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Growth of solutions of some kinds of linear difference equations

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Abstract
In this paper, we investigate the growth and value distribution of solutions of some kinds of linear difference equations, where there may be more than one coefficient having the same maximal order and the same maximal type.

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1 Introduction

Throughout this paper, we use the standard notations and basic results of Nevanlinna's value distribution theory (see [1–4]). In addition, we use the notations $\sigma(f)$, $\sigma_2(f)$ and $\lambda(f-a)$ to denote, respectively, the order, the hyperorder, and the exponent of convergence of the sequence of a-points of a meromorphic function $f(z)$ in the complex plane, where $a \in \mathbb{C} \cup \{\infty\}$. Furthermore, we can get the definition of $\lambda(f-\varphi)$, when a is replaced by a meromorphic function $\varphi(z)$.

Recently, with the research and further development of difference analogs of Nevanlinna's theory, it has been applied more and more widely in the difference field. By this important tool, many scholars investigated the linear difference equation

$$A_k(z)f(z+c_k)+\cdots+A_1(z)f(z+c_1)+A_0(z)f(z)=0, \tag{1.1}$$

where $k \in \mathbb{N}_+$ and c_j, $j=1,\ldots,k$ are distinct nonzero complex constants, and obtained many results on the growth and the exponent of convergence of the sequence of zeros of meromorphic solutions of (1.1). For instance, in [5–7], the authors considered the case when there is exactly only one coefficient of (1.1) having the maximal order; in [8–10], the authors considered the case when there is exactly only one coefficient having the type strictly greater than the others among those having the maximal order; and in [5, 6, 11], the authors considered the case when the coefficients of (1.1) are polynomials.

Further, how about the case when there are more than one coefficient having the same maximal order and the same maximal type?

In 2013, Liu [12] considered the growth and the exponent of convergence of the sequence of small function value points of second order linear difference equations and obtained the following theorem.

Theorem A (see [12]) *Let c_1, c_2 ($\neq c_1$), a be nonzero constants, $h_1(z)$ be a nonzero meromorphic functions with $\sigma(h_1) < 1$, $B(z)$ be a nonzero meromorphic function. If $B(z)$ satisfies any one of the following three conditions*:

(i) $\sigma(B) > 1$ *and* $\delta(\infty, B) > 0$;

(ii) $\sigma(B) < 1$;

(iii) $B(z) = h_0(z)e^{bz}$, *where b is a nonzero constant, $h_0(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(h_0) < 1$,*

then every meromorphic solution $f(z)$ ($\not\equiv 0$) of the difference equation

$$f(z + c_2) + h_1(z)e^{az}f(z + c_1) + B(z)f(z) = 0$$

satisfies $\sigma(f) \geq \max\{\sigma(B), 1\} + 1$. *Further, if $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with* $\sigma(\varphi) < \max\{\sigma(B), 1\} + 1$, *then* $\lambda(f - \varphi) = \sigma(f) \geq \max\{\sigma(B), 1\} + 1$.

Liu and Mao [13] considered the growth of meromorphic solutions of (1.1), where the coefficients may have the same order and the same type, and obtained the following theorem.

Theorem B (see [13]) *Let $A_j(z) = h_j(z)e^{P_j(z)} + D_j(z)$, $j = 0, 1, \ldots, k$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ are polynomials with degree n (≥ 1), $h_j(z)$ ($\not\equiv 0$), $D_j(z)$ are entire functions with order less than n. If a_{jn}, $j = 0, 1, \ldots, k$ are distinct complex numbers, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of* (1.1) *satisfies* $\sigma(f) \geq \max_{0\leq j\leq k}\{\sigma(A_j)\} + 1$.

In this paper, we are concerned with the more general problem than Theorems A and B, and obtain the following results, which extend and improve the previous results.

First, we consider the difference equation (1.1).

Theorem 1.1 *Let $k, n \in \mathbb{N}_+$, $A_j(z) = B_j(z)e^{P_j(z)} + D_j(z)e^{Q_j(z)} + R_j(z)$, $j = 0, 1, \ldots, k$, where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$, $Q_j(z) = b_{jn}z^n + \cdots + b_{j0}$, $j = 0, 1, \ldots, k$ are polynomials with degree n and satisfy $|a_{jn}| \geq |b_{jn}| > 0$, $j = 0, 1, \ldots, k$, $B_j(z)$, $D_j(z)$, $R_j(z)$, $j = 0, 1, \ldots, k$ are meromorphic functions and satisfy* $\max_{0\leq j\leq k}\{\sigma(B_j), \sigma(D_j), \sigma(R_j)\} = \omega < n$, $A_j(z) - R_j(z) \not\equiv 0$, $j = 0, 1, \ldots, k$. *Let c_j, $j = 1, \ldots, k$ be distinct nonzero complex constants. If there exists an $i \in \{0, 1, \ldots, k\}$ such that for all j ($\neq i$), $|a_{in}| \geq |a_{jn}|$, and*

(i) $\arg a_{in} \neq \arg a_{jn}$, *or* $\arg a_{in} = \arg a_{jn}$, $|a_{in}| > |a_{jn}|$

and

(ii) $\arg a_{in} \neq \arg b_{jn}$, *or* $\arg a_{in} = \arg b_{jn}$, $|a_{in}| > |b_{jn}|$

hold simultaneously, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of (1.1) *satisfies* $\sigma(f) \geq n + 1$. *Further, if $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(\varphi) < n + 1$, then for every meromorphic solution $f(z)$ ($\not\equiv 0$) of* (1.1) *with $\sigma_2(f) < 1$, we have* $\lambda(f - \varphi) = \sigma(f) \geq n + 1$.

Remark 1.1 In Theorem 1.1, if there exist some j ($\neq i$) such that $A_j(z) - R_j(z) \equiv 0$, or some of $B_j(z)$ ($j \neq i$), $D_j(z)$, $R_j(z)$, $j = 0, 1, \ldots, k$, are equal to zero, then the corresponding result holds by using a similar proof to the one of Theorem 1.1.

Next, we consider difference operators instead of shift operators in (1.1).

For a nonzero complex constant c, the forward differences $\Delta^k f(z)$, $k \in \mathbb{N}_+$, are defined (see [14]) by

$$\Delta f(z) = \Delta^1 f(z) = f(z + c) - f(z), \qquad \Delta^{k+1} f(z) = \Delta^k f(z + c) - \Delta^k f(z), \quad k \in \mathbb{N}_+.$$

It is shown in [6] that

$$\Delta^k f(z) = \sum_{j=0}^{k} C_k^j (-1)^{k-j} f(z+jc), \qquad f(z+kc) = \sum_{j=0}^{k} C_k^j \Delta^j f(z), \quad k \in \mathbb{N}_+.$$

Then we can obtain the following theorem.

Theorem 1.2 *Suppose that $A_j(z)$, $j = 0,1,\ldots,k$, satisfy the hypotheses of Theorem* 1.1, *and i is also defined as in Theorem* 1.1. *If $i = 0$, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of the difference equation*

$$A_k(z)\Delta^k f(z) + \cdots + A_1(z)\Delta f(z) + A_0(z)f(z) = 0 \tag{1.2}$$

satisfies $\sigma(f) \geq n+1$. Further, if $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(\varphi) < n+1$, then for every meromorphic solution $f(z)$ ($\not\equiv 0$) of (1.2) *with $\sigma_2(f) < 1$, we have $\lambda(f-\varphi) = \sigma(f) \geq n+1$.*

In the end, we can easily get the following corollary.

Corollary 1.1 *Let $a_j \in \mathbb{C}$, $j = 0,1,\ldots,k$, such that $a_0 \neq a_j$, $|a_0| \geq |a_j| \geq 0$, $j = 1,\ldots,k$, and $h_j(z)$ ($\not\equiv 0$), $j = 0,1,\ldots,k$, be meromorphic functions with order less than n, then every meromorphic solution $f(z)$ ($\not\equiv 0$) of the difference equation*

$$\begin{aligned} &h_k(z)e^{a_k z^n}\Delta^k f(z) + h_{k-1}(z)e^{a_{k-1}z^n}\Delta^{k-1}f(z) + \cdots + h_1(z)e^{a_1 z^n}\Delta f(z) \\ &\quad + h_0(z)e^{a_0 z^n}f(z) = 0 \end{aligned}$$

satisfies $\sigma(f) \geq n+1$. Further, if $\varphi(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(\varphi) < n+1$, then for every meromorphic solution $f(z)$ ($\not\equiv 0$) with $\sigma_2(f) < 1$, we have $\lambda(f-\varphi) = \sigma(f) \geq n+1$.

2 Preliminary lemmas

Lemma 2.1 (see [6]) *Let η_1, η_2 be two arbitrary complex numbers, and $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given, then there exists a set $E \subset (0,+\infty)$ with finite logarithmic measure such that for all $r \notin E \cup [0,1]$, we have*

$$\exp\bigl(-r^{\sigma-1+\varepsilon}\bigr) \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\bigl(r^{\sigma-1+\varepsilon}\bigr).$$

Remark 2.1 It follows from Lemma 2.1 that

$$\begin{aligned} \left|\frac{\Delta^i f(z)}{f(z)}\right| &= \left|\frac{\sum_{j=0}^{i} C_i^j (-1)^{i-j} f(z+jc)}{f(z)}\right| \leq \sum_{j=0}^{i} C_i^j \left|\frac{f(z+jc)}{f(z)}\right| \\ &\leq \sum_{j=0}^{i} C_i^j \exp\bigl\{r^{\sigma-1+\varepsilon}\bigr\} = 2^i \exp\bigl\{r^{\sigma-1+\varepsilon}\bigr\}, \quad i \in \mathbb{N}_+. \end{aligned}$$

Lemma 2.2 (see [15]) *Let $f(z)$ be a meromorphic function with $\sigma(f) = \beta < +\infty$, then for any given $\varepsilon > 0$, there exists a set $E \subset [0,+\infty)$ with $mE < +\infty$ such that for all z with $|z| =$*

$r \notin [0,1] \cup E$, $r \to \infty$, we have

$$\exp\{-r^{\beta+\varepsilon}\} \le |f(z)| \le \exp\{r^{\beta+\varepsilon}\}.$$

Lemma 2.3 (see [16]) *Suppose that $P(z) = (\alpha + \beta i)z^n + \cdots$ (α, β are real numbers such that $|\alpha| + |\beta| \neq 0$) is a polynomial with degree n (≥ 1), and $\omega(z)$ ($\not\equiv 0$) is a meromorphic function with $\sigma(\omega) < n$. Set $g(z) = \omega(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P,\theta) = \alpha\cos n\theta - \beta\sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_0 \subset [0,2\pi)$ with linear measure zero, such that for any $\theta \in [0,2\pi)\backslash(H_0 \cup H_1)$, there exists $r_0 = r_0(\theta,\varepsilon)$ (> 0) such that for $|z| = r > r_0$, we have*

(i) *if $\delta(P,\theta) > 0$, then*

$$\exp\{(1-\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1+\varepsilon)\delta(P,\theta)r^n\};$$

(ii) *if $\delta(P,\theta) < 0$, then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\} < |g(re^{i\theta})| < \exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $H_1 = \{\theta \in [0,2\pi) : \delta(P,\theta) = 0\}$ is a finite set.

Remark 2.2 Let $P(z) = az^n + \cdots$ be a polynomial with degree n and $z = re^{i\theta}$, we denote $\delta(P,\theta) = |a|\cos(\arg a + n\theta)$.

Lemma 2.4 (see [17]) *Let $f(z)$ be a nonconstant meromorphic function, $\varepsilon > 0$, and $c \in \mathbb{C}\backslash\{0\}$. If $\zeta = \sigma_2(f) < 1$, then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\zeta-\varepsilon}}\right)$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.5 *Let c_j, $j = 1,\ldots,k$ be distinct nonzero complex constants, $A_j(z)$, $j = 0,1,\ldots,k$, $F(z)$ be meromorphic functions such that $A_k(z)A_0(z)F(z) \not\equiv 0$. If $f(z)$ is a meromorphic solution of the difference equation*

$$A_k(z)f(z+c_k) + \cdots + A_1(z)f(z+c_1) + A_0(z)f(z) = F(z) \tag{2.1}$$

and satisfies $\max\{\sigma(F), \sigma(A_j), j = 0,1,\ldots,k\} = \omega < \sigma(f) = \sigma$ $(0 < \sigma \le \infty)$, $\sigma_2(f) < 1$, then we have

$$\lambda(f) = \sigma(f).$$

Proof We use a similar proof to the one in [12] here. We rewrite (2.1) as

$$\frac{1}{f(z)} = \frac{1}{F(z)}\left(A_k(z)\frac{f(z+c_k)}{f(z)} + \cdots + A_1(z)\frac{f(z+c_1)}{f(z)} + A_0(z)\right). \tag{2.2}$$

By Lemma 2.4, there exists a set $E \subset (1,+\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin E$, we have

$$m\left(r, \frac{f(z+c_j)}{f(z)}\right) = o(T(r,f)), \quad j = 1,\ldots,k.$$

Thus, (2.2) implies that

$$\begin{aligned} T(r,f) &= N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{f}\right) + O(1) \\ &\le N\left(r,\frac{1}{f}\right) + m\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k} m(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f(z+c_j)}{f(z)}\right) + O(1) \\ &\le N\left(r,\frac{1}{f}\right) + T(r,F) + \sum_{j=0}^{k} T(r,A_j) + o\bigl(T(r,f)\bigr), \quad r \notin E. \end{aligned} \tag{2.3}$$

Set $m_l E = \log\delta < \infty$. Since $\sigma(f) = \sigma$, there exists a sequence $\{r_n'\}_{n=1}^{\infty}$ tending to ∞ such that $(\delta+2)r_n' < r_{n+1}'$ and

$$\lim_{r_n'\to\infty} \frac{\log T(r_n',f)}{\log r_n'} = \sigma.$$

We may choose $r_n \in [r_n', (\delta+2)r_n'] \backslash E$, $n = 1, 2, \ldots$. Since

$$\frac{\log T(r_n,f)}{\log r_n} \ge \frac{\log T(r_n',f)}{\log(\delta+2)r_n'} = \frac{\log T(r_n',f)}{\log r_n'(1+\frac{\log(\delta+2)}{\log r_n'})},$$

we have

$$\varliminf_{r_n\to\infty} \frac{\log T(r_n,f)}{\log r_n} \ge \lim_{r_n'\to\infty} \frac{\log T(r_n',f)}{\log r_n'(1+\frac{\log(\delta+2)}{\log r_n'})} = \sigma,$$

that is,

$$\lim_{r_n\to\infty} \frac{\log T(r_n,f)}{\log r_n} = \sigma.$$

Then for sufficiently small ε $(0 < \varepsilon < \frac{\sigma-\omega}{2})$ and sufficiently large r_n, we have

$$T(r_n,f) \ge r_n^{\sigma-\varepsilon}$$

and

$$T(r_n,F) \le r_n^{\omega+\varepsilon}, \qquad T(r_n,A_j) \le r_n^{\omega+\varepsilon}, \quad j = 0,1,\ldots,k.$$

Hence,

$$\frac{T(r_n,F)}{T(r_n,f)} \le r_n^{\omega-\sigma+2\varepsilon} \to 0, \quad r_n \to \infty$$

and

$$\frac{T(r_n,A_j)}{T(r_n,f)} \le r_n^{\omega-\sigma+2\varepsilon} \to 0, \quad r_n \to \infty, j = 0,1,\ldots,k,$$

hold. Then, for sufficiently large r_n, we have

$$T(r_n,F) \le \frac{1}{k+4}T(r_n,f), \qquad o\big(T(r_n,f)\big) \le \frac{1}{k+4}T(r_n,f),$$
$$T(r_n,A_j) \le \frac{1}{k+4}T(r_n,f), \quad j=0,1,\ldots,k.$$

It follows from (2.3) that for sufficiently large r_n,

$$T(r_n,f) \le N\left(r_n,\frac{1}{f}\right) + \frac{k+3}{k+4}T(r_n,f),$$

that is,

$$T(r_n,f) \le (k+4)N\left(r_n,\frac{1}{f}\right).$$

Therefore,

$$\sigma = \lim_{r_n\to\infty}\frac{\log T(r_n,f)}{\log r_n} \le \overline{\lim_{r_n\to\infty}}\frac{\log N(r_n,\frac{1}{f})}{\log r_n} \le \lambda(f),$$

that is, $\lambda(f)=\sigma(f)$. □

Remark 2.3 Noting that

$$m\left(r,\frac{\Delta^i f}{f}\right) \le O\left(\sum_{j=0}^{i} m\left(r,\frac{f(z+jc)}{f(z)}\right)\right), \quad i\in\mathbb{N}_+,$$

we see that if $f(z+c_i)$, $i=1,\ldots,k$, in Lemma 2.5 are replaced, respectively, by $\Delta^i f$, $i=1,\ldots,k$, the corresponding result holds.

3 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Suppose that $f(z)$ ($\not\equiv 0$) is a meromorphic solution of (1.1). If $\sigma(f)=\infty$, then $\sigma(f)\ge n+1$ holds obviously. Now, we suppose $\sigma(f)=\sigma<\infty$. Set

$$I_1=\{j\neq i \mid \arg a_{in}\neq\arg a_{jn} \text{ and } \arg a_{in}\neq\arg b_{jn}\},$$
$$I_2=\big\{j\neq i \mid \arg a_{in}=\arg b_{jn}\neq\arg a_{jn} \text{ and } |a_{in}|>|b_{jn}|\big\},$$
$$I_3=\big\{j\neq i \mid \arg a_{in}=\arg a_{jn}\neq\arg b_{jn} \text{ and } |a_{in}|>|a_{jn}|\big\},$$
$$I_4=\big\{j\neq i \mid \arg a_{in}=\arg b_{jn}=\arg a_{jn} \text{ and } |a_{in}|>|a_{jn}|\ge|b_{jn}|\big\}.$$

It is clear that I_1, I_2, I_3, I_4 do not intersect with each other, and $I=I_1\cup I_2\cup I_3\cup I_4=\{0,1,\ldots,k\}\backslash\{i\}$.

Now, we may choose $\theta_0\in(0,2\pi)$ such that $\cos(\arg a_{in}+n\theta_0)=1$. (If $n=1$ and $\arg a_{in}=0$, then we replace $[0,2\pi)$ in Lemma 2.3 by $[-\frac{\pi}{2},\frac{3\pi}{2})$; if $n\ge 2$, this kind of θ_0 can always be chosen.) Set $z=re^{i\theta}$, $\theta\in[0,2\pi)$.

For $j \in I_1$, there exists sufficiently small ε_1 (> 0) such that for all $\theta \in (\theta_0 - \varepsilon_1, \theta_0 + \varepsilon_1) \subset (0, 2\pi)$, we have

$$\cos(\arg a_{in} + n\theta) > \max_{j \in I_1} \{\cos(\arg a_{jn} + n\theta), \cos(\arg b_{jn} + n\theta), 0\}.$$

Since $|a_{in}| \geq |a_{jn}| \geq |b_{jn}| > 0$, $j \neq i$, we see that

$$\delta(P_i, \theta) > \max_{j \in I_1} \{\delta(P_j, \theta), \delta(Q_j, \theta), 0\}.$$

For $j \in I_2$, there exists sufficiently small ε_2 (> 0) such that for all $\theta \in (\theta_0 - \varepsilon_2, \theta_0 + \varepsilon_2) \subset (0, 2\pi)$, we have

$$\cos(\arg a_{in} + n\theta) = \cos(\arg b_{jn} + n\theta) > \max_{j \in I_2} \{\cos(\arg a_{jn} + n\theta), 0\}.$$

Since $|a_{in}| > |b_{jn}|$, $j \neq i$, and $|a_{in}| \geq |a_{jn}|$, we see that

$$\delta(P_i, \theta) > \max_{j \in I_2} \{\delta(P_j, \theta), \delta(Q_j, \theta)\} > 0.$$

For $j \in I_3$, there exists sufficiently small ε_3 (> 0) such that for all $\theta \in (\theta_0 - \varepsilon_3, \theta_0 + \varepsilon_3) \subset (0, 2\pi)$, we have

$$\cos(\arg a_{in} + n\theta) = \cos(\arg a_{jn} + n\theta) > \max_{j \in I_3} \{\cos(\arg b_{jn} + n\theta), 0\}.$$

Since $|a_{in}| > |a_{jn}| \geq |b_{jn}|$, $j \neq i$, we see that

$$\delta(P_i, \theta) > \max_{j \in I_3} \{\delta(P_j, \theta), \delta(Q_j, \theta)\} > 0.$$

For $j \in I_4$, there exists sufficiently small ε_4 (> 0) such that for all $\theta \in (\theta_0 - \varepsilon_4, \theta_0 + \varepsilon_4) \subset (0, 2\pi)$, we have

$$\cos(\arg a_{in} + n\theta) = \cos(\arg b_{jn} + n\theta) = \cos(\arg a_{jn} + n\theta) > 0.$$

Since $|a_{in}| > |a_{jn}| \geq |b_{jn}|$, we see that

$$\delta(P_i, \theta) > \max_{j \in I_4} \{\delta(P_j, \theta), \delta(Q_j, \theta)\} > 0.$$

Set $\varepsilon_0' = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, then for any $\theta \in (\theta_0 - \varepsilon_0', \theta_0 + \varepsilon_0') \subset (0, 2\pi)$, we have

$$\delta(P_i, \theta) = \delta_1 > \max_{j \in I} \{\delta(P_j, \theta), \delta(Q_j, \theta), 0\} = \delta_2. \tag{3.1}$$

If $|b_{in}| < |a_{in}|$ and $\arg b_{in} = \arg a_{in}$, then for all $\theta \in (\theta_0 - \varepsilon_0', \theta_0 + \varepsilon_0') \subset (0, 2\pi)$, we have

$$\delta(P_i, \theta) > \delta(Q_i, \theta) > 0.$$

If $\arg a_{in} \neq \arg b_{in}$ and $|b_{in}| \leq |a_{in}|$, then there exists ε_0 ($0 < \varepsilon_0 \leq \varepsilon_0'$) such that for all $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \subset (0, 2\pi)$, we have

$$\cos(\arg a_{in} + n\theta) > \max\{\cos(\arg b_{in} + n\theta), 0\}$$

and, correspondingly,

$$\delta(P_i, \theta) > \max\{\delta(Q_i, \theta), 0\}.$$

Therefore, if $b_{in} \neq a_{in}$, and $|b_{in}| \leq |a_{in}|$, then there exists ε_0 (> 0) such that for all $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \subset (0, 2\pi)$, we have

$$\delta(P_i, \theta) = \delta_1 > \max\{\delta(Q_i, \theta), 0\} = \delta_3. \tag{3.2}$$

If $b_{in} = a_{in}$, then $A_i(z) = \tilde{B}_i(z)e^{a_{in}z^n} + R_i(z)$, where $\tilde{B}_i(z)$ is a meromorphic function and satisfies $\sigma(\tilde{B}_i) \leq \max\{\omega, n-1\}$. Since $A_i(z) - R_i(z) \not\equiv 0$, we see that $\tilde{B}_i(z) \not\equiv 0$. Here, $D_i(z) \equiv 0$, and

$$\delta(a_{in}z^n, \theta) = \delta(P_i, \theta) = \delta_1 > 0. \tag{3.3}$$

By Lemma 2.2, for any given ε ($0 < \varepsilon < \frac{1}{2}\min\{\frac{\delta_1-\delta_2}{\delta_1+\delta_2}, \frac{\delta_1-\delta_3}{\delta_1+\delta_3}, n-\omega\}$), there exists a set $E_1 \subset (0, +\infty)$ with finite linear measure, such that for all z satisfying $|z| = r \notin E_1$, $j = 0, 1, \ldots, k$ and $r \to \infty$, we have

$$|B_j(z)| \leq \exp\{r^{\omega+\varepsilon}\}, \qquad |D_j(z)| \leq \exp\{r^{\omega+\varepsilon}\}, \qquad |R_j(z)| \leq \exp\{r^{\omega+\varepsilon}\}. \tag{3.4}$$

By Lemma 2.3 and (3.1)-(3.4), for the above $\varepsilon > 0$, there exists $H_0 \subset [0, 2\pi)$ with linear measure zero and a finite set $H_1 = \bigcup_{i=0}^{k}\{\theta \in [0, 2\pi) | \delta(P_i, \theta) = 0 \text{ or } \delta(Q_i, \theta) = 0\}$, such that for all $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0) \backslash (H_0 \cup H_1)$, there exists $r_0 = r_0(\theta, \varepsilon) > 0$, such that for $r > r_0$, we have

$$\begin{aligned} |A_j(z)| &\leq |B_j(z)e^{P_j(z)}| + |D_j(z)e^{Q_j(z)}| + |R_j(z)| \\ &\leq 2\exp\{(1+\varepsilon)\delta_2 r^n\} + \exp\{r^{\omega+\varepsilon}\} \\ &\leq 3\exp\{(1+\varepsilon)\delta_2 r^n\}\exp\{r^{\omega+\varepsilon}\}, \quad j \neq i \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |A_i(z)| &\geq |B_i(z)e^{P_i(z)}| - |D_i(z)e^{Q_i(z)}| - |R_i(z)| \\ &\geq \exp\{(1-\varepsilon)\delta_1 r^n\} - \exp\{(1+\varepsilon)\delta_3 r^n\} - \exp\{r^{\omega+\varepsilon}\} \\ &\geq \exp\{(1-\varepsilon)\delta_1 r^n\} - 2\exp\{(1+\varepsilon)\delta_3 r^n\}\exp\{r^{\omega+\varepsilon}\}. \end{aligned} \tag{3.6}$$

By Lemma 2.1, for the above $\varepsilon > 0$, there exists a set $E_2 \subset (1, +\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin E_2 \cup [0, 1]$, we have

$$\left|\frac{f(z+c_j)}{f(z+c_i)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}, \quad j \neq i. \tag{3.7}$$

Equation (1.1) gives

$$-A_i(z) = A_k(z)\frac{f(z+c_k)}{f(z+c_i)} + \cdots + A_{i+1}(z)\frac{f(z+c_{i+1})}{f(z+c_i)} + A_{i-1}(z)\frac{f(z+c_{i-1})}{f(z+c_i)} + \cdots + A_0(z)\frac{f(z)}{f(z+c_i)}$$

and, correspondingly,

$$\begin{aligned}\left|A_i(z)\right| \le{}& \left|A_k(z)\right|\left|\frac{f(z+c_k)}{f(z+c_i)}\right| + \cdots + \left|A_{i+1}(z)\right|\left|\frac{f(z+c_{i+1})}{f(z+c_i)}\right| \\ &+ \left|A_{i-1}(z)\right|\left|\frac{f(z+c_{i-1})}{f(z+c_i)}\right| + \cdots + \left|A_0(z)\right|\left|\frac{f(z)}{f(z+c_i)}\right|. \end{aligned} \tag{3.8}$$

It follows from (3.5)-(3.8) that for all $z = re^{i\theta}$, where $\theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash(H_0 \cup H_1)$, $r \notin [0,1] \cup E_1 \cup E_2$ and $r \to \infty$, we have

$$\begin{aligned}&\exp\{(1-\varepsilon)\delta_1 r^n\} - 2\exp\{(1+\varepsilon)\delta_3 r^n\}\exp\{r^{\omega+\varepsilon}\} \\ &\quad\le 3k\exp\{r^{\sigma-1+\varepsilon}\}\exp\{(1+\varepsilon)\delta_2 r^n\}\exp\{r^{\omega+\varepsilon}\}. \end{aligned} \tag{3.9}$$

Since $0 < \varepsilon < \frac{1}{2}\min\{\frac{\delta_1-\delta_2}{\delta_1+\delta_2}, \frac{\delta_1-\delta_3}{\delta_1+\delta_3}, n-\omega\}$, we have

$$(1+\varepsilon)\delta_2 < (1-2\varepsilon)\delta_1, \qquad (1+\varepsilon)\delta_3 < (1-2\varepsilon)\delta_1, \qquad \omega+\varepsilon < n-\varepsilon$$

and

$$\frac{2\exp\{(1+\varepsilon)\delta_3 r^n\}\exp\{r^{\omega+\varepsilon}\}}{\exp\{(1-\varepsilon)\delta_1 r^n\}} < 2\exp\{r^{\omega+\varepsilon} - \varepsilon\delta_1 r^n\} \to 0, \quad r \notin [0,1] \cup E_1 \cup E_2, r \to \infty.$$

Then from (3.9), for sufficiently large r, we have

$$\frac{1}{2}\exp\{(1-\varepsilon)\delta_1 r^n\} \le 3k\exp\{r^{\sigma-1+\varepsilon}\}\exp\{(1+\varepsilon)\delta_2 r^n\}\exp\{r^{\omega+\varepsilon}\},$$

i.e.,

$$\exp\{(1-\varepsilon)\delta_1 r^n - (1+\varepsilon)\delta_2 r^n - r^{\omega+\varepsilon}\} \le 6k\exp\{r^{\sigma-1+\varepsilon}\},$$

i.e.,

$$\exp\left\{\frac{1}{2}\varepsilon\delta_1 r^n\right\} \le 6k\exp\{r^{\sigma-1+\varepsilon}\}. \tag{3.10}$$

Then (3.10) implies $n \le \sigma - 1 + \varepsilon$. Since ε is arbitrary, we have $\sigma(f) = \sigma \ge n+1$.

Therefore, every meromorphic solution $f(z)$ ($\not\equiv 0$) satisfies $\sigma(f) \ge n+1$.

Set $g(z) = f(z) - \varphi(z)$, then $g(z)$ solves the equation

$$\begin{aligned}&A_k(z)g(z+c_k) + \cdots + A_1(z)g(z+c_1) + A_0(z)g(z) \\ &\quad= -A_k(z)\varphi(z+c_k) - \cdots - A_1(z)\varphi(z+c_1) - A_0(z)\varphi(z).\end{aligned}$$

Since $\sigma(\varphi) < n+1$, $\varphi(z)$ ($\not\equiv 0$) does not solve (1.1), that is,

$$A_k(z)\varphi(z+c_k) + \cdots + A_1(z)\varphi(z+c_1) + A_0(z)\varphi(z) \not\equiv 0$$

and

$$\begin{aligned}&\sigma\bigl(A_k(z)\varphi(z+c_k) + \cdots + A_1(z)\varphi(z+c_1) + A_0(z)\varphi(z)\bigr)\\ &\quad\le \max\bigl\{\sigma(A_j), j = 0,1,\dots,k, \sigma(\varphi)\bigr\}\\ &\quad< n+1 \le \sigma(f).\end{aligned}$$

Therefore, by Lemma 2.5, we have $\lambda(g) = \sigma(g)$, *i.e.*, $\lambda(f-\varphi) = \sigma(f)$.

The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 Equation (1.2) gives

$$-A_0(z) = A_k(z)\frac{\Delta^k f(z)}{f(z)} + \cdots + A_1(z)\frac{\Delta f(z)}{f(z)}. \tag{3.11}$$

By Remark 2.1, we have

$$\left|\frac{\Delta^j f(z)}{f(z)}\right| \le O\bigl(\exp\bigl\{r^{\sigma-1+\varepsilon}\bigr\}\bigr), \quad j = 1,\dots,k. \tag{3.12}$$

By combining (3.5), (3.6), (3.12) with (3.11), for all $z = re^{i\theta}$, where $\arg z = \theta \in (\theta_0 - \varepsilon_0, \theta_0 + \varepsilon_0)\backslash(H_0 \cup H_1)$, $|z| = r \notin [0,1] \cup E_1 \cup E_2$ and for sufficiently large r, we have

$$\begin{aligned}&\exp\bigl\{(1-\varepsilon)\delta_1 r^n\bigr\} - 2\exp\bigl\{(1+\varepsilon)\delta_3 r^n\bigr\}\exp\bigl\{r^{\omega+\varepsilon}\bigr\}\\ &\quad\le O\bigl(\exp\bigl\{r^{\sigma-1+\varepsilon}\bigr\}\bigr)\exp\bigl\{(1+\varepsilon)\delta_2 r^n\bigr\}\exp\bigl\{r^{\omega+\varepsilon}\bigr\}.\end{aligned}$$

By using a similar method to the one in the proof of Theorem 1.1, we have $\sigma(f) = \sigma \ge n+1$.

Further, set $g(z) = f(z) - \varphi(z)$, then $g(z)$ solves the equation

$$\begin{aligned}&A_k(z)\Delta^k g(z) + \cdots + A_1(z)\Delta g(z) + A_0(z)g(z)\\ &\quad= -A_k(z)\Delta^k\varphi(z) - \cdots - A_1(z)\Delta\varphi(z) - A_0(z)\varphi(z).\end{aligned}$$

Since $\sigma(\varphi) < n+1$, $\varphi(z)$ ($\not\equiv 0$) does not solve (1.2), that is,

$$A_k(z)\Delta^k\varphi(z) + \cdots + A_1(z)\Delta\varphi(z) + A_0(z)\varphi(z) \not\equiv 0$$

and

$$\sigma\bigl(A_k\Delta^k\varphi + \cdots + A_1\Delta\varphi + A_0\varphi\bigr) \le \max\bigl\{\sigma(A_j), j = 0,1,\dots,k, \sigma(\varphi)\bigr\} < n+1 \le \sigma(f).$$

Therefore, by Lemma 2.5 and Remark 2.3, we have $\lambda(g) = \sigma(g)$, *i.e.*, $\lambda(f-\varphi) = \sigma(f)$.

The proof of Theorem 1.2 is completed. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, and they read and approved the final manuscript.

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