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Successive iteration and positive extremal solutions for nonlinear impulsive q_k-difference equations

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Abstract
This paper investigates the existence of positive extremal solutions for nonlinear impulsive q_k-difference equations via a monotone iterative method. The main result is well illustrated with the aid of an example.

MSC: 34B37; 39A13

Keywords: successive iteration; q_k-difference equation; impulse; positive; extremal solutions

1 Introduction

In recent years, there has been put focus on developing the existence theory for initial and boundary value problems of q-difference equations and inclusions. The pioneer work on the topic dates back to the first quarter of the 20th century. In contrast to the classical definition of the derivative, the concept of q-calculus does not involve the idea of limit. The importance of q-difference equations lies in the fact that these equations are always completely controllable and appear in the q-optimal control problems [1]. The q-analog of continuous variational calculus, known as variational q-calculus, is regarded as a generalization of the continuous variational calculus due to the presence of an extra parameter q, which may be physical or economical in nature. The variational calculus on a q-uniform lattice helps to find the extremum of the functional involved in Lagrange problems of q-Euler equations rather than solving the Euler-Lagrange equation itself [2]. The applications of q-calculus appear in several disciplines such as special functions, supersymmetry, control theory, operator theory, combinatorics, initial and boundary value problems of q-difference equations, *etc.* For details and examples, we refer the reader to the books [3–6] and [7–18]. In a recent paper [17], the authors discussed the existence and uniqueness of solutions for impulsive q_k-difference equations. However, it has been found that the study of q_k-difference equations is still at its initial phase and needs further attention.

Motivated by [17], in this paper, we obtain positive extremal solutions for a new class of nonlinear impulsive q_k-difference equations by the method of successive iterations. Precisely, we investigate the following problem:

$$\begin{cases} D_{q_k}u(t) = f(t, u(t)), & 0 < q_k < 1, t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), & k = 1, 2, \dots, m, \\ u(0) = \lambda u(\eta) + d, & \eta \in J_r, r \in \mathbb{Z}, \end{cases} \tag{1.1}$$

where D_{q_k} are q_k-derivatives ($k = 0, 1, 2, \dots, m$), $f \in C(J \times \mathbb{R}, \mathbb{R}^+)$, $I_k \in C(\mathbb{R}, \mathbb{R}^+)$, $J = [0, T]$, $T > 0$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \backslash \{t_1, t_2, \dots, t_m\}$, $J_r = (t_r, T]$, $0 \le \lambda < 1$, $d \ge 0$, $0 \le r \le m$ and $\triangle u(t_k) = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$, and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ ($k = 1, 2, \dots, m$), respectively.

The paper is organized as follows. In Section 2, we present some preliminary material and prove an auxiliary lemma, which plays a pivotal role in establishing the main result. Section 3 contains the main result, while an illustrative example is presented in Section 4.

2 Preliminaries

Let us fix $J_0 = [0, t_1], J_1 = (t_1, t_2], \dots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, T]$ with $T = t_{m+1}$ and introduce the Banach space

$$PC(J, \mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k), k = 0, 1, \dots, m, \text{and } u(t_k^+) \text{ exist}, k = 1, 2, \dots, m\}$$

with the norm $\|u\| = \sup_{t \in J} |u(t)|$.

Next we outline some basic concepts of the q_k-calculus [17].

For $0 < q_k < 1$ and $t \in J_k$, we define the q_k-derivatives of a real valued continuous function f as

$$D_{q_k}f(t) = \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \qquad D_{q_k}f(t_k) = \lim_{t \to t_k} D_{q_k}f(t). \tag{2.1}$$

Higher order q_k-derivatives are given by

$$D_{q_k}^0 f(t) = f(t), \qquad D_{q_k}^n f(t) = D_{q_k} D_{q_k}^{n-1} f(t), \quad n \in \mathbb{N}, t \in J_k.$$

The q_k-integral of a function f is defined by

$$_{t_k}\mathcal{I}_{q_k} f(t) := \int_{t_k}^t f(s)\, d_{q_k}s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1 - q_k^n)t_k), \quad t \in J_k, \tag{2.2}$$

provided the series converges. If $a \in (t_k, t)$ and f is defined on the interval (t_k, t), then

$$\int_a^t f(s)\, d_{q_k}s = \int_{t_k}^t f(s)\, d_{q_k}s - \int_{t_k}^a f(s)\, d_{q_k}s.$$

Observe that

$$D_{q_k}\big(_{t_k}\mathcal{I}_{q_k} f(t)\big) = D_{q_k} \int_{t_k}^t f(s)\, d_{q_k}s = f(t),$$

$$_{t_k}\mathcal{I}_{q_k}\big(D_{q_k} f(t)\big) = \int_{t_k}^t D_{q_k} f(s)\, d_{q_k}s = f(t),$$

$$ {}_a\mathcal{I}_{q_k}\big(D_{q_k}f(t)\big) = \int_a^t D_{q_k}f(s)\,d_{q_k}s = f(t) - f(a), \quad a \in (t_k, t). $$

In the case $t_k = 0$ and $q_k = q$ in (2.1) and (2.2), then $D_{q_k}f = D_qf$, ${}_{t_k}\mathcal{I}_{q_k}f = {}_0\mathcal{I}_qf$, where D_q and ${}_0\mathcal{I}_q$ are the well-known q-derivative and q-integral of the function $f(t)$ defined by

$$ D_qf(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad {}_0\mathcal{I}_qf(t) = \int_0^t f(s)\,d_qs = \sum_{n=0}^{\infty} t(1-q)q^n f\big(tq^n\big). $$

Lemma 2.1 *For a given $\sigma(t) \in C(J, \mathbb{R})$, a function $u \in PC(J, \mathbb{R})$ is a solution of the linear impulsive q_k-difference equations*

$$ \begin{cases} D_{q_k}u(t) = \sigma(t), & 0 < q_k < 1, t \in J', \\ \triangle u(t_k) = I_k(u(t_k)), & k = 1, 2, \ldots, m, \\ u(0) = \lambda u(\eta) + d, & \eta \in J_r, \end{cases} \tag{2.3} $$

if and only if u satisfies the following impulsive q_k-integral equations:

$$ u(t) = \begin{cases} \int_0^t \sigma(s)\,d_{q_0}s + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta}\sigma(s)\,d_{q_r}s \\ \quad + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i(u(t_i)) + \frac{d}{1-\lambda}, \quad t \in J_0; \\ \int_{t_k}^t \sigma(s)\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)) + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta}\sigma(s)\,d_{q_r}s \\ \quad + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i(u(t_i)) + \frac{d}{1-\lambda}, \quad t \in J_k. \end{cases} \tag{2.4} $$

Proof As argued in [18], the solution of the q_k-difference equations (2.3) can be written as

$$ u(t) = u(0) + \int_{t_k}^t \sigma(s)\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum_{i=1}^{k} I_i\big(u(t_i)\big), \quad t \in J_k. \tag{2.5} $$

Substituting $t = \eta$ in (2.5), we have

$$ u(\eta) = u(0) + \int_{t_r}^{\eta} \sigma(s)\,d_{q_r}s + \sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum_{i=1}^{r} I_i\big(u(t_i)\big), \quad \eta \in J_r. \tag{2.6} $$

Using the nonlocal boundary condition (2.3), we obtain

$$ \begin{aligned} u(t) = {}& \int_{t_k}^t \sigma(s)\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \sum_{i=1}^{k} I_i\big(u(t_i)\big) + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta}\sigma(s)\,d_{q_r}s \\ & + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}}\sigma(s)\,d_{q_i}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i\big(u(t_i)\big) + \frac{d}{1-\lambda}, \quad t \in J_k. \end{aligned} \tag{2.7} $$

Conversely, assume that $u(t)$ satisfies the q_k-integral equation (2.4). Then, by employing the operator D_{q_k} on both sides of (2.4) and applying $t = \eta$, we obtain (2.3). This completes the proof. □

3 Main result

Define a cone $P \subset PC(J,\mathbb{R})$ by

$$P = \{u \in PC(J,\mathbb{R}) : u(t) \geq 0, t \in J\}$$

and an operator $\mathscr{Q} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}(\mathscr{Q}u)(t) = {} & \int_{t_k}^{t} f(s,u(s))\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f(s,u(s))\,d_{q_i}s + \sum_{i=1}^{k} I_i(u(t_i)) \\ & + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta} f(s,u(s))\,d_{q_r}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}} f(s,u(s))\,d_{q_i}s \\ & + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i(u(t_i)) + \frac{d}{1-\lambda}. \end{aligned} \tag{3.1}$$

For the sake of convenience, we introduce some notations and assumptions.

Let $b, c > 0$ be constants and $R = b + c + \frac{d}{1-\lambda}$.

(H_1) $f(t,\cdot)$ is nondecreasing on $J \times [0,R]$, and $f(t,u) \leq \frac{b}{M}$ on $J \times [0,R]$, where $M = T + \frac{\lambda\eta}{1-\lambda}$.
(H_2) $I_k(\cdot)$, $k = 1,2,\dots,m$, are nondecreasing on $[0,R]$, and $I_k(u) \leq \frac{(1-\lambda)c}{m}$ on $[0,R]$.
(H_3) $f(t,0) \not\equiv 0$ on any subinterval of J.

We construct two explicit monotone iterative sequences, which converge to positive extremal solutions of nonlinear impulsive q_k-difference equations (1.1):

$$\begin{aligned} v_{n+1}(t) = {} & \int_{t_k}^{t} f(s,v_n(s))\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f(s,v_n(s))\,d_{q_i}s + \sum_{i=1}^{k} I_i(v_n(t_i)) \\ & + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta} f(s,v_n(s))\,d_{q_r}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}} f(s,v_n(s))\,d_{q_i}s \\ & + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i(v_n(t_i)) + \frac{d}{1-\lambda}, \quad \text{with initial value } v_0(t) = 0, \\ u_{n+1}(t) = {} & \int_{t_k}^{t} f(s,u_n(s))\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f(s,u_n(s))\,d_{q_i}s + \sum_{i=1}^{k} I_i(u_n(t_i)) \\ & + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta} f(s,u_n(s))\,d_{q_r}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}} f(s,u_n(s))\,d_{q_i}s \\ & + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i(u_n(t_i)) + \frac{d}{1-\lambda}, \quad \text{with initial value } u_0(t) = R. \end{aligned} \tag{3.2}$$

Recall that a solution u^* of problem (1.1) is called maximal (minimal) if $u^* \geq (\leq) u$ holds for any solution u of problem (1.1). The maximal and minimal solutions of problem (1.1) are called its extremal solutions.

Theorem 3.1 *If assumptions* (H_1)-(H_3) *hold, then the nonlinear impulsive q_k-difference equations* (1.1) *has positive extremal solutions v^*, u^* in* $(0,R]$, *which can be achieved by* *monotone iterative sequences defined by* (3.2). *Moreover, the following relation holds*:

$$v_0 \le v_1 \le \cdots \le v_n \cdots \le v^* \le \cdots \le u^* \cdots \le u_n \le \cdots \le u_1 \le u_0. \tag{3.3}$$

Proof By Lemma 2.1, one can transform the nonlocal boundary value problem (1.1) to an equivalent fixed point problem: $u = \mathcal{Q}u$. That is, a fixed point of the operator equation $u = \mathcal{Q}u$ is a solution of the problem (1.1).

Obviously, $\mathcal{Q} : P \to P$. By a similar process to that employed in [18], it is easy to show that $\mathcal{Q} : P \to P$ is completely continuous.

Denote a ball $B = \{u \in P, \|u\| \le R\}$. Now, we show that $\mathcal{Q}(B) \subset B$. Then, for $u \in B$, by (H_1) and (H_2), we have

$$\begin{aligned}
\bigl|(\mathcal{Q}u)(t)\bigr| &= \int_{t_k}^{t} f\bigl(s,u(s)\bigr)\,d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} f\bigl(s,u(s)\bigr)\,d_{q_i}s + \sum_{i=1}^{k} I_i\bigl(u(t_i)\bigr) \\
&\quad + \frac{\lambda}{1-\lambda}\int_{t_r}^{\eta} f\bigl(s,u(s)\bigr)\,d_{q_r}s + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}} f\bigl(s,u(s)\bigr)\,d_{q_i}s \\
&\quad + \frac{\lambda}{1-\lambda}\sum_{i=0}^{r} I_i\bigl(u(t_i)\bigr) + \frac{d}{1-\lambda} \\
&\le \frac{b}{M}\left[\int_{t_k}^{t} d_{q_k}s + \sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} d_{q_i}s\right] + \frac{k(1-\lambda)c}{m} \\
&\quad + \frac{\lambda b}{(1-\lambda)M}\left[\int_{t_r}^{\eta} d_{q_r}s + \sum_{i=0}^{r-1}\int_{t_i}^{t_{i+1}} d_{q_i}s\right] + \frac{r\lambda c}{m} + \frac{d}{1-\lambda} \\
&\le \frac{b}{M}\left[t - t_k + \sum_{i=0}^{k-1}(t_{i+1}-t_i)\right] + \frac{k(1-\lambda)c}{m} + \frac{\lambda b}{(1-\lambda)M}\left[\eta - t_r + \sum_{i=0}^{r-1}(t_{i+1}-t_i)\right] \\
&\quad + \frac{r\lambda c}{m} + \frac{d}{1-\lambda} \\
&\le \frac{bT}{M} + (1-\lambda)c + \frac{\eta\lambda b}{(1-\lambda)M} + \lambda c + \frac{d}{1-\lambda} \\
&\le R,
\end{aligned} \tag{3.4}$$

which implies that $\|\mathcal{Q}u\| \le R$. Thus $\mathcal{Q}(B) \subset B$.

Next, let us denote the iterative sequence $v_{n+1}(t) = \mathcal{Q}v_n(t)$ $(n = 0,1,2,\ldots)$ and pick $v_0(t) = 0$. Then $v_1 = \mathcal{Q}v_0 = \mathcal{Q}0$, $\forall t \in J$. In view of $v_0(t) = 0 \in B$ and $\mathcal{Q} : B \to B$, it follows that $v_n \in \mathcal{Q}(B) \subset B$ $(n = 0,1,2,\ldots)$. Thus, we have

$$v_1(t) = (\mathcal{Q}0)(t) \ge 0 = v_0(t), \quad \forall t \in J.$$

Applying the conditions (H_1) and (H_2), it is easy to prove the operator $\mathcal{Q}$ is nondecreasing.

So, we have

$$v_2(t) = (\mathcal{Q}v_1)(t) \ge (\mathcal{Q}v_0)(t) = v_1(t), \quad \forall t \in J.$$

By mathematical induction, one can show that the sequence $\{v_n\}_{n=1}^{\infty}$ satisfies

$$v_{n+1}(t) \geq v_n(t), \quad \forall t \in J, n = 0, 1, 2, \ldots. \tag{3.5}$$

Similarly, we denote the iterative sequence $u_{n+1}(t) = \mathcal{Q}u_n(t)$ ($n = 0, 1, 2, \ldots$) and pick $u_0(t) = R$. Then $u_1 = \mathcal{Q}u_0$. In view of $u_0(t) = R \in B$ and $\mathcal{Q} : B \to B$, $u_n \in \mathcal{Q}(B) \subset B$ ($n = 0, 1, 2, \ldots$). Thus, by (H_1) and (H_2), we obtain

$$\begin{aligned} u_1(t) &= \int_{t_k}^{t} f\bigl(s, u_0(s)\bigr)\, d_{q_k}s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f\bigl(s, u_0(s)\bigr)\, d_{q_i}s + \sum_{i=1}^{k} I_i\bigl(u_0(t_i)\bigr) \\ &\quad + \frac{\lambda}{1-\lambda} \int_{t_r}^{\eta} f\bigl(s, u_0(s)\bigr)\, d_{q_r}s + \frac{\lambda}{1-\lambda} \sum_{i=0}^{r-1} \int_{t_i}^{t_{i+1}} f\bigl(s, u_0(s)\bigr)\, d_{q_i}s \\ &\quad + \frac{\lambda}{1-\lambda} \sum_{i=0}^{r} I_i\bigl(u_0(t_i)\bigr) + \frac{d}{1-\lambda} \\ &\leq \frac{b}{M}\left[\int_{t_k}^{t} d_{q_k}s + \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} d_{q_i}s\right] + \frac{k(1-\lambda)c}{m} \\ &\quad + \frac{\lambda b}{(1-\lambda)M}\left[\int_{t_r}^{\eta} d_{q_r}s + \sum_{i=0}^{r-1} \int_{t_i}^{t_{i+1}} d_{q_i}s\right] + \frac{r\lambda c}{m} + \frac{d}{1-\lambda} \\ &\leq \frac{bT}{M} + (1-\lambda)c + \frac{\eta\lambda b}{(1-\lambda)M} + \lambda c + \frac{d}{1-\lambda} \\ &\leq R = u_0(t), \quad \forall t \in J. \end{aligned}$$

Noting that $\mathcal{Q}$ is nondecreasing, we have

$$u_2(t) = (\mathcal{Q}u_1)(t) \leq (\mathcal{Q}u_0)(t) = u_1(t), \quad \forall t \in J.$$

Again, by mathematical induction, it can be shown that the sequence $\{u_n\}_{n=1}^{\infty}$ satisfies

$$u_{n+1}(t) \leq u_n(t), \quad \forall t \in J, n = 0, 1, 2, \ldots. \tag{3.6}$$

By the complete continuity of the operator $\mathcal{Q}$, the sequences $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are relative compact. It means that $\{v_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ have convergent subsequences $\{v_{n_k}\}_{k=1}^{\infty}$ and $\{u_{n_k}\}_{k=1}^{\infty}$, respectively, and there exist $v^*, u^* \in B$ such that $v_{n_k} \to v^*$, $u_{n_k} \to u^*$ as $k \to \infty$. Using this fact together with (3.5) and (3.6) yields

$$\lim_{n\to\infty} v_n = v^*, \qquad \lim_{n\to\infty} u_n = u^*.$$

In consequence, from the continuity of the operator $\mathcal{Q}$, it follows that $\mathcal{Q}v^* = v^*$, $\mathcal{Q}u^* = u^*$. This means that u^* and v^* are two solutions of problem (1.1).

Finally, we prove that u^* and v^* are positive extremal solutions of problem (1.1) in $(0, R]$.

If $w \in [0, R]$ is any solution of problem (1.1), then $\mathcal{Q}w = w$ and $v_0(t) \leq w(t) \leq u_0(t)$. This implies that $\mathcal{Q}$ is nondecreasing and that

$$v_n(t) \leq w(t) \leq u_n(t), \quad \forall t \in J, n = 0, 1, 2, \ldots. \tag{3.7}$$

Thus, employing (3.5)-(3.7), we obtain

$$v_0 \leq v_1 \leq \cdots \leq v_n \cdots \leq v^* \leq w \leq u^* \cdots \leq u_n \leq \cdots \leq u_1 \leq u_0. \tag{3.8}$$

In view of $f(t,0) \not\equiv 0$, $\forall t \in J$, 0 is not a solution of the problem (1.1). Consequently, it follows from (3.8) that u^* and v^* are positive extremal solutions of the nonlinear impulsive q_k-difference equations (1.1) in $(0,R]$, which can be achieved by monotone iterative sequences given in (3.2).

This completes the proof. □

4 Example

Example 4.1 Consider the impulsive nonlocal boundary value problem of the nonlinear q_k-difference equation

$$\begin{cases} D_{\frac{1}{5+k}} u(t) = f(t,u) = t^3 + \frac{t}{10}u^2, & t \in [0, \frac{1}{2}], t \neq \frac{k}{3+k}, \\ \triangle u(\frac{k}{3+k}) = \arctan u(\frac{k}{3+k}), & k = 1,2, \\ u(0) = \frac{1}{8}u(\frac{1}{4}) + \frac{7}{8}, \end{cases} \tag{4.1}$$

where $q_k = \frac{1}{5+k}$ $(k = 0,1,2)$, $t_k = \frac{k}{3+k}$ $(k = 1,2)$, $m = 2$, $\lambda = \frac{1}{8}$, $\eta = \frac{1}{4}$, $d = \frac{7}{8}$, $M = \frac{15}{28}$, $f(t,u) = t^3 + \frac{t}{10}u^2$, and $I_k(u) = \arctan u$. Taking $b = 4 - \pi$, $c = \pi$, $R = 5$, it is easy to verify that all conditions of Theorem 3.1 hold. Hence, by Theorem 3.1, the problem (4.1) has positive extremal solutions in $(0,5]$, which can be achieved by the monotone iterative sequences given by (3.2).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, RPA, GW, BA, LZ, and AH, contributed to each part of this work equally and read and approved the final version of the manuscript.

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