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Oscillation criteria for a class of nonlinear neutral differential equations

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Abstract
In this paper, we deal with the oscillation of the second order nonlinear neutral differential equations of the form

$$\big(a(t)\big(x(t)+\delta p(t)x(t-\tau)\big)'\big)'+f\big(t,x(t-\sigma)\big)-g\big(t,x(t-\rho)\big)=0.$$

The oscillation criteria for these equations have been obtained. Furthermore, examples are given to illustrate the criteria, respectively.

Keywords: neutral differential equations; bounded oscillation; almost oscillation; bounded almost oscillation

1 Introduction

The differential equations that we study describe many phenomena and dynamical processes in various fields, and they have attracted a great deal of attention of researchers in physical sciences, mathematics, biology, and economy. In addition, these equations play an important role in numerical simulations of nonlinear partial differential equations, queuing problems, and discretization in solid state and quantum physics. For the application, please see [1].

In this paper, we consider the oscillation of second order nonlinear neutral differential equations with mixed type term of the form

$$\big(a(t)\big(x(t)+\delta p(t)x(t-\tau)\big)'\big)'+f\big(t,x(t-\sigma)\big)-g\big(t,x(t-\rho)\big)=0, \tag{1.1}$$

where $\delta=+1$ or -1, $t\ge t_0$, $a(t)$ is a continuously differentiable function, $p(t)$ is a continuous bounded function with $a(t)>0$, $p(t)\ge 0$, $f(t,u)$ and $g(t,v)$ are continuous functions, the constants $\tau,\sigma,\rho\in[0,\infty)$. Denote $\lambda=\max\{\tau,\sigma,\rho\}$, $t_1=t_0+\lambda$, $L^1[t_0,\infty)=\{x(t)\mid\int_{t_0}^{\infty}|x(s)|\,ds<\infty\}$.

The following conditions will be assumed throughout this paper:

(H_1) $\int_t^{\infty}\frac{1}{a(s)}\,ds=\infty$ for all $t\ge t_0$,
(H_2) $\frac{f(t,u)}{u}\ge q(t-\sigma)>0$ for $u\ne 0$ and $0<\frac{g(t,v)}{v}\le r(t-\rho)$ for $v\ne 0$,
(H_3) $0<\frac{f(t,u)}{u}\le q(t-\sigma)$ for $u\ne 0$ and $\frac{g(t,v)}{v}\ge r(t-\rho)>0$ for $v\ne 0$,
(H_4) $\frac{1}{r(t)-q(t)}$ is bounded, where $q,r\in C([t_0,\infty),R^{+})$.

We also assume that $x(t)$ is a nontrivial solution of (1.1). The investigation of oscillatory behavior of solutions of various types of differential equations done by many researchers is motivated by many application problems in physics, biology, ecology, and so on. In particular, an increasing interest in obtaining oscillation criteria for different classes of differential and functional differential equations has been manifested recently. Please see [1–16].

The paper is organized as follows. We will first present criteria for (1.1) when $\delta = +1$ in Section 2 and then for (1.1) when $\delta = -1$ in Section 3. Some examples will be given to illustrate the obtained criteria, respectively. The proofs of the main results are left to Section 4.

2 Statement of the main results when $\delta = +1$

We first rewrite (1.1) as

$$\bigl(a(t)\bigl(x(t)+p(t)x(t-\tau)\bigr)'\bigr)' + f\bigl(t,x(t-\sigma)\bigr) - g\bigl(t,x(t-\rho)\bigr) = 0. \tag{2.1}$$

In this section, four oscillatory criteria will be presented and some illustrated examples will be given.

Theorem 2.1 *Suppose that conditions* (H_1), (H_2) *and* (H_4) *hold*, $q(t) > r(t)$, $r(t)$ *is bounded and* $\sigma \geq \rho$. *Then* (2.1) *is bounded oscillatory.*

Remark 2.2 In [16], the authors considered

$$\bigl(x(t)+p(t)x\bigl(\tau(t)\bigr)\bigr)^{(n)} + q(t)f\bigl(x\bigl(\sigma(t)\bigr)\bigr) = 0$$

and established the criteria for the solution to be oscillatory when $0 \leq p(t) < 1$. Even though this result is about the higher order equations, the generality of our results is not robbed since our equations include a larger class of equations.

Example 1 Consider the differential equation

$$\begin{aligned}&\left(\left(1+\frac{1}{t}\right)\bigl(x(t)+2x(t-2\pi)\bigr)'\right)' + 3\left(1+\frac{1}{t}\right)x(t-2\pi)\\&\quad+3\left(1+\frac{1}{t}\right)x^5(t-2\pi) - \frac{3}{t^2}x\left(t-\frac{\pi}{2}\right) = 0.\end{aligned} \tag{2.2}$$

Viewing (2.2) as (2.1), we have $a(t) = 1 + (1/t)$, $p(t) = 2 > 0$, and

$$q(t) = 3\left(1+\frac{1}{t+2\pi}\right) > r(t) = \frac{3}{(t+\frac{\pi}{2})^2}.$$

Moreover, $\tau = 2\pi$, $\sigma = 2\pi > \rho = \pi/2$, and $r(t)$ is bounded for $t \geq 2\pi$. Note that conditions (H_1), (H_2) and (H_4) are satisfied and, by Theorem 2.1, (2.2) is bounded oscillatory.

Theorem 2.3 *Suppose that conditions* (H_1), (H_2) *and* (H_4) *hold*, $q(t) > r(t)$, $q(t)$, $1/a(t)$ *are bounded and* $\sigma < \rho$. *Then* (2.1) *is almost oscillatory.*

Example 2 Consider the differential equation

$$\big(t\big(x(t)+x(t-\pi)\big)'\big)' + \frac{t\pi}{t-2\pi}x(t-2\pi) - \frac{\pi}{t-\frac{7}{2}\pi}x\left(t-\frac{7}{2}\pi\right) = 0. \tag{2.3}$$

Viewing (2.3) as (2.1), we have $a(t) = t$, $p(t) = 1 > 0$, and

$$q(t) = \frac{(t+2\pi)\pi}{t} > r(t) = \frac{\pi}{t}.$$

Moreover, $\tau = \pi$, $\sigma = 2\pi < \rho = 7\pi/2$, $q(t)$ is bounded for $t \geq 4\pi$. Note that conditions (H_1), (H_2) and (H_4) are satisfied. By Theorem 2.3, (2.3) is almost oscillatory. Indeed, $x(t) = t\sin t$ is an unbounded oscillatory solution of (2.3).

Theorem 2.4 *Suppose that conditions* (H_1), (H_3) *and* (H_4) *hold*, $q(t) < r(t)$, $r(t)$, $1/a(t)$ *are bounded and* $\sigma \geq \rho$. *Then* (2.1) *is bounded almost oscillatory.*

Theorem 2.5 *Suppose that conditions* (H_1), (H_3) *and* (H_4) *hold*, $q(t) < r(t)$, $q(t)$ *is bounded and* $\sigma < \rho$. *Then* (2.1) *is bounded almost oscillatory.*

Example 3 Consider the differential equation

$$\begin{aligned}&\left(\frac{t+1-3\pi}{t-3\pi}\left(x(t)+\frac{t-\pi}{t(t+1-\pi)}x(t-\pi)\right)'\right)'\\&\quad+\frac{2t+2-3\pi}{(t-3\pi)(2t-3\pi)}x\left(t-\frac{3\pi}{2}\right)-x(t-3\pi)=0.\end{aligned} \tag{2.4}$$

Viewing (2.4) as (2.1), we have

$$\begin{aligned}a(t) &= \frac{t+1-3\pi}{t-3\pi},\\ p(t) &= \frac{t-\pi}{t(t+1-\pi)} > 0,\\ q(t) &= \frac{2t+2}{t(2t-3\pi)} < r(t) = 1.\end{aligned}$$

Also,

$$\tau = \pi, \sigma = \frac{3\pi}{2} < \rho = 3\pi \text{ and } q(t) \text{ is bounded for } t \geq 3\pi.$$

We note that conditions (H_1), (H_3) and (H_4) are satisfied and, by Theorem 2.5, (2.4) is bounded almost oscillatory. In fact, $x(t) = (1 + (1/t))\sin t$ is a bounded oscillatory solution of (2.4).

3 Statement of the main results when $\delta = -1$

In this section, we consider (1.1) when $\delta = -1$. So (1.1) becomes

$$\big(a(t)\big(x(t)-p(t)x(t-\tau)\big)'\big)' + f\big(t,x(t-\sigma)\big) - g\big(t,x(t-\rho)\big) = 0. \tag{3.1}$$

Four oscillation criteria have been obtained. In addition, some example will be given to demonstrate the obtained results.

Theorem 3.1 *Suppose that conditions* (H_1), (H_3) *and* (H_4) *hold*, $p(t) \geq 1$, $q(t) < r(t)$, $\sigma \leq \rho$ *and* $r(t)$ *is bounded. Then* (3.1) *is bounded oscillatory.*

Example 4 Consider the differential equation

$$\left(\left(1-\frac{1}{t}\right)\bigl(x(t)-2x(t-\pi)\bigr)'\right)' + \frac{3}{t^2}x\left(t-\frac{\pi}{2}\right) - 3x(t-\pi) = 0. \tag{3.2}$$

Viewing (3.2) as (3.1), we have $\tau = \pi$, $\sigma = \frac{\pi}{2} < \rho = \pi$,

$$\begin{aligned} &a(t) = 1-\frac{1}{t}, \\ &p(t) = 2 > 0, \\ &q(t) = \frac{3}{(t+\frac{\pi}{2})^2} < r(t) = 3 \end{aligned}$$

for $t \geq \pi$. We note that conditions (H_1), (H_3) and (H_4) are satisfied and, by Theorem 3.1, (3.2) is bounded oscillatory.

Theorem 3.2 *Suppose that conditions* (H_1), (H_3) *and* (H_4) *hold*, $q(t) < r(t)$, $\sigma \geq \rho$, $0 \leq p(t) \leq p_1 < 1$ *or* $1 < p_2 \leq p(t)$, $r(t)$ *and* $1/a(t)$ *are bounded. Then* (3.1) *is bounded almost oscillatory.*

Theorem 3.3 *Suppose that conditions* (H_1), (H_2) *and* (H_4) *hold*, $q(t) > r(t)$, $\sigma < \rho$, $0 \leq p(t) \leq p_1 < 1$ *or* $1 < p_2 \leq p(t)$, $q(t)$ *and* $1/a(t)$ *are bounded. Then* (3.1) *is bounded almost oscillatory.*

Example 5 Consider the differential equation

$$\begin{aligned} &\left(\frac{t}{t+1}\left(x(t)-\frac{2(t-2\pi)}{t}x(t-2\pi)\right)'\right)' + \frac{(t^3+t^2-2t-1)(t-\pi)}{2t^2(t+1)^2}x(t-\pi) \\ &\quad - \frac{(2t+1)(t-\frac{3\pi}{2})}{2t(t+1)^2}x\left(t-\frac{3\pi}{2}\right) = 0. \end{aligned} \tag{3.3}$$

Viewing (3.3) as (3.1), we have $\tau = 2\pi$, $\sigma = \pi < \rho = 3\pi/2$,

$$\begin{aligned} &a(t) = \frac{t}{t+1}, \\ &p(t) = \frac{2(t-2\pi)}{t} \geq 1.2 \quad \text{for } t \geq 5\pi, \\ &q(t) = \frac{t((t+\pi)^3+(t+\pi)^2-2t-2\pi-1)}{2(t+\pi)^2(t+\pi+1)^2}, \\ &r(t) = \frac{t(2t+3\pi+1)}{2(t+3\pi/2)(t+3\pi/2+1)^2}. \end{aligned}$$

Clearly, $q(t) > r(t)$ for large t and $1/a(t)$ and $q(t)$ are bounded. We note that conditions (H_1), (H_2) and (H_4) are satisfied and, by Theorem 3.3, (3.3) is bounded almost oscillatory.

Theorem 3.4 *Suppose that conditions* (H_1), (H_2) *and* (H_4) *hold*, $q(t) > r(t)$, $\sigma \geq \rho$, $q(t)$ *is bounded*, $0 \leq p(t) \leq p_1 < 1$, *or* $1/a(t)$ *is bounded and* $1 < p_2 \leq p(t)$. *Then* (3.1) *is bounded almost oscillatory.*

Example 6 Consider the differential equation

$$\left(\frac{t}{t+1}\left(x(t) - \frac{t-2\pi}{2t}x(t-2\pi)\right)'\right)' + \frac{(t^3+t^2-2t-1)(t-2\pi)}{2t^2(t+1)^2}x(t-2\pi) - \frac{(2t+1)(t-\pi/2)}{2t(t+1)^2}x(t-\pi/2) = 0. \tag{3.4}$$

Regarding (3.4) as (3.1), we have $\tau = 2\pi$, $\sigma = 2\pi > \rho = \frac{\pi}{2}$,

$$\begin{aligned} a(t) &= \frac{t}{t+1}, \\ 0 < p(t) &= \frac{t-2\pi}{2t} \leq \frac{1}{2} < 1, \quad t \geq 3\pi, \\ q(t) &= \frac{t((t+2\pi)^3 + (t+2\pi)^2 - 2t - 4\pi - 1)}{2(t+2\pi)^2(t+2\pi+1)^2}, \\ r(t) &= \frac{t(2t+\pi+1)}{2(t+\pi/2)(t+\pi/2+1)^2}. \end{aligned}$$

Clearly, $q(t) > r(t)$ for large enough t and $q(t)$ is bounded. Notice that (H_1), (H_2) and (H_4) are satisfied therefore, by Theorem 3.4, (3.4) is bounded almost oscillatory.

4 Proofs of the main results

Here we will give the proofs of the main results.

Proof of Theorem 2.1 Let $x(t)$ be a bounded non-oscillatory solution. Suppose $x(t)$ is an eventually positive solution. Then there exists $t_2 \geq t_1$ such that $x(t) > 0$ and $x(t-\lambda) > 0$ for $t \geq t_2$. Let

$$z(t) = a(t)\big(x(t) + p(t)x(t-\tau)\big)' - \int_{t-\sigma}^{t-\rho} r(s)x(s)\,ds. \tag{4.1}$$

From (2.1) and (H_2) it follows that

$$z'(t) \leq \big(r(t-\sigma) - q(t-\sigma)\big)x(t-\sigma) < 0, \quad t \geq t_2. \tag{4.2}$$

So $z(t)$ is decreasing, and

$$-\infty \leq \lim_{t\to\infty} z(t) = c < \infty.$$

If $c = -\infty$, from (4.1) and the boundedness of $x(t)$ and $r(t)$, we have

$$\lim_{t\to\infty} a(t)\big(x(t) + p(t)x(t-\tau)\big)' = -\infty.$$

Then there exist $l_1 > 0$ and $t_3 \geq t_2$ such that

$$\big(x(t) + p(t)x(t-\tau)\big)' \leq -\frac{l_1}{a(t)}, \quad t \geq t_3.$$

Integrating both sides of the above inequality, according to (H_1), we obtain

$$\lim_{t\to\infty} \big(x(t) + p(t)x(t-\tau)\big) = -\infty,$$

which contradicts the boundedness of $x(t)$ and $p(t)$. This contradiction shows that $|c| < \infty$, *i.e.*, $z(t)$ is bounded.

From (4.2) it follows that

$$x(t-\sigma) \leq \frac{1}{r(t-\sigma) - q(t-\sigma)} z'(t). \tag{4.3}$$

So $x \in L^1[t_0, \infty)$ by (H_4).

(i) If $c > 0$, from (4.1) we have

$$z(t) \leq a(t)\big(x(t) + p(t)x(t-\tau)\big)', \quad t \geq t_2.$$

Therefore, since $z(t) \to c$ as $t \to \infty$,

$$\big(x(t) + p(t)x(t-\tau)\big)' \geq \frac{c}{a(t)}, \quad t \geq t_2.$$

From (H_1) we have $\lim_{t\to\infty}(x(t) + p(t)x(t-\tau)) = \infty$, which contradicts the boundedness of $x(t)$.

(ii) If $c < 0$, in view of $x \in L^1[t_0, \infty)$, we have

$$\lim_{t\to\infty} \int_{t-\sigma}^{t-\rho} r(s)x(s)\,ds = 0.$$

Then, since $z(t) \to c$ as $t \to \infty$, there exist $\epsilon \in (0, -c)$ and $t_4 \geq t_2$ such that

$$a(t)\big(x(t) + p(t)x(t-\tau)\big)' \leq c + \epsilon < 0, \quad t \geq t_4.$$

Hence, by (H_1) again, we obtain

$$\lim_{t\to\infty} \big(x(t) + p(t)x(t-\tau)\big) = -\infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$.

(iii) If $c = 0$, in view of $z'(t) < 0$, we have $z(t) > 0$. So

$$a(t)\big(x(t) + p(t)x(t-\tau)\big)' > \int_{t-\sigma}^{t-\rho} r(s)x(s)\,ds > 0, \quad t \geq t_2.$$

Since $x(t) + p(t)x(t-\tau)$ is positive and increasing, the integral

$$\int_{t_0}^{\infty} \big(x(t) + p(t)x(t-\tau)\big)\,dt$$

is divergent, a contradiction to $x \in L^1[t_0,\infty)$. The contradictions obtained in the above three cases show that (2.1) has no bounded eventually positive solution. Now suppose that $x(t)$ is a bounded eventually negative solution. Then $x(t-\lambda) < 0$ for some $t_2 > t_1$ and all $t \geq t_2$. From (2.1), (4.1) and (H_2), we have

$$z'(t) \geq \bigl(r(t-\sigma) - q(t-\sigma)\bigr)x(t-\sigma) > 0, \quad t \geq t_2. \tag{4.4}$$

So $z(t)$ is increasing and $-\infty < \lim_{t\to\infty} z(t) = c \leq \infty$. Then an argument parallel to the above also leads to contradictions. Therefore, every bounded solution of (2.1) is oscillatory. □

Proof of Theorem 2.3 Without loss of generality, suppose that $x(t)$ is an eventually positive solution. Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for all $t \geq t_2$. Let

$$z(t) = a(t)\bigl(x(t) + p(t)x(t-\tau)\bigr)' + \int_{t-\rho}^{t-\sigma} q(s)x(s)\,ds. \tag{4.5}$$

From (2.1) it follows that

$$z'(t) \leq \bigl(r(t-\rho) - q(t-\rho)\bigr)x(t-\rho) < 0, \quad t \geq t_2. \tag{4.6}$$

So $z(t)$ is decreasing and

$$-\infty \leq \lim_{t\to\infty} z(t) = c < \infty.$$

If $c = -\infty$, then

$$\lim_{t\to\infty} a(t)\bigl(x(t) + p(t)x(t-\tau)\bigr)' = -\infty.$$

By (H_1), we obtain $\lim_{t\to\infty}(x(t) + p(t)x(t-\tau)) = -\infty$, which contradicts $x(t) + p(t)x(t-\tau) > 0$. Therefore $|c| < \infty$ so $z(t)$ is bounded.

From (4.6) we have

$$x(t-\rho) \leq \frac{1}{r(t-\rho) - q(t-\rho)} z'(t), \quad t \geq t_2 \tag{4.7}$$

so, by (H_4), $x \in L^1[t_0,\infty)$ and $\lim_{t\to\infty} \int_{t-\rho}^{t-\sigma} q(s)x(s)\,ds = 0$. Since $1/a(t)$ is bounded, by (4.5), $(x(t) + p(t)x(t-\tau))'$ is bounded. This implies that $x(t) + p(t)x(t-\tau)$ is uniformly continuous on $[t_1,\infty)$. Note that the property $x \in L^1[t_0,\infty)$ and the boundedness of $p(t)$ imply that $x(t) + p(t)x(t-\tau) \in L^1[t_0,\infty)$. Hence $\lim_{t\to\infty}(x(t) + p(t)x(t-\tau)) = 0$, so $\lim_{t\to\infty} x(t) = 0$. Therefore, every solution x of (2.1) which is not in the class of $o(1)$ as $t \to \infty$ is oscillatory. □

Proof of Theorem 2.4 Without loss of generality, assume that $x(t)$ is a bounded eventually positive solution and $z(t)$ is defined by (4.1). Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for $t \geq t_2$. From (2.1) and (H_3), we have

$$z'(t) \geq \bigl(r(t-\sigma) - q(t-\sigma)\bigr)x(t-\sigma) > 0, \quad t \geq t_2. \tag{4.8}$$

So $z(t)$ is increasing. Then

$$-\infty < \lim_{t\to\infty} z(t) = d \leq \infty.$$

If $\lim_{t\to\infty} z(t) = \infty$, then from (4.1) and the boundedness of $x(t)$ and $r(t)$, we obtain

$$\lim_{t\to\infty} a(t)\big(x(t) + p(t)x(t-\tau)\big)' = \infty.$$

Then there exist $l_2 > 0$ and $t_3 \geq t_2$ such that

$$a(t)\big(x(t) + p(t)x(t-\tau)\big)' \geq l_2, \quad t \geq t_3.$$

From (H_1) it follows that

$$\lim_{t\to\infty} \big(x(t) + p(t)x(t-\tau)\big) = \infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$. So $|d| < \infty$ and $z(t)$ is bounded. From (4.8) we have

$$x(t-\sigma) \leq \frac{1}{r(t-\sigma) - q(t-\sigma)} z'(t), \quad t \geq t_2.$$

Therefore, by (H_4), $x \in L^1[t_0, \infty)$. By the same reasoning as that used in the proof of Theorem 2.3, we have $\lim_{t\to\infty} x(t) = 0$. Therefore, every bounded solution x of (2.1) which is not in the class of $o(1)$ as $t \to \infty$ must be oscillatory. □

Proof of Theorem 2.5 Without loss of generality, suppose that $x(t)$ is a bounded eventually positive solution. Let $z(t)$ be defined by (4.5). Take $t_2 \geq t_1$ such that $x(t-\lambda) > 0$ for $t \geq t_2$. From (2.1) and (H_3), we have

$$z'(t) \geq \big(r(t-\rho) - q(t-\rho)\big)x(t-\rho) > 0, \quad t \geq t_2. \tag{4.9}$$

Hence $z(t)$ is increasing and

$$-\infty < \lim_{t\to\infty} z(t) = d \leq \infty.$$

By using the method similar to that used in the proof of Theorem 2.4, we have $-\infty < d < \infty$. Therefore $z(t)$ is bounded. From (4.9) it follows that

$$x(t-\rho) \leq \frac{1}{r(t-\rho) - q(t-\rho)} z'(t), \quad t \geq t_2.$$

Thus, by (H_4), $x \in L^1[t_0, \infty)$ and $\lim_{t\to\infty} \int_{t-\rho}^{t-\sigma} q(s)x(s)\,ds = 0$. Then it follows from (4.5) that

$$\lim_{t\to\infty} a(t)\big(x(t) + p(t)x(t-\tau)\big)' = d.$$

(i) If $d > 0$, then there exists $t_5 \geq t_2$ such that

$$a(t)\big(x(t) + p(t)x(t-\tau)\big)' \geq \frac{d}{2}, \quad t \geq t_5.$$

From (H_1) we have

$$\lim_{t\to\infty}\big(x(t)+p(t)x(t-\tau)\big)=\infty,$$

which contradicts the boundedness of $x(t)$ and $p(t)$.

(ii) If $d<0$, similar to the case (i), we have

$$\lim_{t\to\infty}\big(x(t)+p(t)x(t-\tau)\big)=-\infty,$$

a contradiction to the boundedness of $x(t)$ and $p(t)$ again. Hence $d=0$, *i.e.*, $\lim_{t\to\infty} z(t)=0$. On the other hand, from (4.9) and $\lim_{t\to\infty} z(t)=0$, we have $z(t)<0$. In view of (4.5), $(x(t)+p(t)x(t-\tau))'<0$, which implies that $x(t)+p(t)x(t-\tau)$ is decreasing. From $x(t)+p(t)x(t-\tau)\in L^1[t_0,\infty)$, we have $\lim_{t\to\infty}(x(t)+p(t)x(t-\tau))=0$. Thus $\lim_{t\to\infty} x(t)=0$. Therefore, every bounded solution x of (2.1) which is not in the class of $o(1)$ as $t\to\infty$ must be oscillatory. □

Proof of Theorem 3.1 Suppose that $x(t)$ is a bounded non-oscillatory solution. Without loss of generality, we assume that $x(t)$ is an eventually positive solution. Let

$$z(t)=a(t)\big(x(t)-p(t)x(t-\tau)\big)'+\int_{t-\rho}^{t-\sigma}q(s)x(s)\,ds. \tag{4.10}$$

From a proof similar to that of Theorem 2.4, we obtain

$$z'(t)>0,\ \lim_{t\to\infty} z(t)=c, |c|<\infty,\ \text{and } x\in L^1[t_0,\infty).$$

(i) If $c>0$, from (4.10) it follows that

$$\lim_{t\to\infty} a(t)\big(x(t)-p(t)x(t-\tau)\big)'=c>\frac{c}{2}.$$

So, for large enough t,

$$\big(x(t)-p(t)x(t-\tau)\big)'\geq\frac{c}{2a(t)}.$$

Hence $\lim_{t\to\infty}(x(t)-p(t)x(t-\tau))=\infty$ by (H_1), which contradicts the boundedness of $x(t)$ and $p(t)$.

(ii) If $c<0$, in view of $\lim_{t\to\infty} z(t)=c$ and $x\in L^1[t_0,\infty)$, there exists $t_6\geq t_1$ such that

$$a(t)\big(x(t)-p(t)x(t-\tau)\big)'\leq\frac{c}{2}<0,\quad t\geq t_6.$$

Hence $\lim_{t\to\infty}(x(t)-p(t)x(t-\tau))=-\infty$ by (H_1), a contradiction to the boundedness of $x(t)$ and $p(t)$ again.

(iii) If $c=0$, in view of $z'(t)>0$, we have $z(t)<0$. Further,

$$\big(x(t)-p(t)x(t-\tau)\big)'<0.$$

We show that $x(t)-p(t)x(t-\tau)>0$. In fact, if there exists $t_7 \geq t_1$ such that $x(t_7)-p(t_7)x(t_7-\tau)<0$, then, for all $t \geq t_7$,

$$x(t)-p(t)x(t-\tau) \leq x(t_7)-p(t_7)x(t_7-\tau)<0.$$

This contradicts $x(t)-p(t)x(t-\tau) \in L^1[t_0,\infty)$. Hence $x(t)-p(t)x(t-\tau)>0$ for all large $t \geq t_1$. From this and the assumption on p, we have $x(t) \geq p(t)x(t-\tau) \geq x(t-\tau)$, which contradicts $x \in L^1[t_0,\infty)$. Thus (3.1) is bounded oscillatory. □

Proof of Theorem 3.2 Without loss of generality, assume that $x(t)$ is a bounded eventually positive solution. By a proof similar to that of Theorem 2.4, we obtain $\lim_{t\to\infty}(x(t)-p(t)x(t-\tau))=0$. Suppose

$$\limsup_{t\to\infty} x(t)=l>0.$$

So there exists a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty} x(t_k)=l>0.$$

(i) If $0 \leq p(t) \leq p_1 < 1$, then we have $(1-p_1)l \leq 0$, which contradicts $l>0$ and $1-p_1>0$.

(ii) If $1<p_2 \leq p(t)$, then we have $0 \leq (1-p_2)l$, which contradicts $l>0$ and $p_2-1>0$.

Therefore, we must have

$$\limsup_{t\to\infty} x(t)=0.$$

Then $\lim_{t\to\infty} x(t)=0$ as $x(t)$ is eventually positive. This shows that (3.1) is bounded almost oscillatory. □

Proof of Theorem 3.3 Without loss of generality, suppose that $x(t)$ is a bounded eventually positive solution. As in the proof of Theorem 2.3, we obtain

$$\lim_{t\to\infty}\bigl(x(t)-p(t)x(t-\tau)\bigr)=0.$$

Then the rest follows from the proof of Theorem 3.2. □

Proof of Theorem 3.4 Without loss of generality, suppose that $x(t)$ is a bounded eventually positive solution. Let

$$z(t)=a(t)\bigl(x(t)-p(t)x(t-\tau)\bigr)'-\int_{t-\sigma}^{t-\rho} r(s)x(s)\,ds. \tag{4.11}$$

By the reasoning similar to that used in the proof of Theorem 2.1, we have $x \in L^1[t_0,\infty)$, $\lim_{t\to\infty} z(t)=c=0$ and $z'(t)<0$. So $z(t)>0$, $(x(t)-p(t)x(t-\tau))'>0$ and $x(t)-p(t)x(t-\tau)$ is increasing. We claim that $x(t)-p(t)x(t-\tau)<0$ for $t \geq t_1$. In fact, if there exists $t_8 \geq t_1$ such that $x(t_8)-p(t_8)x(t_8-\tau) \geq 0$, then $x(t)-p(t)x(t-\tau) \geq x(t_8+1)-p(t_8+1)x(t_8+1-\tau)>0$ for $t \geq t_8+1$, which contradicts $x(t)-p(t)x(t-\tau) \in L^1[t_1,\infty)$. Hence $x(t)-p(t)x(t-\tau)<0$

for all $t \geq t_1$. If $0 \leq p(t) \leq p < 1$ is satisfied, then $x(t) < px(t-\tau)$ for all $t \geq t_1$. This implies that $\lim_{t\to\infty} x(t) = 0$.

If $1/a(t)$ is bounded and $1 < p_2 \leq p(t)$, from the proof of Theorem 3.2, we have $\lim_{t\to\infty}(x(t) - p(t)x(t-\tau)) = 0$ and thus $\lim_{t\to\infty} x(t) = 0$. Therefore, (3.1) is bounded almost oscillatory. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors carried out the proof. All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.

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References

1. Gepreel, K, Shehata, AR: Rational Jacobi elliptic solutions for nonlinear difference differential lattice equation. Appl. Math. Lett. **25**, 1173-1178 (2012)
2. Al-Hamouri, R, Zein, A: Oscillation criteria for certain even order neutral delay differential equations. Int. J. Differ. Equ. **2014**, Article ID 437278 (2014)
3. Baculíková, B, Džurina, J: Oscillation theorems for second order neutral differential equations. Comput. Math. Appl. **61**, 94-99 (2011)
4. Elabbay, EM, Hegazi, AS, Saker, SH: Oscillation of solutions to delay differential equations with positive and negative coefficients. Electron. J. Differ. Equ. **2000**, 13 (2000)
5. Fu, XL, Li, TX, Zhang, CH: Oscillation of second-order damped differential equations. Adv. Differ. Equ. **2013**, Article ID 326 (2013)
6. Gai, MJ, Shi, B, Zhang, DC: Oscillations criteria for second order nonlinear differential equations of neutral type. Appl. Math. J. Chin. Univ. Ser. B **16**(2), 122-126 (2001)
7. Grammatikopoulos, MK, Ladas, G, Meimaridou, A: Oscillation and asymptotic behavior of second order neutral differential equations. Ann. Math. Pures Appl. **148**, 29-40 (1987)
8. Hasanbulli, M, Rogovchenko, YV: Oscillation criteria for second order nonlinear neutral differential equations. Appl. Math. Comput. **215**, 4392-4399 (2010)
9. Karpuz, B, Öcalan, Ö, Özturk, S: Comparison theorem on the oscillation and asymptotics behaviour of higher-order neutral differential equations. Glasg. Math. J. **52**, 107-114 (2010)
10. Li, WT, Agarwal, RP: Interval oscillations criteria for second-order nonlinear differential equations with damping. Comput. Math. Appl. **40**, 217-230 (2000)
11. Li, TX, Rogovchenko, YV, Zhang, CH: Oscillation results for second-order nonlinear neutral differential equations. Adv. Differ. Equ. **2013**, Article ID 336 (2013)
12. Li, HJ, Yeh, CC: Oscillations criteria for second-order neutral delay difference equations. Comput. Appl. Math. **36**(10-12), 123-132 (1998)
13. Qin, HZ, Shang, N, Lu, YM: A note on oscillation criteria of second order nonlinear neutral delay differential equations. Comput. Math. Appl. **56**, 2987-2992 (2008)
14. Ruan, S: Oscillations of second order neutral differential equations. Can. Math. Bull. **36**(4), 485-496 (1993)
15. Xing, GJ, Li, TX, Zhang, CH: Oscillation of higher-order quasi-linear neutral differential equations. Adv. Differ. Equ. **2011**, Article ID 45 (2011)
16. Zhang, QX, Yan, JR, Gao, L: Oscillation behavior of even-order nonlinear neutral differential equations with variable coefficients. Comput. Math. Appl. **59**, 426-430 (2010)