|<2\pi\text{ when }\lambda=1;|z|<|\log\lambda|\text{ when }\lambda\neq1\bigr)\tag{1.10}$$

with, of course,

$$B_n^{(\alpha)}(x) = \mathcal{B}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{B}_n^{(\alpha)}(\lambda) := \mathcal{B}_n^{(\alpha)}(0;\lambda), \tag{1.11}$$

where $\mathcal{B}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Bernoulli numbers of order α.

On the other hand, Luo [6] gave an analogous extension of the generalized Euler polynomials as the so-called Apostol-Euler polynomials of order α.

Definition 1.3 (Luo [6]) The Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ of order α $(\alpha, \lambda \in \mathbb{C})$ are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.12}$$

with, of course,

$$E_n^{(\alpha)}(x) = \mathcal{E}_n^{(\alpha)}(x;1) \quad \text{and} \quad \mathcal{E}_n^{(\alpha)}(\lambda) := \mathcal{E}_n^{(\alpha)}(0;\lambda), \tag{1.13}$$

where $\mathcal{E}_n^{(\alpha)}(\lambda)$ denotes the so-called Apostol-Euler numbers of order α.

On the subject of the Genocchi polynomials $G_n(x)$ and their various extensions, a remarkably large number of investigations have appeared in the literature (see, for example, [7–11]). Moreover, Luo (see [12]) introduced and investigated the Apostol-Genocchi polynomials of (real or complex) order α, which are defined as follows.

Definition 1.4 The Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ $(\lambda \in \mathbb{C})$ of order α $(\alpha \in \mathbb{N})$ are defined by means of the following generating function:

$$\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} \cdot e^{xz} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr) \tag{1.14}$$

with, of course,

$$\begin{aligned} &G_n^{(\alpha)}(x) = \mathcal{G}_n^{(\alpha)}(x;1), \qquad \mathcal{G}_n^{(\alpha)}(\lambda) := \mathcal{G}_n^{(\alpha)}(0;\lambda), \\ &\mathcal{G}_n(x;\lambda) := \mathcal{G}_n^{(1)}(x;\lambda) \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n^{(1)}(\lambda), \end{aligned} \tag{1.15}$$

where $\mathcal{G}_n(\lambda)$, $\mathcal{G}_n^{(\alpha)}(\lambda)$, and $\mathcal{G}_n(x;\lambda)$ denote the so-called Apostol-Genocchi numbers, the Apostol-Genocchi numbers of order α, and the Apostol-Genocchi polynomials, respectively.

Ozden *et al.* [13] investigated the following unification (and generalization) of the generating functions of the three families of Apostol-type polynomials:

$$\frac{2^{1-\kappa} z^{\kappa}}{\beta^b e^z - a^b} e^{xz} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;\kappa,a,b)\frac{z^n}{n!}$$

$$\bigl(|z| < 2\pi \text{ when } \beta = a; |z| < \bigl|b\log(\beta/a)\bigr| \text{ when } \beta \neq a; \kappa, \beta \in \mathbb{C}; a, b \in \mathbb{C} \setminus \{0\}\bigr). \tag{1.16}$$

In [14] Özarslan further gave an extension of the above definition (1.16) as follows:

$$\left(\frac{2^{1-\kappa}z^{\kappa}}{\beta^b e^z - a^b}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)\frac{z^n}{n!}$$

$$\bigl(\alpha \in \mathbb{N}; |z| < 2\pi \text{ when } \beta = a; |z| < \bigl|b\log(\beta/a)\bigr| \text{ when } \beta \neq a; \kappa,\beta \in \mathbb{C}; a,b \in \mathbb{C}\setminus\{0\}\bigr) \tag{1.17}$$

and gave some identities for $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)$.

Recently, Luo and Srivastava [15] further extended the Apostol-type polynomials as follows.

Definition 1.5 (Luo and Srivastava [15]) The generalized Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ of order α $(\alpha,\lambda,\mu;\nu \in \mathbb{C})$ are defined by means of the following generating function:

$$\left(\frac{2^{\mu}z^{\nu}}{\lambda e^z + 1}\right)^{\alpha} e^{xz} = \sum_{n=0}^{\infty} \mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\frac{z^n}{n!} \quad \bigl(|z| < \bigl|\log(-\lambda)\bigr|\bigr). \tag{1.18}$$

By comparing Definition 1.5 with Definitions 1.2, 1.3 and 1.4, we readily find that

$$\mathcal{B}_n^{(\alpha)}(x;\lambda) = (-1)^{\alpha}\mathcal{F}_n^{(\alpha)}(x;-\lambda;0;1) \quad (\alpha \in \mathbb{N}), \tag{1.19}$$

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;0) \quad (\alpha \in \mathbb{C}) \tag{1.20}$$

and

$$\mathcal{G}_n^{(\alpha)}(x;\lambda) = \mathcal{F}_n^{(\alpha)}(x;\lambda;1;1) \quad (\alpha \in \mathbb{N}). \tag{1.21}$$

Furthermore, if we compare the generating functions (1.16), (1.17) and (1.18), we readily see that

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = -\frac{1}{a^b}\mathcal{F}_n^{(1)}\left(x;-\left(\frac{\beta}{a}\right)^b;1-\kappa;\kappa\right), \tag{1.22}$$

$$\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b) = (-1)^{\alpha}\frac{1}{a^{b\alpha}}\mathcal{F}_n^{(\alpha)}\left(x;-\left(\frac{\beta}{a}\right)^b;1-\kappa;\kappa\right). \tag{1.23}$$

More investigations of this subject can be found in [5, 6, 12–22].

The aim of this paper is to give the multiplication formula for the Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ and obtain an explicit representation of $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ in terms of the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$. We study some relations between the family of Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ and the family of Hurwitz zeta functions $\Phi_{\mu}(z,s,a)$. Some special cases also are shown.

2 Multiplication formula for the Apostol-type polynomials

In this section we give a unified multiplication formula for the Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$. We will see that some well-known results are the corresponding special cases of our result.

First we need the following lemmas.

Lemma 2.1 (Multinomial identity [23], p.28, Theorem B) *If $x_1, x_2, \ldots, x_m$ are commuting elements of a ring ($\Longleftrightarrow$ $x_i x_j = x_j x_i$, $1 \le i < j \le m$), then we have for all integers $n \ge 0$:*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{\substack{a_1,a_2,\ldots,a_m \ge 0 \\ a_1+a_2+\cdots+a_m=n}} \binom{n}{a_1, a_2, \ldots, a_m} {x_1}^{a_1} {x_2}^{a_2} \cdots {x_m}^{a_m}, \tag{2.1}$$

the last summation takes place over all positive or zero integers $a_i \ge 0$ such that $a_1 + a_2 + \cdots + a_m = n$, where

$$\binom{n}{a_1, a_2, \ldots, a_m} := \frac{n!}{a_1! a_2! \cdots a_m!},$$

are called multinomial coefficients defined by [23], *p.28, Definition B.*

Lemma 2.2 (Generalized multinomial identity [23], p.41, Eq. (12m)) *If $x_1, x_2, \ldots, x_m$ are commuting elements of a ring ($\Longleftrightarrow$ $x_i x_j = x_j x_i$, $1 \le i < j \le m$), then we have for all real or complex variable α:*

$$(1 + x_1 + x_2 + \cdots + x_m)^\alpha = \sum_{v_1,v_2,\ldots,v_m \ge 0} \binom{\alpha}{v_1, v_2, \ldots, v_m} {x_1}^{v_1} {x_2}^{v_2} \cdots {x_m}^{v_m}, \tag{2.2}$$

the last summation takes place over all positive or zero integers $v_i \ge 0$, where

$$\binom{\alpha}{v_1, v_2, \ldots, v_m} := \frac{\{\alpha\}_{v_1+v_2+\cdots+v_m}}{v_1! v_2! \cdots v_m!} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha - v_1 - v_2 - \cdots - v_m + 1)}{v_1! v_2! \cdots v_m!}$$

are called generalized multinomial coefficients defined by [23], *p.27, Eq.* (10 C'').

Theorem 2.3 (Multiplication formula) *For $\mu, v, r \in \mathbb{N}$ and $v \le 1$, $n, l \in \mathbb{N}_0$, $\alpha, \lambda \in \mathbb{C}$, we have*

$$\begin{aligned} \mathcal{F}_n^{(\alpha)}(rx; \lambda; \mu; v) = {} & r^{n-v\alpha} \sum_{v_1,v_2,\ldots,v_{r-1} \ge 0} \binom{\alpha}{v_1, v_2, \ldots, v_{r-1}} \\ & \times (-\lambda)^m \mathcal{F}_n^{(\alpha)}\left(x + \frac{m}{r}; \lambda^r; \mu; v\right), \quad r \text{ odd}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{F}_n^{(l)}(rx; \lambda; \mu; v) = {} & \frac{(-1)^l 2^{\mu l} r^{n-vl}}{(n+1)_{(1-v)l}} \sum_{\substack{0 \le v_1,v_2,\ldots,v_{r-1} \le l \\ v_1+v_2+\cdots+v_{r-1}=l}} \binom{l}{v_1, v_2, \ldots, v_{r-1}} \\ & \times (-\lambda)^m \mathcal{B}_{n+(1-v)l}^{(l)}\left(x + \frac{m}{r}; \lambda^r\right), \quad r \text{ even}, \end{aligned} \tag{2.4}$$

where $m = v_1 + 2v_2 + \cdots + (r-1)v_{r-1}$.

Proof It is not difficult to show that

$$\frac{1}{\lambda e^z + 1} = -\frac{1 - \lambda e^z + \lambda^2 e^{2z} + \cdots + (-\lambda)^{r-1} e^{(r-1)z}}{(-\lambda)^r e^{rz} - 1}. \tag{2.5}$$

When r is odd, by (1.18) and (2.5) we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{F}_n^{(\alpha)}(rx;\lambda;\mu;\nu)\frac{z^n}{n!}\\
&=\frac{1}{r^{\nu\alpha}}\left(\frac{2^\mu(rz)^\nu}{\lambda^r e^{rz}+1}\right)^\alpha\left(\frac{\lambda^r e^{rz}+1}{\lambda e^z+1}\right)^\alpha e^{rxz}\\
&=\frac{1}{r^{\nu\alpha}}\left(\frac{2^\mu(rz)^\nu}{\lambda^r e^{rz}+1}\right)^\alpha\left(\sum_{k=0}^{r-1}\left(-\lambda e^z\right)^k\right)^\alpha e^{rxz}\\
&=\frac{1}{r^{\nu\alpha}}\sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0}\binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}}(-\lambda)^m\left(\frac{2^\mu(rz)^\nu}{\lambda^r e^{rz}+1}\right)^\alpha e^{(x+\frac{m}{r})rz}\\
&=\sum_{n=0}^{\infty}\left[r^{n-\nu\alpha}\sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0}\binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}}(-\lambda)^m\mathcal{F}_n^{(\alpha)}\left(x+\frac{m}{r};\lambda^r;\mu;\nu\right)\right]\frac{z^n}{n!}.
\end{aligned}\tag{2.6}$$

Comparing the coefficients of $\frac{z^n}{n!}$ on both sides of (2.6), we obtain the assertion (2.3) of Theorem 2.3.

When r is even, we can similarly prove the assertion (2.4) of Theorem 2.3. The proof is complete. □

It follows that we can deduce the well-known formulas from Theorem 2.3.

Letting $\lambda \longmapsto -\lambda$, taking $\mu = 0$ and $\nu = 1$ in (2.3) and (2.4) and noting (1.19), we can obtain the following main result of Luo (see [24], p.380, Theorem 2.1).

Corollary 2.4 *For $r,\alpha \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, the following multiplication formula for the Apostol-Bernoulli polynomials of higher order holds true*:

$$\mathcal{B}_n^{(\alpha)}(rx;\lambda)=r^{n-\alpha}\sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0}\binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}}\lambda^m\mathcal{B}_n^{(\alpha)}\left(x+\frac{m}{r};\lambda^r\right),\tag{2.7}$$

where $m = \nu_1 + 2\nu_2 + \cdots + (r-1)\nu_{r-1}$.

Taking $\mu = 1$ and $\nu = 0$ in (2.3) and (2.4), and noting (1.20), we can obtain the following main result of Luo (see [24], p.385, Theorem 3.1).

Corollary 2.5 *For $r \in \mathbb{N}$, $n,l \in \mathbb{N}_0$, $\alpha,\lambda \in \mathbb{C}$, the following multiplication formula for the Apostol-Euler polynomials of higher order holds true*:

$$\mathcal{E}_n^{(\alpha)}(rx;\lambda)=r^n\sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0}\binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}}(-\lambda)^m\mathcal{E}_n^{(\alpha)}\left(x+\frac{m}{r};\lambda^r\right),\quad r \text{ odd},\tag{2.8}$$

$$\begin{aligned}
\mathcal{E}_n^{(l)}(rx;\lambda)=\frac{(-2)^l r^n}{(n+1)_l}&\sum_{\substack{0\leq\nu_1,\nu_2,\ldots,\nu_{r-1}\leq l\\ \nu_1+\nu_2+\cdots+\nu_{r-1}=l}}\binom{l}{\nu_1,\nu_2,\ldots,\nu_{r-1}}\\
&\times(-\lambda)^m\mathcal{B}_{n+l}^{(l)}\left(x+\frac{m}{r};\lambda^r\right),\quad r \text{ even},
\end{aligned}\tag{2.9}$$

where $m = \nu_1 + 2\nu_2 + \cdots + (r-1)\nu_{r-1}$.

Taking $\mu = \nu = 1$ in (2.3) and (2.4), and noting (1.21), we can obtain the following main result (see [14], p.2462, Corollary 4.6).

Corollary 2.6 *For $\alpha, r \in \mathbb{N}$, $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, the following multiplication formula for the Apostol-Genocchi polynomials of higher order holds true*:

$$\mathcal{G}_n^{(\alpha)}(rx;\lambda) = r^{n-\alpha} \sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0} \binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}} (-\lambda)^m \mathcal{G}_n^{(\alpha)}\left(x+\frac{m}{r};\lambda^r\right), \quad r \text{ odd}, \tag{2.10}$$

$$\begin{aligned} \mathcal{G}_n^{(l)}(rx;\lambda) = (-2)^l r^{n-l} &\sum_{\substack{0\leq \nu_1,\nu_2,\ldots,\nu_{r-1}\leq l\\ \nu_1+\nu_2+\cdots+\nu_{r-1}=l}} \binom{l}{\nu_1,\nu_2,\ldots,\nu_{r-1}} \\ &\times (-\lambda)^m \mathcal{B}_n^{(l)}\left(x+\frac{m}{r};\lambda^r\right), \quad r \text{ even}, \end{aligned} \tag{2.11}$$

where $m = \nu_1 + 2\nu_2 + \cdots + (r-1)\nu_{r-1}$.

Taking $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (2.3), and noting (1.23), we can obtain the following multiplication formulas for the polynomials $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)$ and $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$ defined by (1.16) and (1.17), respectively.

Corollary 2.7 *For $\kappa, \mu, \nu, m, n, l, r \in \mathbb{N}_0$, $\alpha, \lambda \in \mathbb{C}$, we have*

$$\begin{aligned} &\mathcal{Y}_{n,\beta}^{(\alpha)}(rx;\kappa,a,b) \\ &\quad = r^{n-\kappa\alpha} \sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0} \binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}} \left(\frac{\beta}{a}\right)^{bm} a^{(r-1)b\alpha} \mathcal{Y}_{n,\beta}^{(\alpha)}\left(x+\frac{m}{r};\kappa;a;br\right) \qquad (2.12) \\ &\quad = r^{n-\kappa\alpha} \sum_{\nu_1,\nu_2,\ldots,\nu_{r-1}\geq 0} \binom{\alpha}{\nu_1,\nu_2,\ldots,\nu_{r-1}} \left(\frac{\beta}{a}\right)^{bm} a^{(r-1)b\alpha} \mathcal{Y}_{n,\beta^r}^{(\alpha)}\left(x+\frac{m}{r};\kappa;a^r;b\right), \qquad (2.13) \end{aligned}$$

where $m = \nu_1 + 2\nu_2 + \cdots + (r-1)\nu_{r-1}$.

Setting $\alpha = l = 1$ in (2.12) and (2.13), respectively, we have (see [13], p.2786, Theorem 8) the following.

Corollary 2.8 *For $\kappa, \mu, \nu, n, r \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$, we have*

$$\begin{aligned} \mathcal{Y}_{n,\beta}(rx;\kappa,a,b) &= r^{n-\kappa} \sum_{j=0}^{r-1} \left(\frac{\beta}{a}\right)^{bj} a^{(r-1)b} \mathcal{Y}_{n,\beta}\left(x+\frac{j}{r};\kappa;a;br\right) \qquad (2.14) \\ &= r^{n-\kappa} \sum_{j=0}^{r-1} \left(\frac{\beta}{a}\right)^{bj} a^{(r-1)b} \mathcal{Y}_{n,\beta^r}\left(x+\frac{j}{r};\kappa;a^r;b\right). \qquad (2.15) \end{aligned}$$

Remark 2.9 In [14], p.2460, Theorem 4.3, one of the main result of Özarslan is not right, the correct form should be (2.12) and (2.13) of Corollary 2.7.

Remark 2.10 In fact, setting $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (2.3) and noting (1.23), we deduce the multiplication formulas which are right only when r is odd. In the same way as the proof of [24], p.380, Theorem 2.1, we can obtain the multiplication formulas (2.12) and (2.13) of Corollary 2.7.

3 A unified representation in conjunction with the Gauss hypergeometric function

In this section we obtain a unified representation of the Apostol-type polynomials $\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)$ with the Gaussian hypergeometric functions.

Theorem 3.1 *For $\mu,\nu,n,l\in\mathbb{N}_0$, $\lambda\in\mathbb{C}$, we have*

$$\begin{aligned}&\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)\\&\quad=2^{\mu l}(\nu l)!\binom{n}{\nu l}\sum_{k=0}^{n-\nu l}\binom{l+k-1}{k}\binom{n-\nu l}{k}\frac{(-\lambda)^k}{(\lambda+1)^{l+k}}\\&\qquad\times\sum_{m=0}^{k}(-1)^m\binom{k}{m}m^k(x+m)^{n-\nu l-k}{}_2F_1\left(-n+\nu l+k,k;k+1;\frac{m}{m+x}\right),\end{aligned}\tag{3.1}$$

where $F(a,b;c;z)$ denotes Gaussian hypergeometric functions defined by (*see* [2], *p*.44, *Eq*. (4))

$$F(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},\quad |z|<1.\tag{3.2}$$

Proof Letting $\alpha=l\in\mathbb{N}$ in (1.18), we have

$$\sum_{n=0}^{\infty}\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)\frac{z^n}{n!}=\left(\frac{2^\mu z^\nu}{\lambda e^z+1}\right)^l e^{xz}.\tag{3.3}$$

Differentiating both sides of (3.3) with respect to the variable z yields

$$\begin{aligned}\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)&=D_z^n\left[\left(\frac{2^\mu z^\nu}{\lambda e^z+1}\right)^l e^{xz}\right]_{z=0}\\&=2^{\mu l}\sum_{s=0}^{n}\binom{n}{s}x^{n-s}D_z^s\left[z^{\nu l}\left(\lambda e^z+1\right)^{-l}\right]_{z=0}\\&=2^{\mu l}\sum_{s=\nu l}^{n}\binom{n}{s}x^{n-s}(\nu l)!\binom{s}{\nu l}D_z^{s-\nu l}\left[\left(\lambda e^z+1\right)^{-l}\right]_{z=0}\\&=2^{\mu l}\sum_{s=\nu l}^{n}\binom{n}{s}x^{n-s}(\nu l)!\binom{s}{\nu l}D_z^{s-\nu l}\left[\left(\lambda+1+\lambda\left(e^z-1\right)\right)^{-l}\right]_{z=0},\end{aligned}$$

where $D_z=\frac{d}{dz}$ is the differential operator.

Applying the generalized binomial theorem

$$(a+b)^{-\alpha}=\sum_{l=0}^{\infty}\binom{\alpha+l-1}{l}a^{-\alpha-l}(-b)^l\quad\left(\alpha\in\mathbb{C},\left|\frac{b}{a}\right|<1\right)$$

and the generating function of the Stirling numbers of the second kind $S(n,k)$ (see, for details, [23], p.206, Theorem A),

$$\frac{(e^z-1)^k}{k!}=\sum_{n=0}^{\infty}S(n,k)\frac{z^n}{n!},$$

we find that

$$\begin{aligned}
&\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)\\
&\quad=2^{\mu l}\sum_{s=\nu l}^{n}\binom{n}{s}x^{n-s}(\nu l)!\binom{s}{\nu l}\sum_{k=0}^{\infty}\binom{l+k-1}{k}(\lambda+1)^{-l-k}(-\lambda)^k D_z^{s-\nu l}\bigl[\bigl(e^z-1\bigr)^k\bigr]_{z=0}\\
&\quad=2^{\mu l}\sum_{s=\nu l}^{n}\binom{n}{s}x^{n-s}(\nu l)!\binom{s}{\nu l}\sum_{k=0}^{s-\nu l}\binom{l+k-1}{k}(-\lambda)^k(\lambda+1)^{-l-k}k!S(s-\nu l,k).
\end{aligned}$$

Noting (see [2], p.58, Eq. (20))

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

and the well-known combinatorial identity

$$\binom{n}{k}\binom{k}{s}=\binom{n}{s}\binom{n-s}{n-k},$$

we readily obtain

$$\begin{aligned}
&\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu)\\
&\quad=2^{\mu l}\sum_{s=\nu l}^{n}\binom{n}{s}x^{n-s}(\nu l)!\binom{s}{\nu l}\sum_{k=0}^{s-\nu l}\binom{l+k-1}{k}\\
&\qquad\times(-\lambda)^k(\lambda+1)^{-l-k}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^{s-\nu l}\\
&\quad=2^{\mu l}(\nu l)!\binom{n}{\nu l}\sum_{k=0}^{n-\nu l}\sum_{s=k+\nu l}^{n}\binom{n-\nu l}{n-s}\binom{l+k-1}{k}\frac{(-\lambda)^k x^{n-s}}{(\lambda+1)^{l+k}}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^{s-\nu l}\\
&\quad=2^{\mu l}(\nu l)!\binom{n}{\nu l}\sum_{k=0}^{n-\nu l}\sum_{s=0}^{n-k-\nu l}\binom{n-\nu l}{n-s-\nu l-k}\binom{l+k-1}{k}\\
&\qquad\times\frac{(-\lambda)^k x^{n-s-k-\nu l}}{(\lambda+1)^{l+k}}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^{s+k}\\
&\quad=2^{\mu l}(\nu l)!\binom{n}{\nu l}\sum_{k=0}^{n-\nu l}\binom{l+k-1}{k}\frac{(-\lambda)^k x^{n-k-\nu l}}{(\lambda+1)^{l+k}}\sum_{m=0}^{k}(-1)^{k-m}\binom{k}{m}m^k\\
&\qquad\times\sum_{s=0}^{n-k-\nu l}\binom{n-\nu l}{n-s-\nu l-k}\left(\frac{m}{x}\right)^s.
\end{aligned}$$

Noting that (in view of $\binom{n}{k} = 0$ when $k > n$ or $k < 0$)

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{\infty} \binom{n}{k},$$

and combining the definition of the Gaussian hypergeometric function

$${}_2F_1(a,b;c;z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

we obtain

$$\begin{aligned}\mathcal{F}_n^{(l)}(x;\lambda;\mu;\nu) &= 2^{\mu l}(\nu l)!\binom{n}{\nu l} \sum_{k=0}^{n-\nu l} \binom{l+k-1}{k}\binom{n-\nu l}{k} \\ &\quad \times \frac{(-\lambda)^k x^{n-k-\nu l}}{(\lambda+1)^{l+k}} \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k \\ &\quad \times {}_2F_1\left(-n+\nu l+k, 1; k+1; -\frac{m}{x}\right). \end{aligned} \tag{3.4}$$

Applying the Pfaff-Kummer hypergeometric transformation [25], p.559, Eq. (15.3.4),

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \quad \left(c \notin \mathbb{Z}_0^- : \left|\arg(1-z)\right| \leq \pi - \epsilon \ (0 < \epsilon < \pi)\right),$$

to (3.4), we arrive at the desired equation, (3.1). This completes our proof. □

Below we show some special cases of (3.1).

Letting $\lambda \longmapsto -\lambda$, taking $\mu = 0$ and $\nu = 1$ in (3.1) and noting (1.19), we easily obtain the following explicit formula for the Apostol-Bernoulli polynomials:

$$\begin{aligned}\mathcal{B}_n^{(l)}(x;\lambda) &= l!\binom{n}{l} \sum_{k=0}^{n-l} \binom{l+k-1}{k}\binom{n-l}{k} \frac{\lambda^k}{(\lambda-1)^k} \\ &\quad \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k (x+m)^{n-l-k} {}_2F_1\left(-n+l+k, k; k+1; \frac{m}{m+x}\right), \end{aligned} \tag{3.5}$$

with $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C} \setminus \{1\}$, which is just the main result of Luo and Srivastava (see [5], p.294, Theorem 1).

Taking $\mu = 1$ and $\nu = 0$ in (3.1) and noting (1.20), we can obtain the following explicit formula for the Apostol-Euler polynomials:

$$\begin{aligned}\mathcal{E}_n^{(l)}(x;\lambda) &= 2^l \sum_{k=0}^{n} \binom{l+k-1}{k}\binom{n}{k} \frac{(-\lambda)^k}{(\lambda+1)^{l+k}} \\ &\quad \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k (x+m)^{n-k} {}_2F_1\left(-n+k, k; k+1; \frac{m}{m+x}\right), \end{aligned} \tag{3.6}$$

with $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C} \setminus \{-1\}$, which is just the main result of Luo (see [6], p.920, Theorem 1).

Taking $\mu = \nu = 1$ in (3.1), and noting (1.21), we can obtain the following explicit representation of the generalized Apostol-Genocchi polynomials:

$$\mathcal{G}_n^{(l)}(x;\lambda) = 2^l l! \binom{n}{l} \sum_{k=0}^{n-l} \binom{l+k-1}{k} \binom{n-l}{k} \frac{(-\lambda)^k}{(\lambda+1)^{l+k}} \\ \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k (x+m)^{n-l-k} {}_2F_1\left(-n+l+k, k; k+1; \frac{m}{m+x}\right), \tag{3.7}$$

with $n, l \in \mathbb{N}_0$, $\lambda \in \mathbb{C} \setminus \{-1\}$, which is just one of the results of Luo and Srivastava (see [15], p.5708, Theorem 1).

Taking $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (3.1), and noting (1.23), we deduce the following well-known formula:

$$\mathcal{Y}_{n,\beta}^{(l)}(x;\kappa,a,b) = 2^{l(1-\kappa)}(l\kappa)! \binom{l+k-1}{k} \binom{n}{l\kappa} \sum_{k=0}^{n-l\kappa} \binom{n-l\kappa}{k} \frac{\beta^{bk}}{(\beta^b - a^b)^{k+1}} \\ \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k (x+m)^{n-k-l\kappa} \\ \times {}_2F_1\left(-n+l\kappa+k, k; k+1; \frac{m}{m+x}\right), \tag{3.8}$$

with $n, l, \kappa \in \mathbb{N}_0$, $\beta \in \mathbb{C}$, $a, b \in \mathbb{C} \setminus \{0\}$, $\beta \neq a$, which is just main result of Özarslan (see [14], p.2454, Theorem 2.1).

Further setting $l = 1$ in (3.8) we deduce the following formula for $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$:

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = 2^{1-\kappa}\kappa! \binom{n}{\kappa} \sum_{k=0}^{n-\kappa} \binom{n-\kappa}{k} \frac{\beta^{bk}}{(\beta^b - a^b)^{k+1}} \\ \times \sum_{m=0}^{k} (-1)^m \binom{k}{m} m^k (x+m)^{n-k-\kappa} \\ \times {}_2F_1\left(-n+\kappa+k, k; k+1; \frac{m}{m+x}\right). \tag{3.9}$$

4 Some explicit relationships between the generalized Apostol-type polynomials and generalized Hurwitz-Lerch zeta function

A general Hurwitz-Lerch zeta function $\Phi(z,s,a)$ defined by (*cf.*, *e.g.*, [2], p.121, *et seq.*)

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \\ \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1\right) \tag{4.1}$$

contains, as *special* cases, not only the Hurwitz (or generalized) zeta function $\zeta(s,a)$ defined by (*cf.* [26], p.249 and [4], p.88)

$$\zeta(s,a) := \Phi(1,s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad \left(\Re(s) > 1; a \in\notin \mathbb{Z}_0^-\right) \tag{4.2}$$

and the Riemann zeta function $\zeta(s)$,

$$\zeta(s) := \Phi(1,s,1) = \zeta(s,1) = \frac{1}{2^s-1}\zeta\left(s,\frac{1}{2}\right) \quad \bigl(\Re(s) > 1; a \notin \mathbb{Z}_0^-\bigr), \tag{4.3}$$

and the Lerch zeta function:

$$\ell_s(\xi) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\xi}}{n^s} = e^{2\pi i\xi}\Phi\bigl(e^{2\pi i\xi}, s, 1\bigr) \quad \bigl(\xi \in \mathbb{R}; \Re(s) > 1\bigr), \tag{4.4}$$

but also such other functions as the polylogarithm function:

$$\begin{aligned}&\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\Phi(z,s,1)\\ &\quad\bigl(s \in \mathbb{C} \text{ when } |z| < 1; \Re(s) > 1 \text{ when } |z| = 1\bigr)\end{aligned} \tag{4.5}$$

and the Lipschitz-Lerch zeta function (*cf.* [2], p.122, Eq. 2.5(11)):

$$\begin{aligned}&\phi(\xi,a,s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\xi}}{(n+a)^s} = \Phi\bigl(e^{2\pi i\xi}, s, a\bigr) =: L(\xi,s,a)\\ &\quad\bigl(a \in \mathbb{C} \setminus \mathbb{Z}_0^-; \Re(s) > 0 \text{ when } \xi \in \mathbb{R} \setminus \mathbb{Z}; \Re(s) > 1 \text{ when } \xi \in \mathbb{Z}\bigr),\end{aligned} \tag{4.6}$$

which was first studied by Rudolf Lipschitz (1832-1903) and Matyáš Lerch (1860-1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions.

A family of the Hurwitz-Lerch zeta functions $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a)$ defined by (see *e.g.* [27], p.727, Eq. (8))

$$\begin{aligned}&\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}\\ &\quad\bigl(\mu \in \mathbb{C}; a,\nu \in \mathbb{C} \setminus \mathbb{Z}_0^-; \rho,\sigma \in \mathbb{R}^+; \rho < \sigma \text{ when } s,z \in \mathbb{C};\\ &\quad\rho = \sigma \text{ and } s \in \mathbb{C} \text{ when } |z| < 1; \rho = \sigma \text{ and } \Re(s-\mu+\nu) > 1 \text{ when } |z| = 1\bigr),\end{aligned} \tag{4.7}$$

contains, as *special* cases, not only the Hurwitz-Lerch zeta function

$$\Phi_{\nu,\nu}^{(\sigma,\sigma)}(z,s,a) = \Phi_{\mu,\nu}^{(0,0)}(z,s,a) = \Phi(z,s,a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{4.8}$$

and the Lipschitz-Lerch zeta function $\phi(\xi,a,s) := \Phi(e^{2\pi i\xi},s,a)$, but also the following generalized Hurwitz-Lerch zeta functions introduced and studied earlier by Goyal and Laddha [28], p.100, Eq. (1.5):

$$\Phi_{\mu,1}^{(1,1)}(z,s,a) = \Phi_{\mu}(z,s,a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{n!} \frac{z^n}{(n+a)^s}, \tag{4.9}$$

which, are called the *Goyal-Laddha-Hurwitz-Lerch zeta functions*.

Below we give an explicit relationship between the Apostol-type polynomials $\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)$ and the Hurwitz-Lerch zeta function $\Phi_\mu(z,s,a)$.

Theorem 4.1 *For $n,\nu,\alpha\in\mathbb{N}_0$; $-1<\lambda\leqq 1$; $n\geqq\nu\alpha$; $x\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $\mu\in\mathbb{C}$, the relationship*

$$\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)=2^{\mu\alpha}(\nu\alpha)!\binom{n}{\nu\alpha}\Phi_\alpha(-\lambda,\nu\alpha-n,x) \tag{4.10}$$

holds true.

Proof Applying the generalized binomial theorem

$$(1+w)^{-\alpha}=\sum_{r=0}^{\infty}\binom{\alpha+r-1}{r}(-w)^r\quad\bigl(|w|<1\bigr)$$

in (1.18), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{F}_n^{(\alpha)}(x;\lambda;\mu;\nu)\frac{z^n}{n!}&=\left(\frac{2^\mu z^\nu}{\lambda e^z+1}\right)^\alpha e^{xz}\\&=2^{\mu\alpha}z^{\nu\alpha}\bigl(1+\lambda e^z\bigr)^{-\alpha}e^{xz}\\&=2^{\mu\alpha}z^{\nu\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\lambda)^k e^{(k+x)z}\\&=\sum_{n=0}^{\infty}\left[2^{\mu\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}(-\lambda)^k(k+x)^n\right]\frac{z^{n+\nu\alpha}}{n!}\\&=\sum_{n=\nu\alpha}^{\infty}\left[2^{\mu\alpha}(\nu\alpha)!\binom{n}{\nu\alpha}\sum_{k=0}^{\infty}\frac{(\alpha)_k}{k!}\frac{(-\lambda)^k}{(k+x)^{\nu\alpha-n}}\right]\frac{z^n}{n!}.\end{aligned} \tag{4.11}$$

Noting (4.9), (4.10) follows. □

Below we see that (4.10) implies some well-known results.

Let $\lambda\longmapsto-\lambda$, taking $\mu=0$ and $\nu=1$ in (4.10) and noting (1.19), we can obtain an explicit relation between the Apostol-Bernoulli polynomials and the Hurwitz-Lerch zeta function:

$$\mathcal{B}_n^{(l)}(x;\lambda)=(-n)_l\Phi_l(\lambda,l-n,x)\quad\bigl(n,l\in\mathbb{N};n\geqq l;|\lambda|<1;x\in\mathbb{C}\setminus\mathbb{Z}_0^-\bigr). \tag{4.12}$$

The above result is just one of the main results of Garg *et al.* (see [29], p.809).

Clearly, we have the following relation between the Apostol-Bernoulli polynomials and the Hurwitz-Lerch zeta function:

$$\mathcal{B}_n(x;\lambda)=-n\Phi(\lambda,1-n,x)\quad\bigl(n\in\mathbb{N};|\lambda|\leqq 1;x\in\mathbb{C}\setminus\mathbb{Z}_0^-\bigr), \tag{4.13}$$

which is just the result of Apostol (see [3]).

Taking $\lambda=1$ in (4.13), we obtain the following well-known relationship between the Bernoulli polynomials and Hurwitz zeta function (see [26], p.264, Theorem 12.13):

$$B_n(x)=-n\zeta(1-n,x)\quad(n\in\mathbb{N}). \tag{4.14}$$

Taking $x = 0$ in (4.14), we obtain the following the well-known relationship between the Bernoulli numbers and the Riemann zeta function (see [26], p.266, Theorem 12.16):

$$B_n = -n\zeta(1-n) \quad (n \in \mathbb{N}). \tag{4.15}$$

Taking $\mu = 1$ and $\nu = 0$ in (4.10) and noting (1.20), we can obtain the following result of Luo (see [30], p.339, Theorem 2.1):

$$\mathcal{E}_n^{(\alpha)}(x;\lambda) = 2^\alpha \Phi_\alpha(-\lambda, -n, x). \tag{4.16}$$

Further taking $\alpha = 1$ in (4.16), we have the following relation between the Apostol-Euler polynomials and the Hurwitz-Lerch zeta function:

$$\mathcal{E}_n(x;\lambda) = 2\Phi(-\lambda, -n, x) \quad \bigl(n \in \mathbb{N}; -1 < \lambda \leqq 1; x \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr). \tag{4.17}$$

Taking $\lambda = 1$ in (4.17), we can obtain the following well-known relation between the Euler polynomials and the L-function:

$$E_n(x) = 2L(-n, x), \tag{4.18}$$

where the L-function is defined by

$$L(s,x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s} \quad \bigl(\Re(s) > 1; x \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr). \tag{4.19}$$

From (4.19), we further obtain the following well-known relation between the Euler numbers and the l-function:

$$E_n = 2l(-n), \tag{4.20}$$

where the l-function is defined by

$$l(s) := \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad \Re(s) > 0. \tag{4.21}$$

Taking $\mu = \nu = 1$ in (4.10), and noting (1.21), we can deduce the following relation between the Apostol-Genocchi polynomials and Hurwitz-Lerch zeta function (see [31], p.124, Corollary 4.2):

$$\mathcal{G}_n^{(l)}(x;\lambda) = \{n\}_l 2^l \Phi_l(-\lambda, l-n, x) \quad \bigl(n, l \in \mathbb{N}; n \geqq l; |\lambda| \leqq 1;\ x \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr) \tag{4.22}$$

and

$$\mathcal{G}_n(x;\lambda) = 2n\Phi(-\lambda, 1-n, x) \quad \bigl(n \in \mathbb{N}; |\lambda| \leqq 1; x \in \mathbb{C} \setminus \mathbb{Z}_0^-\bigr). \tag{4.23}$$

Taking $\lambda = -(\frac{\beta}{a})^b$, $\mu = 1-\kappa$, $\nu = \kappa$ in (4.10), and noting (1.23), we can deduce the following relations between the polynomials $\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b)$, $\mathcal{Y}_{n,\beta}(x;\kappa,a,b)$, and the (generalized)

Hurwitz zeta functions [13, 14, 18]:

$$\mathcal{Y}_{n,\beta}^{(\alpha)}(x;\kappa,a,b) = (-1)^{\alpha}\frac{2^{(1-\kappa)\alpha}(\kappa\alpha)!}{a^{b\alpha}}\binom{n}{\kappa\alpha}\Phi_{\alpha}\left(\left(\frac{\beta}{a}\right)^{b},\kappa\alpha-n,x\right) \tag{4.24}$$

and

$$\mathcal{Y}_{n,\beta}(x;\kappa,a,b) = -\frac{2^{(1-\kappa)}(\kappa)!}{a^{b}}\binom{n}{\kappa}\Phi\left(\left(\frac{\beta}{a}\right)^{b},\kappa-n,x\right). \tag{4.25}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally in writing this paper, and they read and approved the final manuscript.

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Acknowledgements
The authors sincerely thank the referees for their valuable suggestions and comments. The present investigation was supported by *Natural Science Foundation Project of Chongqing*, *China* under Grant CSTC2011JJA00024, *Research Project of Science and Technology of Chongqing Education Commission*, *China* under Grant KJ120625, *Fund of Chongqing Normal University*, *China* under Grants 10XLR017 and 2011XLZ07.

Received: 12 February 2015 Accepted: 20 April 2015 Published online: 01 May 2015

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