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Existence of the uniform attractors for a non-autonomous modified Swift-Hohenberg equation

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Abstract

The paper is concerned with the non-autonomous modified Swift-Hohenberg equation $u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(x,t)$. It is shown that a uniform attractor exists in H_0^2 when the external force only satisfies the translation bounded condition instead of translation compactness. In order to overcome the difficulty caused by the critical nonlinearity terms u^3 and the parameter b belonging to the real set $\mathbb{R}$, we take advantage of the Gagliardo-Nirenberg inequality several times.

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Keywords: modified Swift-Hohenberg equation; uniform attractor; condition (C^*); uniform condition (C)

1 Introduction

The Swift-Hohenberg type equations arise in the study of convective hydrodynamics, plasma confinement in toroidal devices, viscous film flow, and bifurcating solutions of the Navier-Stokes equations; see [1]. The long-time behavior, bifurcation, and the pattern selections of the solution for this equation have been investigated in [2–4].

We are concerned with the following non-autonomous modified Swift-Hohenberg equation:

$$u_t + \Delta^2 u + 2\Delta u + au + b|\nabla u|^2 + u^3 = g(x,t), \quad \text{in } \Omega \times \mathbb{R}_\tau, \tag{1.1}$$

$$u = \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times \mathbb{R}_\tau, \tag{1.2}$$

$$u(x,\tau) = u_\tau(x), \quad \text{in } \Omega, \tag{1.3}$$

where $\mathbb{R}_\tau = [\tau, +\infty)$, Ω is an open connected bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, a and b are arbitrary constants, $u_t = \frac{\partial u}{\partial t}$, and g is an external forcing term with $g(x,t) \in L^2_{C^*}(\mathbb{R}; X)$. If $b = 0$ and $g \equiv 0$, then (1.1) is the usual Swift-Hohenberg equation. The system (1.1)-(1.3) with $g \equiv 0$ was proposed by the authors in [5] as a pattern formation system with two unbounded spatial directions that is near the onset to instability. Polat studied the existence of global attractors for the problem (1.1) when $g \equiv 0$ in [6] and Song *et al.* improved the result by showing that the system possesses a global attractor in

H^k spaces in [7]. Recently, the authors established the existence of pullback attractors for (1.1)-(1.3) in [8]. To the best of our knowledge, the existence of the uniform attractors for a non-autonomous modified Swift-Hohenberg equation has not yet been considered; it is presently our concern.

In the last two decades, the dynamical systems and their attractors have been extensively studied, please refer to [9–13] and references therein. As we know, the most general methods dealing with the non-autonomous dynamical systems were presented by Chepyzhov and Vishik in their work in [14], and these methods make the general theory of autonomous systems applicable, but it is unsatisfactory that it can only be used in handling the problems with translation compact symbols. In 2005, in [15], the authors presented a new notation and obtained their abstract results by means of the methods introduced in [10] to deal with 2D Navier-Stokes equations with a normal external force in $L^2_{\mathrm{loc}}(\mathbb{R}, L^2)$ which is translation bounded but not translation compact. In the sequel, in [16], a new class of time-dependent external forces in $L^2_{C^*}(\mathbb{R}, X)$ was presented, where $L^2_{C^*}(\mathbb{R}, X)$ denotes the set of all functions satisfying condition (C^*) (see below Definition 2.4) and the functions in $L^2_{C^*}(\mathbb{R}, X)$ are translation bounded but not translation compact in $L^2_{\mathrm{loc}}(\mathbb{R}, L^2)$. Moreover, the authors proved the existence of uniform attractors for the weakly damped non-autonomous hyperbolic equations. Motivated by [15, 16], in the present paper, we illustrate the existence of uniform attractors for a non-autonomous modified Swift-Hohenberg equation (1.1)-(1.3) using the techniques in [15, 16].

This paper is organized as follows: in Section 2, we give some basic definitions and abstract results concerning the uniform attractors for non-autonomous dynamical systems. In Section 3, we will show the uniformly bounded absorbing set and uniform attractors in H_0^2.

2 Non-autonomous systems and their attractors

In this subsection, we iterate some basic definitions and abstract results concerning the uniform attractors for non-autonomous dynamical systems in [15, 16], which are important to get our main results.

With the usual notation, we denote $H = L^2(\Omega)$, and endow H with the standard scalar product and norm $(\cdot,\cdot)$, $\|\cdot\|$. For simplicity, we denote $V = H_0^2(\Omega)$ with norm $\|u\|_{H_0^2(\Omega)} = \|\triangle u\|$, and write $\|\cdot\|_{m,p}$ and $\|\cdot\|_p$ as the norm of $W^{m,p}(\Omega)$ and $L^p(\Omega)$, respectively.

We let an operator $A = \triangle^2$ and λ be the first eigenvalues of A; by the Poincaré inequality, we have

$$\|\triangle u\|^2 \geq \lambda \|u\|^2, \quad \forall u \in V. \tag{2.1}$$

Let E be a Banach space, and let a two-parameter family of mappings $\{U(t,\tau)\} = \{U(t,\tau) | t \geq \tau, \tau \in \mathbb{R}\}$ act on E:

$$U(t,\tau) : E \to E, \quad t \geq \tau, \tau \in \mathbb{R}.$$

Definition 2.1 Let Σ be a parameter set. $\{U_\sigma(t,\tau) | t \geq \tau, \tau \in \mathbb{R}\}$, $\sigma \in \Sigma$ is said to be a family of processes in Banach space E, if for each $\sigma \in \Sigma$, $\{U_\sigma(t,\tau)\}$ is a process, that is, the two-parameter family of mappings $\{U_\sigma(t,\tau)\}$ from E to E satisfy

$$U_\sigma(t,s) \circ U_\sigma(s,\tau) = U_\sigma(t,\tau), \quad \forall t \geq s \geq \tau, \tau \in \mathbb{R}, \tag{2.2}$$

$$U_\sigma(\tau,\tau) = \mathrm{Id} \text{ is the identity operator}, \quad \tau \in \mathbb{R}, \tag{2.3}$$

where Σ is called the symbol space and $\sigma \in \Sigma$ is the symbol.

A set $B_0 \subset E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any $\tau \in \mathbb{R}$ and $B \in \mathcal{B}(E)$, there exists $t_0 = t_0(\tau, B) \geq \tau$ such that $\bigcup_{\sigma\in\Sigma} U_\sigma(t,\tau)B \subset B_0$ for all $t \geq t_0$. A set $Y \subset E$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, if for any fixed $\tau \in \mathbb{R}$ and every $B \in \mathcal{B}(E)$,

$$\lim_{t\to\infty} \sup_{\sigma\in\Sigma} \operatorname{dist}_E\bigl(U_\sigma(t,\tau)B, Y\bigr) = 0, \tag{2.4}$$

where $\mathcal{B}(E)$ is the set of all bounded subset of E.

Assumption I Let $\{T(h)|h \geq 0\}$ be a family of operators acting on Σ and satisfy:

(i) $T(h)\Sigma = \Sigma$, $\forall h \in \mathbb{R}^+$;

(ii) translation identity:

$$U_\sigma(t+h, \tau+h) = U_{T(h)\sigma}(t,\tau), \quad \forall \sigma \in \Sigma, t \geq \tau, \tau \in \mathbb{R}, h \geq 0. \tag{2.5}$$

Definition 2.2 A family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$ is said to be satisfying uniformly (w.r.t. $\sigma \in \Sigma$) condition (C) if for any fixed $\tau \in \mathbb{R}$, $B \in \mathcal{B}(E)$, and $\varepsilon > 0$, there exist a $t_0 = t_0(\tau, B, \varepsilon) \geq \tau$ and a finite dimensional subspace E_m of E such that

(i) $P_m(\bigcup_{\sigma\in\Sigma} \bigcup_{t\geq t_0} U_\sigma(t,\tau)B)$ is bounded;

(ii) $\|(I-P_m)(\bigcup_{\sigma\in\Sigma} \bigcup_{t\geq t_0} U_\sigma(t,\tau)x)\|_E \leq \varepsilon$, $\forall x \in B$,

where $\dim E_m = m$ and $P_m : E \to E_m$ is a bounded projector.

Theorem 2.3 *Let Σ be a complete metric space, and Assumption* I *holds. Then a family of processes $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, possess the compact uniform (w.r.t. $\sigma \in \Sigma$) attractor $\mathcal{A}_\Sigma$ in E satisfying*

$$\mathcal{A}_\Sigma = \omega_{0,\Sigma}(B_0) = \omega_{\tau,\Sigma}(B_0), \quad \forall \tau \in \mathbb{R},$$

if it:

(i) *has a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set B_0*;

(ii) *satisfies uniformly (w.r.t. $\sigma \in \Sigma$) condition* (C).

Moreover, if E is a uniformly convex Banach space, then the converse is true.

Remark 1 Theorem 2.3 is true without any continuous assumption on $\{U_\sigma(t,\tau)\}$, $\sigma \in \Sigma$, and $\{T(t)\}_{t\geq 0}$.

Definition 2.4 Let X be a Hilbert space. A function $g \in L^2_{\mathrm{loc}}(\mathbb{R}; X)$ is said to satisfy condition (C*) if for any $\varepsilon > 0$, there exists a finite dimensional subspace X_1 of X such that

$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \bigl\|(I-P_m)g(x,s)\bigr\|_X^2 \,\mathrm{d}s < \varepsilon,$$

where $P_m : X \to X_1$ is the canonical projector.

Denote by $L^2_{C^*}(\mathbb{R};X)$ the set of all functions satisfying condition (C^*).

Lemma 2.5 *If $h \in L^2_{C^*}(\mathbb{R};X)$, then for any $\varepsilon > 0$ and $\tau \in \mathbb{R}$, we have*

$$\sup_{t\geq\tau}\int_\tau^t e^{-\alpha(t-s)}\left\|(I-P_m)h(s)\right\|_X^2\,\mathrm{d}s < \varepsilon,$$

where P_m is the same as in Definition 2.4 *and α is a positive constant.*

Lemma 2.6 [9, 14] (Gagliardo-Nirenberg inequality) *Let Ω be an open, bounded domain of the Lipschitz class in $\mathbb{R}^n$. Assume that $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $r \geq 1$, $0 < \theta \leq 1$, and that*

$$k - \frac{n}{p} \leq \theta\left(m - \frac{n}{q}\right) + (1-\theta)\frac{n}{r}.$$

Then the following inequality holds:

$$\|u\|_{k,p} \leq C(\Omega)\|u\|_r^{1-\theta}\|u\|_{m,q}^\theta.$$

3 Uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set and uniform (w.r.t. $\sigma \in \Sigma$) attractor in V

For the existence of the solutions for (1.1)-(1.3), since the time-dependent term introduces no essential complications, we directly give the following results of the existence and uniqueness of solution without proof. In fact, the proof is based on the Faedo-Galerkin approximation approaches; see [9] for the details.

Theorem 3.1 *If g and u_τ are given satisfying $g \in L^2_{\mathrm{loc}}(\mathbb{R};H)$, $u_\tau \in V$, then* (1.1)-(1.3) *have a unique solution*

$$u(t) \in C(\mathbb{R}_\tau;V), \qquad \partial_t u \in C(\mathbb{R}_\tau;H).$$

We now give a fixed external force g_0 in $L^2_b(\mathbb{R};X)$ and define the symbol space $\mathcal{H}(g_0)$ for (1.1)-(1.3). Let a fixed symbol $\sigma_0(s) = g_0(s) = g_0(\cdot,s)$ satisfy condition (C^*) in $L^2_{\mathrm{loc}}(\mathbb{R};X)$. That is, the family of translations $\{g_0(s+h), h \in \mathbb{R}\}$ form a function set satisfying condition (C^*). Therefore

$$\mathcal{H}(\sigma_0) = \mathcal{H}(g_0) = \left[g_0(x,s+h)|h \in \mathbb{R}\right]_{L^{2,w}_{\mathrm{loc}}(\mathbb{R};X)},$$

where [ ] denotes the closure of a set in a topological space $L^{2,w}_{\mathrm{loc}}(\mathbb{R};X)$.

Thus, for any $g(x,t) \in \mathcal{H}(g_0)$, the problem (1.1)-(1.3) with g instead of g_0 possesses a corresponding process $\{U_g(t,\tau)\}$ acting on V.

Proposition 3.2 [14, 16] *If X is a reflexive separable, then*

(1) *for all $g_1 \in \mathcal{H}(\phi)$, $\|g_1\|^2_{L^2_b(\mathbb{R};X)} \leq \|g\|^2_{L^2_b(\mathbb{R};X)}$*;
(2) *the translation group $\{T(t)\}$ is weakly continuous on $\mathcal{H}(g)$*;
(3) *$T(t)\mathcal{H}(g) = \mathcal{H}(g)$ for all $t \in \mathbb{R}$.*

Therefore, the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, $U_g(t,\tau): E \to E$, $t \geq \tau$, $\tau \in \mathbb{R}$ are defined. Furthermore, the translation semigroup $\{T(h)|h \in \mathbb{R}^+\}$ satisfies $\forall h \in \mathbb{R}^+$, $T(h)\mathcal{H}(g_0) = \mathcal{H}(g_0)$, and the following translation identity holds:

$$U_g(t+h,\tau+h) = U_{T(h)g}(t,\tau), \quad \forall g \in \mathcal{H}(g_0), t \geq \tau, \tau \in \mathbb{R}, h \geq 0.$$

For (1.1)-(1.3), we give a fixed external force $g_0 \in L^2_{C^*}(\mathbb{R};H)$ and $\mathcal{H}(\sigma_0) = \mathcal{H}(g_0) = [g_0(x,s+h)|h \in \mathbb{R}]_{L^{2,w}_{\mathrm{loc}}(\mathbb{R};H)}$.

For convenience, hereafter we denote by c an arbitrary positive constant, which may be different from line to line and even in the same line.

3.1 *A priori* estimates

Theorem 3.3 *If $g_0 \in L^2_b(\mathbb{R};H)$, $u_\tau \in V$, then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(h_0)$ corresponding to the problem* (1.1)-(1.3) *has a bounded uniformly* (*w.r.t.* $g \in \mathcal{H}(h_0)$) *absorbing set B_0 in V.*

Proof The proof of this results follows from the following two steps.

Step 1: Taking the scalar products in H of (1.1) with u, we find

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \|\triangle u(t)\|^2 &= -\|u(t)\|_4^4 + 2\|\nabla u(t)\|^2 - a\|u(t)\|^2 \\ &\quad - b\int_\Omega |\nabla u(t)|^2 u(t)\,\mathrm{d}x + \big(g(t),u(t)\big), \end{aligned} \tag{3.1}$$

and the Hölder and Poincaré inequalities give

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \|\triangle u(t)\|^2 &\leq -2\|u(t)\|_4^4 + 4\|\nabla u(t)\|^2 + 2|a|\|u(t)\|^2 \\ &\quad + 2|b|\int_\Omega |\nabla u(t)|^2|u(t)|\,\mathrm{d}x + \frac{\|g(t)\|^2}{\lambda}. \end{aligned} \tag{3.2}$$

We use the Gagliardo-Nirenberg inequality with $k=1$, $n=p=r=m=q=2$, $\theta=\frac{1}{2}$ and get

$$4\|\nabla u(t)\|^2 \leq c\|u(t)\|\|\triangle u(t)\| \leq c\|u(t)\|^2 + \frac{1}{4}\|\triangle u(t)\|^2. \tag{3.3}$$

Similarly, by the Hölder inequality, the Gagliardo-Nirenberg inequality with $k=1$, $n=m=q=2$, $p=r=4$, $0<\theta<\frac{1}{2}$, and the Young inequality, it follows that

$$\begin{aligned} 2|b|\int_\Omega |\nabla u(t)|^2|u(t)|\,\mathrm{d}x &\leq 2|b|\|\nabla u(t)\|_4^2\|u(t)\| \\ &\leq c\|\triangle u(t)\|^{2\theta}\|u(t)\|_4^{2(1-\theta)}\|u(t)\| \\ &\leq c\|\triangle u(t)\|^{2\theta}\|u(t)\|_4^{3-2\theta} \\ &\leq \frac{1}{4}\|\triangle u(t)\|^2 + c\|u(t)\|_4^{\frac{3-2\theta}{1-\theta}}. \end{aligned} \tag{3.4}$$

Substituting (3.3), (3.4) into (3.2), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \frac{1}{2}\|\triangle u(t)\|^2 \leq -2\|u(t)\|_4^4 + c\|u(t)\|^2 + c\|u(t)\|_4^{\frac{3-2\theta}{1-\theta}} + \frac{\|g(t)\|^2}{\lambda}. \tag{3.5}$$

Since $3 < \frac{3-2\theta}{1-\theta} < 4$ $(0 < \theta < \frac{1}{2})$, there exists $M > 0$ such that

$$-2\|u(t)\|_4^4 + c\|u(t)\|^2 + c\|u(t)\|_4^{\frac{3-2\theta}{1-\theta}} \le -2\|u(t)\|_4^4 + c\|u(t)\|_4^2 + c\|u(t)\|_4^{\frac{3-2\theta}{1-\theta}} \le M. \quad (3.6)$$

Thus we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \frac{1}{2}\|\triangle u(t)\|^2 \le M + \frac{\|g(t)\|^2}{\lambda}. \quad (3.7)$$

Again in line with the Poincaré inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2 \le M + \frac{\|g(t)\|^2}{\lambda}. \quad (3.8)$$

From Proposition 3.2, recall that

$$\|g\|_{L_b^2}^2 \le \|g_0\|_{L_b^2}^2, \quad (3.9)$$

and set $\delta = \frac{\lambda}{2}$, then by the Gronwall lemma,

$$\|u(t)\|^2 \le \|u_\tau\|^2 e^{-\delta(t-\tau)} + \left(1 + \delta^{-1}\right)\left(M + \frac{\|g_0\|_{L_b^2}^2}{2\delta}\right). \quad (3.10)$$

Now, multiplying (3.10) by $e^{\delta t}$ and integrating it over (τ, t), we get

$$\begin{aligned}\int_\tau^t e^{\delta s}\|u(s)\|^2\,\mathrm{d}s &\le \int_\tau^t e^{\delta\tau}\|u_\tau\|^2\,\mathrm{d}s + \int_\tau^t e^{\delta s}\left(1+\delta^{-1}\right)\left(M + \frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)\mathrm{d}s \\ &\le (t-\tau)e^{\delta\tau}\|u_\tau\|^2 + \frac{e^{\delta t}(1+\delta)}{\delta^2}\left(M + \frac{\|g_0\|_{L_b^2}^2}{2\delta}\right). \quad (3.11)\end{aligned}$$

Analogously, multiplying (3.7) by $e^{\delta t}$ and integrating it over (τ, t) and together with (3.11), we derive that

$$\begin{aligned}\int_\tau^t e^{\delta s}\|\triangle u(s)\|^2\,\mathrm{d}s &\le 2e^{\delta\tau}\|u_\tau\|^2 + 2\delta\int_\tau^t e^{\delta s}\|u(s)\|^2\,\mathrm{d}s + \frac{2M}{\delta}e^{\delta t} \\ &\quad + \frac{1}{\delta}\int_\tau^t e^{\delta s}\|g(s)\|^2\,\mathrm{d}s \\ &\le 2\big(1+\delta(t-\tau)\big)e^{\delta\tau}\|u_\tau\|^2 + \frac{2e^{\delta t}}{\delta}\left((2+\delta)M + \frac{1+\delta}{2\delta}\|g_0\|_{L_b^2}^2\right) \\ &\quad + \frac{1}{\delta}\int_\tau^t e^{\delta s}\|g_0\|_{L_b^2}^2\,\mathrm{d}s. \quad (3.12)\end{aligned}$$

Step 2: Multiplying (1.1) by $\triangle^2 u$ in H we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u(t)\|^2 + \|\triangle^2 u(t)\|^2 + a\|\triangle u(t)\|^2 + 2\big(\triangle u(t), \triangle^2 u(t)\big) \\ &\quad + b\big(|\nabla u(t)|^2, \triangle^2 u(t)\big) + \big(u(t)^3, \triangle^2 u(t)\big) = \big(g(t), \triangle^2 u(t)\big). \quad (3.13)\end{aligned}$$

Using the Hölder and Young inequalities, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u(t)\|^2 + \|\triangle^2 u(t)\|^2 \le -2|a|\|\triangle u(t)\|^2 + 16\|\triangle u(t)\|^2 + 4|b|^2\|\nabla u(t)\|_4^4 + 4\|u(t)\|_6^6 + 4\|g(t)\|^2. \tag{3.14}$$

The Gagliardo-Nirenberg inequality with $k = 1$, $p = 4$, $n = m = q = r = 2$, $\theta = \frac{1}{4}$ yields

$$4|b|^2\|\nabla u(t)\|_4^4 \le c\|u(t)\|^3\|\triangle u(t)\| \le \frac{\lambda}{4}\|\triangle u(t)\|^2 + c\|u(t)\|^6.$$

The Gagliardo-Nirenberg inequality with $k = 0$, $p = 6$, $n = m = q = r = 2$, $\theta = \frac{1}{6}$ gives

$$4\|u(t)\|_6^6 \le c\|u(t)\|^5\|\triangle u(t)\| \le \frac{\lambda}{4}\|\triangle u(t)\|^2 + c\|u(t)\|^{10}.$$

Substituting these estimates into (3.14), and combining with the Poincaré inequalities, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\triangle u(t)\|^2 + \delta\|\triangle u(t)\|^2 \le c\big(\|\triangle u(t)\|^2 + \|u(t)\|^6 + \|u(t)\|^{10} + \|g(t)\|^2\big).$$

Multiplying this by $(t-\tau)e^{\delta t}$ and integrating it over (τ, t), we derive that

$$(t-\tau)e^{\delta t}\|\triangle u(t)\|^2 \le c\biggl[\int_\tau^t \big(1 + (s-\tau)\big)e^{\delta s}\|\triangle u(s)\|^2\,\mathrm{d}s + \int_\tau^t (s-\tau)e^{\delta s}\|g(s)\|^2\,\mathrm{d}s + \int_\tau^t (s-\tau)e^{\delta s}\big(\|u(s)\|^6 + \|u(s)\|^{10}\big)\,\mathrm{d}s\biggr]$$

and hence

$$\begin{aligned}\|\triangle u(t)\|^2 &\le c\left(1 + \frac{1}{t-\tau}\right)\int_\tau^t e^{\delta(s-t)}\|\triangle u(s)\|^2\,\mathrm{d}s + c\int_\tau^t e^{\delta(s-t)}\|g(s)\|^2\,\mathrm{d}s \\ &\quad + c\int_\tau^t e^{\delta(s-t)}\|u(s)\|^6\,\mathrm{d}s + c\int_\tau^t e^{\delta(s-t)}\|u(s)\|^{10}\,\mathrm{d}s \\ &:= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.15}$$

Now, we estimate the terms on the right-hand side of (3.15):

$$\begin{aligned}I_2 &= c\int_\tau^t e^{\delta(s-t)}\|g(s)\|^2\,\mathrm{d}s \\ &\le c\left(\int_{t-1}^t e^{\delta(s-t)}\|g(s)\|^2\,\mathrm{d}s + \int_{t-2}^{t-1} e^{\delta(s-t)}\|g(s)\|^2\,\mathrm{d}s + \cdots\right) \\ &\le c\big(1 + e^{-\delta} + e^{-2\delta} + \cdots\big)\|g\|_{L_b^2}^2 \\ &\le c\big(1 + \delta^{-1}\big)\|g\|_{L_b^2}^2 \\ &\le c\big(1 + \delta^{-1}\big)\|g_0\|_{L_b^2}^2.\end{aligned} \tag{3.16}$$

By (3.12), we get

$$\begin{aligned}
I_1 &= c\left(1+\frac{1}{t-\tau}\right)e^{-\delta t}\int_\tau^t e^{\delta s}\left\|\triangle u(s)\right\|^2\,\mathrm{d}s \\
&\le c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta(t-\tau)\right)e^{-\delta(t-\tau)}\|u_\tau\|^2 \\
&\quad + c\left(1+\frac{1}{t-\tau}\right)\left((2+\delta)M+\frac{1+\delta}{2\delta}\|g_0\|_{L_b^2}^2\right) \\
&\quad + c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta^{-1}\right)\|g_0\|_{L_b^2}^2 \\
&\le c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta(t-\tau)\right)e^{-\delta(t-\tau)}\|u_\tau\|^2 + c\left(1+\frac{1}{t-\tau}\right)(2+\delta)M \\
&\quad + c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta^{-1}\right)\|g_0\|_{L_b^2}^2. \qquad (3.17)
\end{aligned}$$

By (3.10) and the Hölder inequalities for the sum, we have

$$\begin{aligned}
I_3 &= c\int_\tau^t e^{\delta(s-t)}\left\|u(s)\right\|^6\,\mathrm{d}s \\
&\le ce^{-\delta t}\int_\tau^t e^{\delta s}\left(\|u_\tau\|^2 e^{-\delta(s-\tau)}+\left(1+\delta^{-1}\right)\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)\right)^3\,\mathrm{d}s \\
&\le ce^{-\delta t}\int_\tau^t e^{\delta s}e^{-3\delta(s-\tau)}\|u_\tau\|^6\,\mathrm{d}s + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^3 e^{-\delta t}\int_\tau^t e^{\delta s}\,\mathrm{d}s \\
&\le ce^{-\delta(t-\tau)}\|u_\tau\|^6\int_\tau^t e^{-2\delta(s-\tau)}\,\mathrm{d}s + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^3 e^{-\delta t}\left(e^{\delta t}-e^{\delta\tau}\right) \\
&\le c(t-\tau)e^{-\delta(t-\tau)}\|u_\tau\|^6 + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^3, \qquad (3.18)
\end{aligned}$$

noting that we used the fact $e^{-2\delta(s-\tau)}\le 1$ for $s\in[\tau,t]$ in the last inequality.

In addition,

$$\begin{aligned}
I_4 &= c\int_\tau^t e^{\delta(s-t)}\left\|u(s)\right\|^{10}\,\mathrm{d}s \\
&\le ce^{-\delta t}\int_\tau^t e^{\delta s}\left(\|u_\tau\|^2 e^{-\delta(s-\tau)}+\left(1+\delta^{-1}\right)\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)\right)^5\,\mathrm{d}s \\
&\le ce^{-\delta t}\int_\tau^t e^{\delta s}e^{-5\delta(s-\tau)}\|u_\tau\|^{10}\,\mathrm{d}s + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^5 e^{-\delta t}\int_\tau^t e^{\delta s}\,\mathrm{d}s \\
&\le ce^{-\delta(t-\tau)}\|u_\tau\|^{10}\int_\tau^t e^{-4\delta(s-\tau)}\,\mathrm{d}s + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^5 e^{-\delta t}\left(e^{\delta t}-e^{\delta\tau}\right) \\
&\le c(t-\tau)e^{-\delta(t-\tau)}\|u_\tau\|^{10} + c\left(M+\frac{\|g_0\|_{L_b^2}^2}{2\delta}\right)^5. \qquad (3.19)
\end{aligned}$$

Collecting all inequalities (3.15)-(3.19),

$$
\begin{aligned}
\|\triangle u(t)\|^2 \leq{}& c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta(t-\tau)\right)e^{-\delta(t-\tau)}\|u_\tau\|^2 \\
&+ c(t-\tau)e^{-\delta(t-\tau)}\left(\|u_\tau\|^6+\|u_\tau\|^{10}\right) \\
&+ c\left(1+\frac{1}{t-\tau}\right)(2+\delta)M + c\left(1+\frac{1}{t-\tau}\right)\left(1+\delta^{-1}\right)\|g_0\|^2_{L^2_b} \\
&+ c\left(M+\frac{\|g_0\|^2_{L^2_b}}{2\delta}\right)^3 + c\left(M+\frac{\|g_0\|^2_{L^2_b}}{2\delta}\right)^5 \\
\leq{}& c\left[\left(1+\frac{1}{t-\tau}\right)\left(1+\delta(t-\tau)\right)e^{-\delta(t-\tau)}\|u_\tau\|^2\right. \\
&+ (t-\tau)e^{-\delta(t-\tau)}\left(\|u_\tau\|^6+\|u_\tau\|^{10}\right) \\
&\left.+ \left(1+\frac{1}{t-\tau}\right)\left(1+\|g_0\|^2_{L^2_b}\right) + \left(M+\frac{\|g_0\|^2_{L^2_b}}{2\delta}\right)^3 + \left(M+\frac{\|g_0\|^2_{L^2_b}}{2\delta}\right)^5\right]. \qquad (3.20)
\end{aligned}
$$

Choosing $(1+\frac{1}{t-\tau})(1+\delta(t-\tau))e^{-\delta(t-\tau)}\|u_\tau\|^2 + (t-\tau)e^{-\delta(t-\tau)}(\|u_\tau\|^6+\|u_\tau\|^{10}) + (1+\frac{1}{t-\tau})(1+\|g_0\|^2_{L^2_b}) \leq (M+\frac{\|g_0\|^2_{L^2_b}}{2\delta})^3 + (M+\frac{\|g_0\|^2_{L^2_b}}{2\delta})^5$ for $t>\tau$, we deduce that there exists a time $t_0 = t_0(\delta, \|g_0\|^2_{L^2_b}, \|u_\tau\|) > \tau$ such that $B_0 = \{u : \|\triangle u(t)\|^2 \leq \rho^2\}$, where $\rho^2 = c[(M+\frac{\|g_0\|^2_{L^2_b}}{2\delta})^3 + (M+\frac{\|g_0\|^2_{L^2_b}}{2\delta})^5]$, for $t \geq t_0$, *i.e.*, B_0 is the uniformly (w.r.t. $\sigma \in \Sigma$) absorbing ball for the process $\{U_\sigma(t,\tau)\}$ in V. □

3.2 Uniform attractor in *V*

Now we prove the existence of compact uniform (w.r.t. $h \in \mathcal{H}(g_0)$) attractor for the problem (1.1)-(1.3) with the external forces $g_0 \in L^2_{C^*}(\mathbb{R};H)$ in V.

Theorem 3.4 *If $g_0(x,t) \in L^2_{C^*}(\mathbb{R};H)$, then the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$ corresponding to the problem* (1.1)-(1.3) *possess a compact uniform* (*w.r.t.* $g \in \mathcal{H}(g_0)$) *attractor $\mathcal{A}_{\mathcal{H}(g_0)}$ in V satisfying*

$$
\mathcal{A}_{\mathcal{H}(g_0)} = \omega_{0,\mathcal{H}(g_0)}(B_0) = \omega_{\tau,\mathcal{H}(g_0)}(B_0), \qquad (3.21)
$$

where B_0 is the uniformly (*w.r.t.* $h \in \mathcal{H}(g_0)$) *absorbing set in V.*

Proof By Theorem 2.3, we need only to verify that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, satisfy uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) condition (C).

Since A^{-1} is a continuous compact operator in H, by the classical spectral theorem, there exists a sequence $\{\lambda_j\}_{j=1}^\infty$ with

$$
0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots, \quad \lambda_j \to \infty, \text{as } j \to \infty,
$$

and a family of elements $\{\omega_j\}_{j=1}^\infty$ of V which are orthonormal in H with

$$
A\omega_j = \lambda_j \omega_j, \quad \forall j \in \mathbb{N}.
$$

Let $H_m = \operatorname{span}\{\omega_1, \omega_2, \dots, \omega_m\}$, and $P_m : H \to H_m$ be an orthogonal projector. For any $u \in V$, we write

$$u = P_m u + (I - P_m)u \triangleq u_1 + u_2.$$

Taking the scalar product of (1.1) with $\triangle^2 u_2$ in H and using the Young inequality, similar to the estimates (3.14), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big\|\triangle u_2(t)\big\|^2 + \big\|\triangle^2 u_2(t)\big\|^2 \\
&\quad\le c\big(\big\|\triangle u_2(t)\big\|^2 + \big\|\nabla u_2(t)\big\|_4^4 + \big\|u_2(t)\big\|_6^6 + \big\|(I - P_m)g(t)\big\|^2\big) \\
&\quad\le c\big(\big\|\triangle u_2(t)\big\|^2 + \big\|\triangle u_2(t)\big\|^4 + \big\|\triangle u_2(t)\big\|^6 + \big\|(I - P_m)g(t)\big\|^2\big) \\
&\quad\le M_1 + c\big\|(I - P_m)g(t)\big\|^2,
\end{aligned} \tag{3.22}$$

where $M_1 = c(\rho^2 + \rho^4 + \rho^6)$. Thus, we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\triangle u_2(t)\big\|^2 + \lambda_m\big\|\triangle u_2(t)\big\|^2 \le M_1 + c\big\|(I - P_m)g(t)\big\|^2. \tag{3.23}$$

By the Gronwall lemma,

$$\big\|\triangle u_2(t)\big\|^2 \le \big\|\triangle u_2(\tau)\big\|^2 e^{-\lambda_m(t-\tau)} + \frac{M_1}{\lambda_m} + c\int_\tau^t e^{-\lambda_m(t-s)}\big\|(I - P_m)g(s)\big\|^2\,\mathrm{d}s. \tag{3.24}$$

Since $g \in L^2_{C^*}(\mathbb{R}; H)$, by Lemma 2.5, for any $\varepsilon > 0$, there exists an m large enough such that

$$c\int_\tau^t e^{-\lambda_m(t-s)}\big\|(I - P_m)g(s)\big\|^2\,\mathrm{d}s \le \frac{\varepsilon}{3}, \quad \forall g \in \mathcal{H}(h_0), \forall t \ge \tau. \tag{3.25}$$

Let $t_1 = \tau + \frac{1}{\lambda_m}\ln\frac{3\rho^2}{\varepsilon}$. Then we conclude that

$$\big\|\triangle u_2(\tau)\big\|^2 e^{-\lambda_m(t-\tau)} \le \rho^2 e^{-\lambda_m(t-\tau)} \le \frac{\varepsilon}{3}, \quad \forall t \ge t_1. \tag{3.26}$$

Obviously, we can choose $\varepsilon > 0$ such that

$$\frac{M_1}{\lambda_m} \le \frac{\varepsilon}{3}. \tag{3.27}$$

Therefore, combining with (3.24)-(3.27) leads to

$$\big\|\triangle u_2(t)\big\|^2 \le \varepsilon, \quad \forall t \ge t_1, \forall g \in \mathcal{H}(g_0),$$

which indicates that the family of processes $\{U_g(t,\tau)\}$, $g \in \mathcal{H}(g_0)$, satisfies uniformly (w.r.t. $g \in \mathcal{H}(g_0)$) condition (C) in V. The proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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